# A POWER FUNCTION WITH A FIXED FINITE GAP EVERYWHERE

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Abstract. We give an application of the extender based Radin forcing to cardinal arithmetic. Assuming $\kappa$ is a large enough cardinal we construct a model satisfying $2^\kappa = \kappa^{+n}$ together with $2^\lambda = \lambda^{+n}$ for each cardinal $\lambda < \kappa$, where $0 < n < \omega$. The cofinality of $\kappa$ can be set arbitrarily or $\kappa$ can remain inaccessible.

When $\kappa$ remains an inaccessible, $V_\kappa$ is a model of ZFC satisfying $2^\lambda = \lambda^{+n}$ for all cardinals $\lambda$.

## 1. Introduction

Investigation of the power function is as old as set theory itself. Already Georg Cantor [3] proposed CH (that is $2^{\aleph_0} = \aleph_1$). As is well known, Cantor was not able to prove his hypothesis. When Kurt Gödel [15] introduced his constructible universe $L$, the first of set theory's *inner models*, he was able to prove that GCH (that is $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals $\alpha$) is consistent with ZFC. That is, it is not 'unsafe' to assume GCH. At least not more so than ZFC. Still, strictly speaking, the power function behavior was not determined. We note that CH is a *local* hypothesis on the power function while GCH is a *global* one. This difference is quite important from current day point of view.

When Paul Cohen [4] ushered Forcing into set theory, he was able to show, among other things, that CH is independent of ZFC. In fact, his technique showed that if $\lambda$ is a regular cardinal we can make $2^\lambda$ almost any cardinal satisfying very weak restrictions. This result, more or less, rendered the local behavior of the power function on regular cardinals 'uninteresting': Practically everything goes.

Somewhat before Forcing technology came into scene Dana Scott [24] proved that if $\lambda$ is a measurable cardinal violating GCH then GCH is violated below $\lambda$ on a measure 1 set. Hence large cardinal axioms impose some structure on the power function. We note that the above result of Cohen does not take any additional structures into consideration. That is, for example, if $\lambda$ is measurable and we enlarge its power set we lose its measurability in the generic extension.

Combining many instances of Cohen's construction in a clever way, William Easton [8] gave a global result: We can set the power function on all regular cardinals

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to practically everything. In Easton's model the size of the power set of the singular cardinals is the lowest possible. This behavior became known as the Singular Cardinal Hypothesis.

Generating a gap on a singular cardinal needed advanced methods. The first result in this direction was a combination of methods by Jack Silver and Karel Prikry. Silver [27, 1] showed how to trade a supercompact for a measurable violating GCH. Prikry [22] showed how to change the cofinality of a measurable cardinal to $\omega$ without collapsing cardinals. Hence we have the first example of a singular strong limit cardinal violating GCH. We note that this violation is rather 'far away'.

Menachem Magidor [16, 17] showed, using a supercompact cardinal, that it is possible that GCH will fail on the first singular strong limit cardinal. Moreover, starting with a huge cardinal we can have the first failure of GCH at a singular strong limit.

At this point the general impression was that there can be arbitrary behavior on the singular cardinals. Then Silver [28] gave the following surprising result: If $\lambda$ is a singular cardinal of uncountable cofinality with GCH holding below $\lambda$ then GCH holds at $\lambda$. Improving on the above, Fred Galvin and András Hajnal [10] showed that the behavior of the power function on a singular cardinal of uncountable cofinality, $\lambda$, is tightly linked to the behavior of the power function below $\lambda$.

As can be seen, the methods to get a gap on a singular cardinal started with some large cardinal. Ronald Jensen [7] proved that this is a necessary starting point. Specifically he proved that if $\neg 0^\dagger$ then SCH.

With these results the investigation of the power function has transformed its form to the current day view: The behavior of the power function on the singular cardinal is linked to the existence of large cardinals. Our aim is to find restrictions on the power function where there are ones. When there are no restrictions we should find equiconsistency results between existence of large cardinals and possible behaviors of the power function.

We outline some of the known facts.

Hugh Woodin was first to use hyper measurable cardinals for $\neg$SCH results. Continuing at the same level, Moti Gitik and Magidor [12] presented a forcing notion, that used a strong cardinal $\lambda$, to blow up $2^\lambda$ to whatever size prescribed together with making $\operatorname{cf}\lambda = \omega$ and keeping GCH below $\lambda$. Hence without some further assumptions we can not restrict the size of the power set of singulars of countable cofinality. Indeed a modification of Gitik and Magidor's forcing to $\aleph_\omega$ gives Magidor's original result: GCH below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for $\alpha < \omega_1$. (albeit from a considerably lower large cardinal assumption). It is still not known if it is possible to get such a model with $2^{\aleph_\omega} \geq \aleph_{\omega_1}$.

On the other hand, we do have restrictions at $\aleph_\omega$. Saharon Shelah showed that $2^{\aleph_\omega} < \min(\aleph_{(2^{\aleph_0})^+}, \aleph_{\omega_4})$.

An interesting twist, connected with the above, is a work of Gitik and Bill Mitchell [14] showing that if there is no inner model with a strong cardinal then $2^{\aleph_\omega} < \aleph_{\omega_1}$. Hence, if Shelah's bound, $\aleph_{\omega_4}$, is optimal we need a stronger large cardinal to approach this bound than the ones used to get up to $\aleph_{\omega_1}$.

Mitchell [20] showed how to get high order measurable cardinals from $\neg$SCH and later on Gitik, building on results of Mitchell, Shelah and Woodin, had pinpoint $\neg$SCH to be equiconsistent with $o(\kappa) = \kappa^{++}$.

Just by looking at these few results — there are many more known and we still lack a lot of information — we can see that the situation on the singular cardinals is much more complicated than the situation on the regular cardinals. Of course getting a full result as in Easton's one is beyond our reach at this point. There are, however, several results concerning the global behavior on all the cardinals.

Matt Foreman and Woodin [9], starting from a supercompact, constructed a model in which GCH fails everywhere. In their model the gap between $\lambda$ and $2^\lambda$ is infinite and is not fixed for all $\lambda$. We know that we can not have a fixed infinite gap for all $\lambda$. A later unpublished modification of the construction, due to Woodin, gave a model satisfying $2^\lambda = \lambda^{++}$ for all cardinals $\lambda$. A referee of our thesis brought to our attention that Woodin generalized the construction and for each $1 < n < \omega$ got a model having a power function with a fixed gap $n$. Unfortunately, this result, also, was never published.

James Cummings [5], starting from a $\mathcal{P}^3(\kappa)$-hypermeasurable constructed a model satisfying $2^\lambda = \lambda^{++}$ for $\lambda$ limit cardinals and GCH everywhere else.

As the basic idea of our work is a generalization of Gitik and Magidor's one we elaborate more on it. Until Gitik and Magidor's work the major theme in generating a gap on a singular cardinal was as follows. The power set of a large cardinal is blown up sacrificing some of its size but not all of it. (That is, starting from a supercompact we are left with a measurable). Then one of the known forcing notions for changing cofinality is applied to this cardinal retaining the gap on the size of the power set. Gitik and Magidor forcing does *both* tasks in *one* step. In essence they found a method to add many Prikry sequences at once using an extender of a prescribed size. Hence in one step they blow up the size of the power set and change the cofinality to $\omega$.

While Prikry forcing can change the cofinality of a measurable cardinal to $\omega$ we have Magidor forcing [18] that adds a new club of a prescribed order type, $\alpha$, using a coherent sequence of measures. Miri Segal [25] combined this forcing with the extender idea of Gitik and Magidor, hence she was able to add a prescribed number of $\alpha$-sequences. That is in one step the power set is blown up and the cofinality is changed to $\operatorname{cf}\alpha$. Of course her starting point was a coherent sequence of extenders.

The marriage of Radin forcing [23, 21] with extenders was done by us [19]. Radin forcing adds to Magidor's one a new ingredient. It enables us to add a club to a large cardinal $\kappa$ while keeping $\kappa$ inaccessible (and even more). Indeed this behavior remains possible when we add many Radin sequences using an extender. This gave us the possibility to control the power function on a club while keeping the cardinal inaccessible. All this in *one* step. The different properties of Radin forcing are related to the length of the measure sequence used to define the forcing. In our extender based Radin forcing we start from an extender sequence and the properties are controlled by the length of the extender *sequence* (controlling the properties of $\kappa$) and the length of the extender (controlling $2^\kappa$).

This paper is a step in the investigation of the global behavior of the power function. The forcing we present should be viewed as a template enabling the construction of models with many different power functions. We stress that our main point is *not* a backwards Easton iteration for blowing up power sets cardinal by cardinal, followed by choosing cardinals after their power set was enlarged. We go the other way: We choose cardinals and then we blow up their power set. I.e., we do not blow up the power set of cardinals which are of no interest for us.

As a specific example we show that for each $1 < n < \omega$ it is possible to have a power function satisfying $2^\lambda = \lambda^{+n}$ for all cardinals $\lambda$. It is possible to get this behavior assuming there is $\kappa$ that is $\kappa^{+n+1}$-strong. (A somewhat weaker assumption is enough). In section 12 we describe the general behavior possible with our technique and give more examples.

Our starting point is the extender based Radin forcing. As mentioned earlier it enables us to control the behavior of the power function on a club of $\kappa$ while keeping $\kappa$ inaccessible. Hence $V_\kappa$ of the generic extension is a model of ZFC with a club of cardinals satisfying a prescribed power function.

In this work we control also the cardinals that are not in the club. We either set their power to the prescribed value or collapse them.

The specific example we construct is $2^\lambda = \lambda^{+3}$ everywhere. Taking any $n < \omega$ is exactly the same. Of course other behaviors are possible with this method.

So we start with $\kappa$ large enough as witnessed by $j{:}V \to M$. We construct from $j$ extender based Radin forcing that sets $2^\kappa = \kappa^{+3}$, shoots a club through $\kappa$ and for each $\lambda$ in the club we have $2^\lambda = \lambda^{+3}$. We are careful to make sure that $\kappa$ remains inaccessible in the generic extension. In order to control the behavior outside the club we add along the Radin sequence generated by the normal measure other forcing notions. That is if $\mu_1, \mu_2$ are 2 successive points in this Radin sequence then we force with $\mathrm{Col}(\mu_1^{+6}, \mu_2) \times \mathrm{C}(\mu_1^{+4}, \mu_2^{+}) \times \mathrm{C}(\mu_1^{+5}, \mu_2^{++}) \times \mathrm{C}(\mu_1^{+6}, \mu_2^{+3})$ as defined in some inner model, where $\mathrm{Col}(\tau, \lambda) = \{f{:}A \to \lambda \mid A \subset \tau,\ |A| < \tau\}$, $\mathrm{C}(\tau, \lambda) = \{f{:}A \to 2 \mid A \subset \lambda,\ |A| < \tau\}$. Note that we actually collapse cardinals in the Radin generic sequence. In order to allow for a Prikry like condition we need a generic filter from which the above forcing conditions will come. That is we will have $I \in V$ that is $\mathrm{Col}(\kappa^{+6}, j(\kappa)) \times \mathrm{C}(\kappa^{+4}, j(\kappa)^{+}) \times \mathrm{C}(\kappa^{+5}, j(\kappa)^{++}) \times \mathrm{C}(\kappa^{+6}, j(\kappa)^{+3})$ (as defined in some inner model of $M$) generic over $M$. It was pointed out by Woodin that a generic filter for forcing of this type can be generated through the normal measure (e.g. if $j$ is witnessing that $\kappa$ is $\kappa^{+3}$-strong and not $\kappa^{+4}$-strong then the normal measure generates a $\mathrm{Col}(\kappa^{+4}, j(\kappa))_M$-generic filter over $M$. However, it does not generate a $\mathrm{Col}(\kappa^{+3}, j(\kappa))_M$-generic filter over $M$.).

At this stage $V_\kappa$ of the generic extension almost satisfies our requirements. The only problem is that we do not have a gap of 3 on $\mu_1^{+}$, $\mu_1^{++}$, $\mu_1^{+3}$. Of course the naive approach is to add also $\mathrm{C}(\mu_1^{+}, \mu_1^{+4}) \times \mathrm{C}(\mu_1^{++}, \mu_1^{+5}) \times \mathrm{C}(\mu_1^{+3}, \mu_1^{+6})$ along the normal Radin sequence. However, in order to have the Prikry condition we need a $\mathrm{C}(\kappa^{+}, \kappa^{+4}) \times \mathrm{C}(\kappa^{++}, \kappa^{+5}) \times \mathrm{C}(\kappa^{+3}, \kappa^{+6}))_M$-generic over $M$. As this forcing is 'below' the extender length, hence 'sees' much of $V$, we do not have such a generic. One solution to this problem is to force such a generic into $V$. We did a similar thing in [13] and the amount of technical difficulties we had to overcome was overwhelming. (And to a large extent blurred the simple idea). A second solution, adopted here, is to do a preparation forcing making a gap of 3 on $\lambda^{+}$, $\lambda^{++}$, $\lambda^{+3}$ for each $\lambda < \kappa$ inaccessible. By making sure that the normal Radin sequence pass only through inaccessibles we will get the prescribed behavior.

This work clarified to the author many points from the simpler [19]. As a result many of the proofs appearing here, in more complex setting, are simpler than the ones appearing in [19].

The reader should be fluent with forcing technology and large cardinals methods. We assume that [12] is known. Especially the 'nice extender' built there. Knowledge of [19] makes reading of this work easier.

The structure of this work is as follows. We start from a universe $V^*$ satisfying GCH that has a suitably large cardinal $\kappa$ as witnessed by an elementary embedding $j$. In section 2, taken almost verbatim from [19], we define extender sequences. In section 3 we construct $V$ as a generic extension of $V^*$ with a changed power function on the 3 first successors of inaccessible cardinals below $\kappa$. We locate in $V$ a generic filter that is used later to change the power function on the next 3 cardinals and collapse all others. In section 4 we incorporate the generic filters located in the previous section into the definition of extender sequence. This revised extender sequence is the one we use in the rest of the paper. In section 5 we define our Modified Extender Based Radin Forcing, $P_{\bar{E}}$. In section 6 we give some basic properties of the just defined forcing notion. Section 7 is dedicated to the proof of the homogeneity of dense open subsets of $P_{\bar{E}}$. This property plays a central role in later analysis. In section 8 we use the homogeneity to prove Prikry's condition for $P_{\bar{E}}$. Section 9 is used to show how to get generic filter over elementary submodels. In section 10 we show what cardinals are not collapsed by $P_{\bar{E}}$ and what their power is. Section 11 is just the statement of the consistency theorem we proved. Section 12 is a list of points for later research and indication of preliminary work we have.

## 2. Extender sequences

Suppose we have an elementary embedding $j{:}V^* \to M^* \supset V_\lambda^{V^*}$, $\text{crit}(j) = \kappa$. The value of $\lambda$ is determined later, according to the different applications we will have.

Construct from $j$ a nice extender as in [12]:

$$E(0) = \langle \langle E_\alpha(0) \mid \alpha \in \mathcal{A}\rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha,\ \alpha,\beta \in \mathcal{A}\rangle\rangle.$$

We remind the reader what are the properties of this extender:

(1) $\mathcal{A} \subseteq |V_\lambda^{V^*}| \setminus \kappa$,
(2) $|\mathcal{A}| = |V_\lambda^{V^*}|$,
(3) $\langle \mathcal{A}, \leq_{\mathcal{A}}\rangle$ is a $\kappa^+$-directed partial order,
(4) $\forall \alpha,\beta \in \mathcal{A}\ \beta \geq_{\mathcal{A}} \alpha \implies \pi_{\beta,\alpha}{:}V_\kappa \to V_\kappa$,
(5) $\forall \alpha \in \mathcal{A}\ E_\alpha(0)$ is a measure on $\kappa$,
(6) $\forall \alpha,\beta \in \mathcal{A}\ \beta \geq_{\mathcal{A}} \alpha \implies \forall X \subseteq \kappa$
$$\pi^{-1}_{\beta,\alpha} X \in E_\beta(0) \iff X \in E_\alpha(0),$$
(7) $\kappa \in \mathcal{A}$,
(8) $\forall \alpha \in \mathcal{A}\ \kappa \leq_{\mathcal{A}} \alpha$. We write $\pi_{\alpha,0}$ instead of $\pi_{\alpha,\kappa}$,
(9) $\forall \alpha,\beta \in \mathcal{A}\ \forall \nu < \kappa\ \nu^0 = \pi_{\alpha,0}(\nu) = \pi_{\beta,0}(\nu)$,
(10) $\forall \alpha,\beta \in \mathcal{A}\ \beta \geq_{\mathcal{A}} \alpha \implies \forall \nu < \kappa\ \pi_{\beta,0}(\nu) = \pi_{\alpha,0}(\pi_{\beta,\alpha}(\nu))$,
(11) $\forall \alpha,\beta,\gamma \in \mathcal{A}\ \gamma \geq_{\mathcal{A}} \beta \geq_{\mathcal{A}} \alpha \implies \exists X \in E_\gamma(0)\ \forall \nu \in X\ \pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$.

If, for example, we need $|E(0)| = \kappa^{+3}$ then, under GCH, we require $\lambda = \kappa + 3$. A typical big set in this extender concentrates on singletons.

If $j$ is not sufficiently closed, then $E(0) \notin M^*$ and the construction stops. We set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(0)\rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 1. ($\mathrm{l}(\bar{E}_\alpha) = 1$)

If, on the other hand, $E(0) \in M^*$ we can construct for each $\alpha \in \operatorname{dom} E(0)$ the following ultrafilter

$$A \in E_{\langle \alpha, E(0) \rangle}(1) \iff \langle \alpha, E(0) \rangle \in j(A).$$

Such an $A$ concentrates on elements of the form $\langle \xi, e(0) \rangle$ where $e(0)$ is an extender on $\xi^0$ and $\xi \in \operatorname{dom} e(0)$. Note that $e(0)$ concentrates on singletons below $\xi^0$. If, for example, $|E(0)| = \kappa^{+3}$ then on a large set we have $|e(0)| = (\xi^0)^{+3}$.

We define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ as

$$\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}(\langle \xi, e(0) \rangle) = \langle \pi_{\beta,\alpha}(\xi), e(0) \rangle.$$

From this definition we get

$$j(\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle})(\langle \beta, E(0) \rangle) = \langle \alpha, E(0) \rangle.$$

Hence we have here an extender

$$E(1) = \langle \langle E_{\langle \alpha, E(0) \rangle}(1) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle} \mid \beta \geq_{\mathcal{A}} \alpha,\ \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

Note that the difference between $\pi_{\beta,\alpha}$ and $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ is quite superficial. We can define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ in a uniform way for both extenders. Just project the first element of the argument using $\pi_{\beta,\alpha}$.

If $\langle E(0), E(1) \rangle \notin M^*$ then the construction stops. In this case we set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(0), E(1) \rangle.$$

We say that $\bar{E}_\alpha$ is an extender sequence of length 2. ($\mathrm{l}(\bar{E}_\alpha) = 2$.)

If $\langle E(0), E(1) \rangle \in M^*$ then we construct the extender $E(2)$ in the same way as we constructed $E(1)$ from $E(0)$.

The above special case being worked out we continue with the general case. Assume we have constructed

$$\langle E(\tau') \mid \tau' < \tau \rangle.$$

If $\langle E(\tau') \mid \tau' < \tau \rangle \notin M^*$ then our construction stops here. We set

$$\forall \alpha \in \mathcal{A}\ \bar{E}_\alpha = \langle \alpha, E(\tau') \mid \tau' < \tau \rangle.$$

and we say that $\bar{E}_\alpha$ is an extender sequence of length $\tau$. ($\mathrm{l}(\bar{E}_\alpha) = \tau$.)

If, on the other hand, $\langle E(\tau') \mid \tau' < \tau \rangle \in M^*$ then we construct

$$A \in E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}(\tau) \iff \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle \in j(A).$$

Defining $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}$ using the first coordinate as before gives the needed projection.

We are quite casual in writing the indices of the projections and ultrafilters. By this we mean that we sometimes write $\pi_{\beta,\alpha}$ when we should have written $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}$ and $E_\alpha(\tau)$ when we should have written $E_{\langle \alpha, E(0), \dots, E(\tau'), \dots, \mid \tau' < \tau \rangle}(\tau)$.

With this abuse of notation, the projection we just defined satisfies

$$j(\pi_{\beta,\alpha})(\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle) = \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle.$$

So we have the extender

$$E(\tau) = \langle \langle E_\alpha(\tau) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha,\ \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

We let the construction run until it stops due to the extender sequence not being in $M^*$.

**Definition 2.1.** $\bar{\mu}$ is an extender sequence if there is an elementary embedding $j{:}V^* \to M^*$ such that $\bar{\nu}$ is an extender sequence generated as above and $\bar{\mu} = \bar{\nu}\restriction\tau$ for $\tau \leq \mathrm{l}(\bar{\nu})$. $\kappa(\bar{\mu})$ is the ordinal at the beginning of the sequence. (i.e. $\kappa(\bar{E}_\alpha) = \alpha$) $\kappa^0(\bar{\mu})$ is $(\kappa(\bar{\mu}))^0$. (i.e. $\kappa^0(\bar{E}_\alpha) = \kappa$)

That is, we do not have to construct the extender sequence until it is not in $M^*$. We can stop anywhere on the way.

**Definition 2.2.** If $\kappa^0(\bar{\mu}_1) < \cdots < \kappa^0(\bar{\mu}_n)$ then the sequence of extender sequences $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle$ is called $^0$-increasing.

**Definition 2.3.** Let $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle$ be $^0$-increasing. An extender sequences $\bar{\mu}$ is called permitted to $\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle$ if $\kappa^0(\bar{\mu}_n) < \kappa^0(\bar{\mu})$.

**Definition 2.4.** We say $A \in \bar{E}_\alpha$ if $\forall \xi < \mathrm{l}(\bar{E}_\alpha)\ A \in E_\alpha(\xi)$.

**Definition 2.5.** $\bar{E} = \langle \bar{E}_\alpha \mid \alpha \in \mathcal{A} \rangle$ is an extender sequence system if there is an elementary embedding $j{:}V^* \to M^*$ such that all $\bar{E}_\alpha$ are extender sequences generated from $j$ as prescribed above and $\forall \alpha, \beta \in \mathcal{A}\ \mathrm{l}(\bar{E}_\alpha) = \mathrm{l}(\bar{E}_\beta)$. This common length is called the length of the system, $\mathrm{l}(\bar{E})$. We write $\bar{E}(\bar{\mu})$ for the extender sequence system to which $\bar{\mu}$ belongs (i.e. $\bar{E}(\bar{E}_\alpha) = \bar{E}$).

We point out that there is a $\kappa^+$-directed partial order on the system $\bar{E}$ inherited from $\mathcal{A}$. That is $\bar{E}_\beta \geq_{\bar{E}} \bar{E}_\alpha \iff \beta \geq_{\mathcal{A}} \alpha$. Of course, this implies that there is $\min \bar{E}$, namely $\bar{E}_\kappa$. From now on we use only the order $\geq_{\bar{E}}$ (even for $\mathcal{A}$) and we write $\operatorname{dom} \bar{E}$ for $\mathcal{A}$.

$\bar{E}_\alpha$ is the generalization of the measure on the $\alpha$ coordinate in Gitik-Magidor forcing [12].

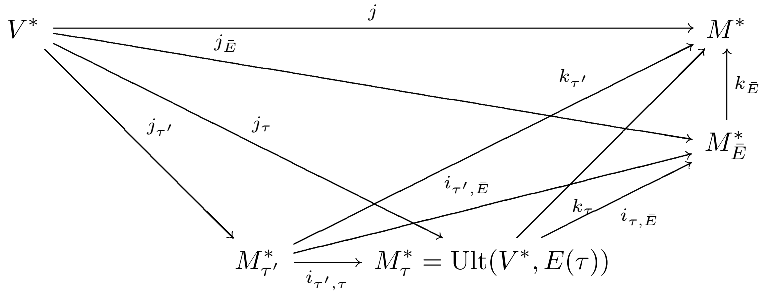
## 3. Preparation forcing

Suppose that we have GCH and an elementary embedding $j{:}V^* \to M^* \supset V^*_{\kappa+3}$, $\mathrm{crit}(j) = \kappa$. (If we need to get, say, $2^\kappa = \kappa^{+5}$ then we should start with $j{:}V^* \to M^* \supset V^*_{\kappa+5}$.) Construct from $j$ an extender sequence system, $\bar{E}$, and define the following embeddings $\forall \tau' < \tau < \mathrm{l}(\bar{E})$

$$\begin{aligned} & j_\tau{:}V^* \to M^*_\tau \simeq \mathrm{Ult}(V^*, E(\tau)), \\ & k_\tau(j_\tau(f)(\bar{E}_\alpha\restriction\tau)) = j(f)(\bar{E}_\alpha\restriction\tau), \\ & i_{\tau',\tau}(j_{\tau'}(f)(\bar{E}_\alpha\restriction\tau')) = j_\tau(f)(\bar{E}_\alpha\restriction\tau'), \\ & \langle M^*_{\bar{E}}, i_{\tau,\bar{E}} \rangle = \lim \mathrm{dir} \langle \langle M^*_\tau \mid \tau < \mathrm{l}(\bar{E}) \rangle, \langle i_{\tau',\tau} \mid \tau' \leq \tau < \mathrm{l}(\bar{E}) \rangle \rangle. \end{aligned} \tag{3.0.1}$$

We restrict $\mathrm{l}(\bar{E})$ by demanding $\forall \tau < \mathrm{l}(\bar{E})\ \bar{E}\restriction\tau \in M^*_\tau$.

These embeddings give rise to the following commutative diagram




The critical points of the embeddings originating in $V^*$ is of course $\kappa$. The critical points of the embeddings originating in the other models are $\kappa^{+4}$ as computed in that models, namely:

$$\begin{aligned}
&\operatorname{crit} i_{\tau',\tau} = \operatorname{crit} k_{\tau'} = \operatorname{crit} i_{\tau',\bar E} = (\kappa^{+4})_{M^*_{\tau'}},\\
&\operatorname{crit} k_\tau = \operatorname{crit}(i_{\tau,\bar E}) = (\kappa^{+4})_{M^*_\tau},\\
&\operatorname{crit} k_{\bar E} = (\kappa^{+4})_{M^*_{\bar E}}.
\end{aligned}$$

All of the models catches $V^{M^*}_{\kappa^{+3}} = V^*_{\kappa^{+3}}$ hence compute $\kappa^{+3}$ to be the same ordinal in all models. The larger $\tau$ is the more resemblance there is between $M^*_\tau$ and $M^*$, and hence with $V^*$ towards $V^*_{\kappa^{+4}}$. This can be observed by noting that

$$(\kappa^{+4})_{M^*_{\tau'}} < j_{\tau'}(\kappa) < (\kappa^{+4})_{M^*_\tau} < j_\tau(\kappa) < (\kappa^{+4})_{M^*_{\bar E}} \leq (\kappa^{+4})_{M^*} \leq \kappa^{+4}.$$

The last weak inequalities can become equalities if $\mathrm{l}(\bar E) = \kappa^{+4}$. Otherwise they would remain sharp inequalities.

Note that in general $\mathrm{l}(\bar E)$ can be very large. However, the requirement $\bar E{\restriction}\tau \in M^*_\tau$ imposes limit on $\mathrm{l}(\bar E)$. In our case to $\mathrm{l}(\bar E) \leq \kappa^{+4}$. In this work we are more strict than that and require $\mathrm{l}(\bar E) < \kappa^{+4}$. In fact, if $\mathrm{l}(\bar E) = \kappa^{+4}$ then the forcing notion we define using it, $P_{\bar E}$, is isomorphic to the forcing defined with $\bar E{\restriction}\tau$ for some $\tau < \kappa^{+4}$. For the specific result we are aiming to it is enough to have $\mathrm{l}(\bar E) = \kappa^+$.

We use the following, quite standard, notation

$$\begin{aligned}
&\mathrm{C}(\lambda,\mu) = \{f \mid f{:}A \to 2,\ A \in [\mu]^{<\lambda}\},\\
&\mathrm{Col}(\lambda,\mu) = \{f \mid f{:}A \to \mu,\ A \in [\lambda]^{<\lambda}\}.
\end{aligned}$$

Note that there is a change from previous works of this type. We use $\mathrm{Col}(\lambda,\mu)$ and not the Levy collapse $\mathrm{Col}(\lambda,{<}\mu)$. As will be seen (much) later this helped us a lot.

Now that the setting is clear and before we start with the preparation, a note is in order. The main point of this work is the forcing $P_{\bar E}$ described in section 5 and the technicalities of the current section are somewhat 'off track'. A reader willing to accept a somewhat weaker result than the one we stated can do without the preparation. For example, to prove

**Theorem.** *If there is $\bar E$ such that $|\bar E| = \kappa^{+3}$, $\operatorname{cf}\mathrm{l}(\bar E) > \kappa$ then there is a model containing a (class) club $C$ such that the cardinals in the model are $\{\mu^+, \mu^{++}, \mu^{+3}, \mu^{+4} \mid \mu \in C\} \cup \lim C$ and the power function is*

$$2^\lambda = \begin{cases} \mu^{+3} & \mu \in \lim C,\ \lambda \in \{\mu, \mu^+, \mu^{++}\} \\ \lambda^+ & \textit{otherwise} \end{cases}.$$

it is enough to locate in $V^*$ a $\mathrm{Col}(\kappa^{+4}, j_{\bar{E}}(\kappa))_{M^*_{\bar{E}}}$-generic filter over $M^*_{\bar{E}}$ and to jump to the next section (skimming through the definitions of subsection 3.5). Similarly we can prove

**Theorem.** *If there is $\bar{E}$ such that $|\bar{E}| = \kappa^{+3}$, $\operatorname{cf} \mathrm{l}(\bar{E}) > \kappa$ then there is a model containing a (class) club $C$ such that the cardinals in the model are $\{\mu^+, \dots, \mu^{+6} \mid \mu \in C\} \cup \lim C$ and the power function is*

$$2^\lambda = \begin{cases} \mu^{+3} & \mu \in \lim C,\ \lambda \in \{\mu, \mu^+, \mu^{++}\} \\ \lambda^{+3} & \mu \in C,\ \lambda \in \{\mu^{+4}, \mu^{+5}, \mu^{+6}\} \\ \lambda^+ & \textit{otherwise} \end{cases} .$$

it is enough to find in $V^*$ a $(\mathrm{Col}(\kappa^{+6}, j_{\bar{E}}(\kappa)) \times \mathrm{C}(\kappa^{+4}, j_{\bar{E}}(\kappa)^+) \times \mathrm{C}(\kappa^{+5}, j_{\bar{E}}(\kappa)^{++}) \times \mathrm{C}(\kappa^{+6}, j_{\bar{E}}(\kappa)^{+3}))_{M^*_{\bar{E}}}$-generic filter over $M^*_{\bar{E}}$ and to jump to the next section (again, skimming through the definitions of subsection 3.5). It is easy to construct both generic filters by going through $\mathrm{Ult}(V^*, E_\kappa(0))$.

The above said, we continue to the preparation. It will give us a very rough approximation of the power set function we seek. For each $\nu \leq \kappa$ an inaccessible, we will have

$$2^{(\nu^+)} = \nu^{+4},$$
$$2^{(\nu^{++})} = \nu^{+5},$$
$$2^{(\nu^{+3})} = \nu^{+6}.$$

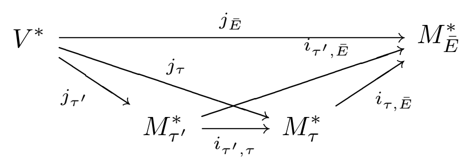
In the generic extension, for each $\tau < \mathrm{l}(\bar{E})$ there is a generic filter for the forcing notion

$$(\mathrm{Col}(\kappa^{+6}, j_\tau(\kappa)) \times \mathrm{C}(\kappa^{+6}, j_\tau(\kappa)^{+3}) \times \mathrm{C}(\kappa^{+5}, j_\tau(\kappa)^{++}) \times \mathrm{C}(\kappa^{+4}, j_\tau(\kappa)^{+}))_{M^*_\tau[G_{<\kappa}]}$$

over $M^*_\tau[G^\tau][H^\tau]$, where $G_{<\kappa}$ $G^\tau$, $H^\tau$ will be defined later.

### 3.1. Reverse Easton forcing for pulling in the needed generic filters.

We make a reverse Easton forcing and lift the diagram



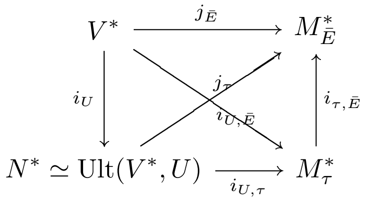

We define the following reverse Easton iteration $\langle P_\nu, \dot{Q}_\nu \mid \nu \leq \kappa \rangle$: When $\nu$ is accessible $\dot{Q}_\nu = \dot{1}$ and when $\nu$ is inaccessible

$$\dot{Q}_\nu = \mathrm{C}(\widehat{\nu}^+, \widehat{\nu}^{+4}) \times \mathrm{C}(\widehat{\nu}^{++}, \widehat{\nu}^{+5}) \times \mathrm{C}(\widehat{\nu}^{+3}, \widehat{\nu}^{+6}).$$

We set $\dot{P}_{>\nu}$ to be the forcing notion name satisfying

$$P_\kappa = P_\nu * \dot{Q}_\nu * \dot{P}_{>\nu}.$$

We factor through the normal ultrafilter to get




$$\begin{aligned} & U = E_\kappa(0), \\ & i_U{:}V^* \to N^* \simeq \mathrm{Ult}(V^*, U), \\ & i_{U,\tau}(i_U(f)(\kappa)) = j_\tau(f)(\kappa), \\ & i_{U,\bar E}(i_U(f)(\kappa)) = j_{\bar E}(f)(\kappa). \end{aligned}$$

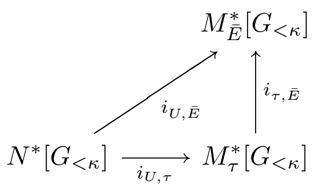
$N^*$ catches $V^*$ only up to $V^*_{\kappa+1}$ and we have

$$\kappa^+ < \mathrm{crit}\, i_U = \mathrm{crit}\, i_{U,\tau} = \mathrm{crit}\, i_{U,\bar E} = (\kappa^{++})_{N^*} < i_U(\kappa) < \kappa^{++}.$$

For later convenience we set

$$\begin{aligned} & \langle P^U_\nu, \dot Q^U_\nu \mid \nu \leq i_U(\kappa)\rangle = i_U(\langle P_\nu, \dot Q_\nu \mid \nu \leq \kappa\rangle), \\ & P^U_{i_U(\kappa)} = P^U_\kappa * \dot Q^U_\kappa * \dot P^U_{>\kappa}, \\ & \langle P^\tau_\nu, \dot Q^\tau_\nu \mid \nu \leq j_\tau(\kappa)\rangle = j_\tau(\langle P_\nu, \dot Q_\nu \mid \nu \leq \kappa\rangle), \\ & P^\tau_{j_\tau(\kappa)} = P^\tau_\kappa * \dot Q^\tau_\kappa * \dot P^\tau_{>\kappa}, \\ & \langle P^{\bar E}_\nu, \dot Q^{\bar E}_\nu \mid \nu \leq j_{\bar E}(\kappa)\rangle = j_{\bar E}(\langle P_\nu, \dot Q_\nu \mid \nu \leq \kappa\rangle), \\ & P^{\bar E}_{j_{\bar E}(\kappa)} = P^{\bar E}_\kappa * \dot Q^{\bar E}_\kappa * \dot P^{\bar E}_{>\kappa}. \end{aligned}$$

We note that $P_\kappa = P^U_\kappa = P^\tau_\kappa = P^{\bar E}_\kappa$. Let $G_{<\kappa}$ be $P_\kappa$-generic over $V^*$. As $\mathrm{crit}(i_{U,\bar E}) = \mathrm{crit}(i_{U,\tau}) = (\kappa^{++})_{N^*}$ and $i''_{U,\bar E} G_{<\kappa} = i''_{U,\tau} G_{<\kappa} = G_{<\kappa}$ we have the lifting




At stage $\kappa$ the forcing notions in $V^*[G_{<\kappa}]$ and $M^*_{\bar E}[G_{<\kappa}]$ are different. We need 3.4 in order to see that we can read a generic over $M^*_{\bar E}[G_{<\kappa}]$ from a generic over $V^*[G_{<\kappa}]$.

**Proposition 3.1.** *If $G$ is $\mathrm{C}(\kappa^+, \kappa^{+4})_{V^*[G_{<\kappa}]}$-generic over $V^*[G_{<\kappa}]$ then it is also $\mathrm{C}(\kappa^+, \kappa^{+4})_{M^*_{\bar E}[G_{<\kappa}]}$-generic over $M^*_{\bar E}[G_{<\kappa}]$.*

*Proof.* We begin by noting that $\mathrm{C}(\kappa^+, \kappa^+)_{M^*_{\bar E}[G_{<\kappa}]} = \mathrm{C}(\kappa^+, \kappa^+)_{V^*[G_{<\kappa}]}$ and if $A \in M^*_{\bar E}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^+)_{M^*_{\bar E}[G_{<\kappa}]}$ then it is also a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^+)_{V^*[G_{<\kappa}]}$.

Suppose $A \in M^*_{\bar E}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^{+4})_{M^*_{\bar E}[G_{<\kappa}]}$. Let $X = \bigcup\{\mathrm{dom}\, p \mid p \in A\}$. As $|A| \leq \kappa^+$, we have $|X| \leq \kappa^+$. Of course $A$ is also a maximal anti-chain of $\mathrm{C}(\kappa^+, X)_{M^*_{\bar E}[G_{<\kappa}]}$. For simplicity let us assume $|X| = \kappa^+$. Choose $f \in M^*_{\bar E}[G_{<\kappa}]$, such that $f{:}X \xrightarrow[\text{onto}]{1-1} \kappa^+$. This $f$ induces a natural isomorphism $\mathrm{C}(\kappa^+, X)_{M^*_{\bar E}[G_{<\kappa}]} \simeq \mathrm{C}(\kappa^+, \kappa^+)_{M^*_{\bar E}[G_{<\kappa}]}$. So $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^+)_{M^*_{\bar E}[G_{<\kappa}]}$. Hence $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^+)_{V^*[G_{<\kappa}]}$. Now we can go back in $V^*[G_{<\kappa}]$. That is $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, X)_{V^*[G_{<\kappa}]}$, hence $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^+, \kappa^{+4})_{V^*[G_{<\kappa}]}$. ☐

**Proposition 3.2.** *If $G$ is $\mathrm{C}(\kappa^{++},\kappa^{+5})_{V^*[G_{<\kappa}]}$-generic over $V^*[G_{<\kappa}]$ then it is also $\mathrm{C}(\kappa^{++},\kappa^{+5})_{M^*_{\bar{E}}[G_{<\kappa}]}$-generic over $M^*_{\bar{E}}[G_{<\kappa}]$.*

*Proof.* We begin by noting that $\mathrm{C}(\kappa^{++},\kappa^{++})_{M^*_{\bar{E}}[G_{<\kappa}]} = \mathrm{C}(\kappa^{++},\kappa^{++})_{V^*[G_{<\kappa}]}$ and if $A \in M^*_{\bar{E}}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{++})_{M^*_{\bar{E}}[G_{<\kappa}]}$ then it is also a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{++})_{V^*[G_{<\kappa}]}$.

Suppose $A \in M^*_{\bar{E}}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{+5})_{M^*_{\bar{E}}[G_{<\kappa}]}$. Let $X = \bigcup\{\operatorname{dom} p \mid p \in A\}$. As $|A| \leq \kappa^{++}$, we have $|X| \leq \kappa^{++}$. Of course $A$ is also a maximal anti-chain of $\mathrm{C}(\kappa^{++},X)_{M^*_{\bar{E}}[G_{<\kappa}]}$. For simplicity let us assume $|X| = \kappa^{++}$. Choose $f \in M^*_{\bar{E}}[G_{<\kappa}]$, such that $f{:}X \xrightarrow[\text{onto}]{1-1} \kappa^{++}$. This $f$ induces a natural isomorphism $\mathrm{C}(\kappa^{++},X)_{M^*_{\bar{E}}[G_{<\kappa}]} \simeq \mathrm{C}(\kappa^{++},\kappa^{++})_{M^*_{\bar{E}}[G_{<\kappa}]}$. So $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{++})_{M^*_{\bar{E}}[G_{<\kappa}]}$. Hence $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{++})_{V^*[G_{<\kappa}]}$. Now we can go back in $V^*[G_{<\kappa}]$. That is $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},X)_{V^*[G_{<\kappa}]}$, hence $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{++},\kappa^{+5})_{V^*[G_{<\kappa}]}$. □

**Proposition 3.3.** *If $G$ is $\mathrm{C}(\kappa^{+3},\kappa^{+6})_{V^*[G_{<\kappa}]}$-generic over $V^*[G_{<\kappa}]$ then it is also $\mathrm{C}(\kappa^{+3},\kappa^{+6})_{M^*_{\bar{E}}[G_{<\kappa}]}$-generic over $M^*_{\bar{E}}[G_{<\kappa}]$.*

*Proof.* We begin by noting that $\mathrm{C}(\kappa^{+3},\kappa^{+3})_{M^*_{\bar{E}}[G_{<\kappa}]} = \mathrm{C}(\kappa^{+3},\kappa^{+3})_{V^*[G_{<\kappa}]}$ and if $A \in M^*_{\bar{E}}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+3})_{M^*_{\bar{E}}[G_{<\kappa}]}$ then it is also a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+3})_{V^*[G_{<\kappa}]}$.

Suppose $A \in M^*_{\bar{E}}[G_{<\kappa}]$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+6})_{M^*_{\bar{E}}[G_{<\kappa}]}$. Let $X = \bigcup\{\operatorname{dom} p \mid p \in A\}$. As $|A| \leq \kappa^{+3}$, we have $|X| \leq \kappa^{+3}$. Of course $A$ is also a maximal anti-chain of $\mathrm{C}(\kappa^{++},X)_{M^*_{\bar{E}}[G_{<\kappa}]}$. For simplicity let us assume $|X| = \kappa^{+3}$. Choose $f \in M^*_{\bar{E}}[G_{<\kappa}]$, such that $f{:}X \xrightarrow[\text{onto}]{1-1} \kappa^{+3}$. This $f$ induces a natural isomorphism $\mathrm{C}(\kappa^{+3},X)_{M^*_{\bar{E}}[G_{<\kappa}]} \simeq \mathrm{C}(\kappa^{+3},\kappa^{+3})_{M^*_{\bar{E}}[G_{<\kappa}]}$. So $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+3})_{M^*_{\bar{E}}[G_{<\kappa}]}$. Hence $f''A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+3})_{V^*[G_{<\kappa}]}$. Now we can go back in $V^*[G_{<\kappa}]$. That is $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},X)_{V^*[G_{<\kappa}]}$, hence $A$ is a maximal anti-chain of $\mathrm{C}(\kappa^{+3},\kappa^{+5})_{V^*[G_{<\kappa}]}$. □

**Corollary 3.4.** *If $G$ is $(\mathrm{C}(\kappa^{+},\kappa^{+4}) \times \mathrm{C}(\kappa^{++},\kappa^{+5}) \times \mathrm{C}(\kappa^{+3},\kappa^{+6})_{V^*[G_{<\kappa}]}$-generic over $V^*[G_{<\kappa}]$ then it is also $(\mathrm{C}(\kappa^{+},\kappa^{+4}) \times \mathrm{C}(\kappa^{++},\kappa^{+5}) \times \mathrm{C}(\kappa^{+3},\kappa^{+6}))_{M^*_{\bar{E}}[G_{<\kappa}]}$-generic over $M^*_{\bar{E}}[G_{<\kappa}]$.*

Choosing $H$, a $\dot{Q}_\kappa[G_{<\kappa}]$-generic filter over $V^*[G_{<\kappa}]$, is done with some caution. For this we set $S_1 = \mathrm{C}(\kappa^{+},\kappa^{+4})_{V^*[G_{<\kappa}]}$, $S_2 = (\mathrm{C}(\kappa^{++},\kappa^{+5}) \times \mathrm{C}(\kappa^{+3},\kappa^{+6}))_{V^*[G_{<\kappa}]}$. That is $\dot{Q}_\kappa[G_{<\kappa}] = S_1 \times S_2$. In the same fashion we factor $\dot{Q}^{\bar{E}}_\kappa[G_{<\kappa}]$: $S^{\bar{E}}_1 = \mathrm{C}(\kappa^{+},\kappa^{+4})_{M^*_{\bar{E}}[G_{<\kappa}]}$, $S^{\bar{E}}_2 = (\mathrm{C}(\kappa^{++},\kappa^{+5}) \times \mathrm{C}(\kappa^{+3},\kappa^{+6}))_{M^*_{\bar{E}}[G_{<\kappa}]}$. We set $S^U_1 = \mathrm{C}(\kappa^{+},\kappa^{+4})_{N^*[G_{<\kappa}]}$, $S^U_2 = (\mathrm{C}(\kappa^{++},\kappa^{+5}) \times \mathrm{C}(\kappa^{+3},\kappa^{+6}))_{N^*[G_{<\kappa}]}$. Then $i_{U,E}(S^U_1 \times S^U_2) = S^{\bar{E}}_1 \times S^{\bar{E}}_2$. $N^*[G_{<\kappa}]$ contains only $\kappa^+$ anti-chains of $S^U_2$ and $S^U_2$ is a $\kappa^+$-closed forcing notion in $V^*[G_{<\kappa}]$. Hence, there is a decreasing sequence $\langle p_\xi \mid \xi < \kappa^+\rangle \subset N^*[G_{<\kappa}]$ that gives rise to an $S^U_2$-generic filter over $N^*[G_{<\kappa}]$. The crucial point is $\langle i_{U,\bar{E}}(p_\xi) \mid \xi < \kappa^+\rangle \subset S^{\bar{E}}_2$, and thus $\langle i_{U,\bar{E}}(p_\xi) \mid \xi < \kappa^+\rangle \subset S_2$. As $V^*[G_{<\kappa}] \vDash \ulcorner S_2$ is $\kappa^{++}$-closed$\urcorner$ there is $p \in S_2$ such that $\forall \xi < \kappa^+\ p \leq i_{U,\bar{E}}(p_\xi)$.

We take $H_2$ to be $S_2$-generic over $V^*[G_{<\kappa}]$ with $p \in H_2$. Let $H_1$ be $S_1$-generic over $V^*[G_{<\kappa}][H_2]$. We set $H = H_1 \times H_2$. Then $H$ is $\dot{Q}_\kappa[G_{<\kappa}]$-generic over $V^*[G_{<\kappa}]$.

We show that, in $V^*[G_{<\kappa}][H]$, for all $\tau < \mathrm{l}(\bar{E})$ there are filters $G^\tau$, $H^\tau$, $G^{\bar{E}}$, $H^{\bar{E}}$ such that

(1) $G^\tau * H^\tau$ is $P^\tau_{j_\tau(\kappa)} * \dot{Q}^\tau_{j_\tau(\kappa)}$-generic over $M^*_\tau$,
(2) $j_\tau''(G_{<\kappa} * H) \subseteq G^\tau * H^\tau$,
(3) $G^{\bar{E}} * H^{\bar{E}}$ is $P^{\bar{E}}_{j_{\bar{E}}(\kappa)} * \dot{Q}^{\bar{E}}_{j_{\bar{E}}(\kappa)}$-generic over $M^*_{\bar{E}}$,
(4) $j_{\bar{E}}''(G_{<\kappa} * H) \subseteq G^{\bar{E}} * H^{\bar{E}}$,
(5) $\forall \tau' < \tau\ i''_{\tau',\tau}(G^{\tau'} * H^{\tau'}) \subseteq G^\tau * H^\tau$,
(6) $i''_{\tau,\bar{E}}(G^\tau * H^\tau) \subseteq G^{\bar{E}} * H^{\bar{E}}$.

Under these conditions the lifting of $i_{\tau',\tau}$, $i_{\tau,\bar{E}}$, $j_\tau$, $j_{\bar{E}}$ is possible. Moreover, the lifting $j_\tau$ is generated by an extender that continues $E(\tau)$.

By 3.4, $G^{\bar{E}}_\kappa = H \cap \dot{Q}^{\bar{E}}_\kappa[G_{<\kappa}]$ is $\dot{Q}^{\bar{E}}_\kappa[G_{<\kappa}]$-generic over $M^*_{\bar{E}}[G_{<\kappa}]$. For later purpose we do the factoring $G^{\bar{E}}_\kappa = H^{\bar{E}}_1 \times H^{\bar{E}}_2$ such that $H^{\bar{E}}_1 \subseteq H_1$, $H^{\bar{E}}_2 \subseteq H_2$. Note that $\langle i_{U,\bar{E}}(p_\xi) \mid \xi < \kappa^+ \rangle \subset H^{\bar{E}}_2$.

For each $\tau < \mathrm{l}(\bar{E})$ we set $G^\tau_\kappa$ to be the filter generated by $i^{-1\prime\prime}_{\tau,\bar{E}} G^{\bar{E}}_\kappa$. We claim that $G^\tau_\kappa$ is $\dot{Q}^\tau_\kappa[G_{<\kappa}]$-generic over $M^*_\tau[G_{<\kappa}]$. So, let $A \in M^*_\tau[G_{<\kappa}]$ be maximal anti-chain in $\dot{Q}^\tau_\kappa[G_{<\kappa}]$.

Then $i_{\tau,\bar{E}}(A) \in M^*_{\bar{E}}[G_{<\kappa}]$ is a maximal anti-chain in $\dot{Q}^{\bar{E}}_\kappa[G_{<\kappa}]$. As $\mathrm{crit}(i_{\tau,\bar{E}}) = (\kappa^{+4})_{M^*_\tau}$ we have $i_{\tau,\bar{E}}(A) = i''_{\tau,\bar{E}} A$. Hence there is $a \in A$ such that $i_{\tau,\bar{E}}(a) \in i_{\tau,\bar{E}}(A) \cap G^{\bar{E}}_\kappa$. By its definition $a \in A \cap G^\tau_\kappa$.

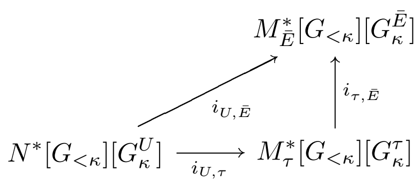
Let $G^U_\kappa$ be the filter generated by $i^{-1\prime\prime}_{U,\bar{E}} G^{\bar{E}}_\kappa$. We claim that $G^U_\kappa$ is $\dot{Q}^U_\kappa[G_{<\kappa}]$-generic over $N^*[G_{<\kappa}]$. For the purpose of the proof we let $G^U_\kappa = H^U_1 \times H^U_2$ so that $i''_{U,\bar{E}} H^U_1 \subseteq H^{\bar{E}}_1$, $i''_{U,\bar{E}} H^U_2 \subseteq H^{\bar{E}}_2$. We start by showing that $H^U_2$ is $Q^U_2$-generic over $N^*[G_{<\kappa}]$.

As $\{i_{U,\bar{E}}(p_\xi) \mid \xi < \kappa^+\} \subseteq H^{\bar{E}}_2$ we get $\{p_\xi \mid \xi < \kappa^+\} \subseteq H^U_2$. The sequence $\{p_\xi \mid \xi < \kappa^+\}$ generates an $S^U_2$-generic filter over $N^*[G_{<\kappa}]$. Hence $H^U_2$ is $S^U_2$-generic over $N^*[G_{<\kappa}]$.

We show that $H^U_1$ is $S^U_1$-generic over $N^*[G_{<\kappa}][H^U_2]$. As $N^*[G_{<\kappa}] \vDash \ulcorner S^U_2$ is $\kappa^{++}$-closed and $S^U_1$ is $\kappa^{++}$c.c.$\urcorner$ it is enough to show genericity over $N^*[G_{<\kappa}]$. So, let $A \in N^*[G_{<\kappa}]$ be a maximal anti-chain in $S^U_1$.

Then $i_{U,\bar{E}}(A) \in M^*[G_{<\kappa}]$ is a maximal anti-chain in $S^{\bar{E}}_1$. This time we get $i_{U,\bar{E}}(A) = i''_{U,\bar{E}} A$ for free. So there is $i_{U,\bar{E}}(a) \in H^{\bar{E}}_1 \cap i_{U,\bar{E}}(A)$ and by its definition $a \in H^U_1$. Hence $H^U_1 \cap A \neq \emptyset$. With this we showed that $G^U_\kappa = H^U_1 \times H^U_2$ is $\dot{Q}^U_\kappa[G_{<\kappa}]$-generic over $N^*[G_{<\kappa}]$.

Of course $i''_{U,\tau} G^U_\kappa \subseteq G^\tau_\kappa$, hence we have the lifting




We set

$$R_U = (\mathrm{Col}(\kappa^{+6}, i_U(\kappa)) \times \mathrm{C}(\kappa^{+4}, i_U(\kappa)^{+}) \times \mathrm{C}(\kappa^{+5}, i_U(\kappa)^{++}) \times \\ \mathrm{C}(\kappa^{+6}, i_U(\kappa)^{+3}))_{N^*[G_{<\kappa}]}.$$

We are going to find $I_U$, an $R_U$-generic filter, over $N^*[G_{<\kappa}]$. We do not force with $I_U$. Anticipating its later usage in the definition of $P_{\bar{E}}$ we need it to be in $M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$. For this we work as follows.

As $U \in M^*_{\bar{E}}$ and $\forall \tau < \mathrm{l}(\bar{E})\ E(\tau) \in M^*_{\bar{E}}$ we have the following diagram

$$\begin{array}{ccc} M^*_{\bar{E}} & & \\ {\scriptstyle i^{\bar{E}}_U}\downarrow & \searrow{\scriptstyle j^{\bar{E}}_\tau} & \\ N^{*\bar{E}} & \xrightarrow[i^{\bar{E}}_{U,\tau}]{} & M^{*\bar{E}}_\tau \end{array} \qquad \begin{array}{l} U = E_\kappa(0), \\ i^{\bar{E}}_U{:}M^*_{\bar{E}} \to N^{*\bar{E}} \simeq \mathrm{Ult}(M^*_{\bar{E}}, U), \\ j^{\bar{E}}_\tau{:}M^*_{\bar{E}} \to M^{*\bar{E}}_\tau \simeq \mathrm{Ult}(M^*_{\bar{E}}, E(\tau)), \\ i^{\bar{E}}_{U,\tau}(i^{\bar{E}}_U(f)(\kappa)) = j^{\bar{E}}_\tau(f)(\kappa), \end{array}$$

As $V^{V^*}_{\kappa+3} = V^{M^*_{\bar{E}}}_{\kappa+3}$ we get that the following are generic extensions: $M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$, $N^{*\bar{E}}[G_{<\kappa}][G^N_\kappa]$. We set

$$R^{\bar{E}}_U = (\mathrm{Col}(\kappa^{+6}, i^{\bar{E}}_U(\kappa)) \times \mathrm{C}(\kappa^{+4}, i^{\bar{E}}_U(\kappa)^{+}) \times \mathrm{C}(\kappa^{+5}, i^{\bar{E}}_U(\kappa)^{++}) \times \\ \mathrm{C}(\kappa^{+6}, i^{\bar{E}}_U(\kappa)^{+3}))_{N^{*\bar{E}}[G_{<\kappa}]}.$$

$R_U$, $R^{\bar{E}}_U$ and their anti-chains are coded in $V^{N^*[G_{<\kappa}]}_{i_U(\kappa)+3}$, $V^{N^{*\bar{E}}[G_{<\kappa}]}_{i^{\bar{E}}_U(\kappa)+3}$ respectively. $V^{N^*[G_{<\kappa}]}_{i_U(\kappa)+3}$, $V^{N^{*\bar{E}}[G_{<\kappa}]}_{i^{\bar{E}}_U(\kappa)+3}$ are determined by $V^{V^*}_{\kappa+3}$, $V^{M^*_{\bar{E}}}_{\kappa+3}$ (and $U$, of course). As $U \in M^*_{\bar{E}}$ and $V^{V^*}_{\kappa+3} = V^{M^*_{\bar{E}}}_{\kappa+3}$ we get that $R_U = R^{\bar{E}}_U$ and each anti-chain of $R_U$ appearing in $N^*[G_{<\kappa}]$ is also an anti-chain of $R^{\bar{E}}_U$ appearing in $N^{*\bar{E}}[G_{<\kappa}]$.

Hence, if $I_U$ is $R^{\bar{E}}_U$-generic filter over $N^{*\bar{E}}[G_{<\kappa}]$ then it is also $R_U$-generic filter over $N^*[G_{<\kappa}]$. Construction of such $I_U$ is done as follows. We set

$$\begin{aligned} R^U_1 &= \mathrm{Col}(\kappa^{+6}, i^{\bar{E}}_U(\kappa))_{N^{*\bar{E}}[G_{<\kappa}]}, \\ R^U_2 &= \mathrm{C}(\kappa^{+6}, i^{\bar{E}}_U(\kappa)^{+3})_{N^{*\bar{E}}[G_{<\kappa}]}, \\ R^U_3 &= \mathrm{C}(\kappa^{+5}, i^{\bar{E}}_U(\kappa)^{++})_{N^{*\bar{E}}[G_{<\kappa}]}, \\ R^U_4 &= \mathrm{C}(\kappa^{+4}, i^{\bar{E}}_U(\kappa)^{+})_{N^{*\bar{E}}[G_{<\kappa}]}, \end{aligned}$$

so that $R^U = R^U_1 \times R^U_4 \times R^U_3 \times R^U_2$.

We claim that there is $I^U_1 \in M^*_{\bar{E}}[G_{<\kappa}]$ which is $R^U_1$-generic over $N^{*\bar{E}}[G_{<\kappa}]$. This is immediate due to $R^U_1$ being $\kappa^+$-closed in $M^*_{\bar{E}}[G_{<\kappa}]$ and $N^{*\bar{E}}[G_{<\kappa}]$ containing only $\kappa^+$ maximal anti-chains of $R^U_1$.

The next step is to show that there is $I^U_2 \in M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$ which is $R^U_2$-generic over $N^{*\bar{E}}[G_{<\kappa}][I^U_1]$. In $M^*_{\bar{E}}[G_{<\kappa}]$, $R^U_2 \simeq \mathrm{C}(\kappa^+, \kappa^{+3})_{M^*_{\bar{E}}[G_{<\kappa}]}$. As $G^{\bar{E}}_\kappa$ is generic over $M^*_{\bar{E}}[G_{<\kappa}]$ and $I^U_1 \in M^*_{\bar{E}}[G_{<\kappa}]$ we get that there is $I^U_2 \in M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$ which is $R^U_2$-generic over $N^{*\bar{E}}[G_{<\kappa}][I^U_1]$.

Locating $I^U_3 \in M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$ which is $R^U_3$-generic over $N^{*\bar{E}}[G_{<\kappa}][I^U_1 \times I^U_2]$ is done as follows. As $N^{*\bar{E}}[G_{<\kappa}] \vDash \ulcorner R^U_1 \times R^U_2$ is $\kappa^{+6}$-closed$\urcorner$, finding $I^U_3$ which is generic over $N^{*\bar{E}}[G_{<\kappa}]$ is enough to ensure genericity over $N^{*\bar{E}}[G_{<\kappa}][I^U_1 \times I^U_2]$. In

$M^*_{\bar{E}}[G_{<\kappa}]$, $R^U_3 \simeq \mathrm{C}(\kappa^+, \kappa^{++})_{M^*_{\bar{E}}[G_{<\kappa}]}$. Once more, as $G^{\bar{E}}_\kappa$ is generic over $M^*_{\bar{E}}[G_{<\kappa}]$, there is $I^U_3 \in M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa]$ which $R^U_3$-generic over $N^{*\bar{E}}[G_{<\kappa}]$.

$I^U_4$ is constructed in the same way.

Let us set $I_U = I^U_1 \times I^U_2 \times I^U_3 \times I^U_4$. The above yield that $I_U$ is $R_U$-generic over $N^{*\bar{E}}[G_{<\kappa}]$, hence over $N^*[G_{<\kappa}]$. As $N^*[G_{<\kappa}] \vDash \ulcorner R^U$ is $\kappa^{+4}$-closed$\urcorner$, $G^U_\kappa$ is generic over $N^*[G_{<\kappa}][I_U]$. Hence $I_U$ is $R_U$-generic over $N^*[G_{<\kappa}][G^U_\kappa]$. We need to construct $G^U_{>\kappa}$ with some care in order to allow $I_U$ to be generic over $N^*[G_{<\kappa}][G^U_\kappa][G^U_{>\kappa}]$.

For this we look at $\dot{P}^U_{>\kappa}[G_{<\kappa} * G^U_\kappa]$ in $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2]$. This forcing, as seen by $V^*[G_{<\kappa}][H]$, is $\kappa^+$-closed of size $\kappa^+$. Hence it is isomorphic to $\mathrm{C}(\kappa^+, \kappa^+)_{V^*[G_{<\kappa}]}$. The main point is that we do not need all of $H$ in order to see $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2]$. Namely, we set $\lambda = (\kappa^{+4})_{M^*_{\bar{E}}}$ and factor $\dot{Q}_\kappa[G_{<\kappa}]$ as $(C(\kappa^+, \lambda) \times C(\kappa^{++}, \kappa^{+5}) \times C(\kappa^{+3}, \kappa^{+6})) \times \mathrm{C}(\kappa^+, \kappa^{+4} \setminus \lambda)$. Then we factor $H$ appropriately as $H' \times H''$. Then $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2]$ is definable inside $V^*[G_{<\kappa}][H']$. So, $\dot{P}^U_{>\kappa}[G_{<\kappa} * G^U_\kappa]$ is embeddable in $\mathrm{C}(\kappa^+, \kappa^{+4} \setminus \lambda)$ and all of its dense sets appearing in $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2]$ are coded in $V^*[G_{<\kappa}][H']$. Hence, there is $G^U_{>\kappa} \in V^*[G_{<\kappa}][H'][H'']$ which is $\dot{P}^U_{>\kappa}[G_{<\kappa} * G^U_\kappa]$-generic over $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2]$.

We consider $R^U_3 \times R^U_4$ in $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2][G^U_{>\kappa}]$. Evidently each anti chain of $R^U_3 \times R^U_4$ in $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2][G^U_{>\kappa}]$ already appears in $N^*[G_{<\kappa}][G^U_\kappa]$. Hence $I^U_3 \times I^U_4$ is $R^U_3 \times R^U_4$-generic over $N^*[G_{<\kappa}][G^U_\kappa][I^U_1][I^U_2][G^U_{>\kappa}]$.

Of course, all of this means that $I_U$ is $R_U$-generic over $N^*[G_{<\kappa}][G^U_\kappa][G^U_{>\kappa}]$.

We set $G^U = G_{<\kappa} * G^U_\kappa * G^U_{>\kappa}$. Then $G^U$ is $P^U_{i_U(\kappa)}$-generic over $N^*$. As $i''_U G_{<\kappa} = G_{<\kappa}$ we get that $i''_U G_{<\kappa} \subseteq G^U$, hence we have the lifting $i_U{:}V^*[G_{<\kappa}] \to N^*[G^U]$. Note that $i_U$ is defined in $V^*[G_{<\kappa}][H]$ and it is the natural embedding defined by a $V^*[G_{<\kappa}]$-ultrafilter extending $U$.

Let $H^U$ be the filter generated by $i''_U H$. This definition is possible as $H \subset V^*[G_{<\kappa}]$ and we just lifted $i_U$ to $V^*[G_{<\kappa}]$. We claim that $H^U$ is $\dot{Q}^U_{i_U(\kappa)}[G^U]$-generic over $N^*[G^U]$. Let, then, $D \in N^*[G^U]$ be dense open in $\dot{Q}^U_{i_U(\kappa)}[G^U]$.

Then there is $f \in V^*[G_{<\kappa}]$ such that $i_U(f)(\kappa) = D$. On a big set, in $V^*[G_{<\kappa}][H]$ sense, $f(\nu)$ is a dense open subset of $\dot{Q}_\kappa[G_{<\kappa}]$. $\dot{Q}_\kappa[G_{<\kappa}]$ is $\kappa^+$-closed in $V^*[G_{<\kappa}]$ hence $D^* = \bigcap_{\nu<\kappa} f(\nu) \in V^*[G_{<\kappa}]$ is dense open in it and by its definition $i_U(D^*) \subseteq i_U(f)(\kappa)$. Choose $p \in D^* \cap H$. $i_U(p) \in i_U(D^*) \cap i''_U H$. Hence, $D \cap H^U \neq \emptyset$.

So we can lift $i_U$ to $i_U{:}V^*[G_{<\kappa}][H] \to N^*[G^U][H^U]$. This embedding is definable inside $V^*[G_{<\kappa}][H]$.

We note that $I_U$ is $R_U$-generic over $N^*[G^U][H^U]$ as $N^*[G^U] \vDash \ulcorner \dot{Q}^U_{i_U(\kappa)}[G^U]$ is $i_U(\kappa)^+$-closed$\urcorner$, hence adds to new anti-chains to $R_U$.

Let $G^\tau_{>\kappa}$ be the filter generated by $i''_{U,\tau} G^U_{>\kappa}$. We claim that $G^\tau_{>\kappa}$ is $\dot{P}^\tau_{>\kappa}[G_{<\kappa} * G^\tau_\kappa]$-generic over $M^*_\tau[G_{<\kappa}][G^\tau_\kappa]$. So, let $D \in M^*_\tau[G_{<\kappa}][G^\tau_\kappa]$ be dense open in $\dot{P}^\tau_{>\kappa}[G_{<\kappa} * G^\tau_\kappa]$.

Let $\dot{D} \in M^*_\tau$ be a $P_\kappa * \dot{Q}^\tau_\kappa$-name for $D$. Then there is $f \in V^*$ such that $j_\tau(f)(\bar{E}_\alpha \restriction \tau) = \dot{D}$. Hence $\{\bar{\nu} \Vdash_{P_{\kappa^0(\bar{\nu})} * \dot{Q}_{\kappa^0(\bar{\nu})}} \ulcorner f(\bar{\nu})$ is dense open in $\dot{P}_{>\kappa^0(\bar{\nu})}\urcorner\} \in E_\alpha(\tau)$. Since $M^*_\tau[G_{<\kappa}][G^\tau_\kappa] \vDash \ulcorner \dot{P}^\tau_{>\kappa}[G_{<\kappa} * G^\tau_\kappa]$ is $\kappa^{+4}$ closed$\urcorner$ we have

$$\{\bar{\nu} \Vdash_{P_{\kappa^0(\bar{\nu})} * \dot{Q}_{\kappa^0(\bar{\nu})}} \ulcorner \dot{P}_{>\kappa^0(\bar{\nu})} \text{ is } \kappa^0(\bar{\nu})^{+4}\text{-closed}\urcorner\} \in E_\alpha(\tau).$$

Since $\{\mu \mid |\{\bar{\nu} \mid \mu = \kappa^0(\bar{\nu})\}| \leq \mu^{+3}\} \in E_\kappa(0)$ we can define a function $f^*$ such that $f^*(\mu)$ is a $P_\mu * \dot{Q}_\mu$-name satisfying $\Vdash_{P_\mu * \dot{Q}_\mu} \ulcorner f^*(\mu) = \bigcap\{f(\bar{\nu}) \mid \mu = \kappa^0(\bar{\nu})\}\urcorner$. So we have

$$\{\mu \mid \Vdash_{P_\mu * \dot{Q}_\mu} \ulcorner f^*(\mu) \text{ is a dense open subset of } P_\mu * \dot{Q}_\mu \urcorner\} \in E_\kappa(0).$$

Hence

$$M^*_\tau \vDash \ulcorner \Vdash_{P_\kappa * \dot{Q}^\tau_\kappa} \ulcorner j_\tau(f^*)(\kappa) \subseteq \dot{D} \text{ is dense open in } \dot{P}^\tau_{>\kappa} \urcorner\urcorner,$$
$$N^* \vDash \ulcorner \Vdash_{P_\kappa * \dot{Q}^U_\kappa} \ulcorner i_U(f^*)(\kappa) \text{ is dense open in } \dot{P}^U_{>\kappa} \urcorner\urcorner.$$

So there is $i_U(g)(\kappa) \in i_U(f^*)(\kappa)[G_{<\kappa}][G^U_\kappa] \cap G^U_{>\kappa}$. We get $D \cap G^\tau_{>\kappa} \neq \emptyset$ by noticing that $j_\tau(g)(\kappa) \in j_\tau(f^*)(\kappa)[G_{<\kappa}][G^\tau_\kappa] \cap i''_{U,\tau} G^U_{>\kappa}$.

Let $G^\tau = G_{<\kappa} * G^\tau_\kappa * G^\tau_{>\kappa}$. $j''_\tau G_{<\kappa} = G_{<\kappa}$, so $j''_\tau G_{<\kappa} \subseteq G^\tau$. Hence we can lift $j_\tau$ to $j_\tau{:}V^*[G_{<\kappa}] \to M^*_\tau[G^\tau]$. We note that this lift is defined in $V^*[G_{<\kappa}][H]$ and it is the natural embedding of a $V^*[G_{<\kappa}]$-extender continuing $E(\tau)$.

Let $H^\tau$ be the filter generated by $j''_\tau H$. We claim that $H^\tau$ is $\dot{Q}^\tau_{j_\tau(\kappa)}[G^\tau]$-generic over $M^*_\tau[G^\tau]$. Let $D \in M^*_\tau[G^\tau]$ be dense open in $\dot{Q}^\tau_{j_\tau(\kappa)}[G^\tau]$.

Then there is $f \in V^*[G_{<\kappa}]$ such that $j_\tau(f)(\bar{E}_\alpha \restriction \tau) = D$. As $A = \{\bar{\nu} \mid f(\bar{\nu}) \text{ is dense open in } \dot{Q}_\kappa[G_{<\kappa}]\} \in E_\alpha(\tau)$ and $\dot{Q}_\kappa[G_{<\kappa}]$ is $\kappa^+$-closed we get that $D^* = \bigcap_{\bar{\nu} \in A} f(\bar{\nu}) \in V^*[G_{<\kappa}]$ is dense open in $\dot{Q}_\kappa[G_{<\kappa}]$. So there is $p \in D^* \cap H$. Hence $j_\tau(p) \in D^* \cap j''_\tau H$. Yielding, $D \cap H^\tau \neq \emptyset$.

So we can do the lift $j_\tau{:}V^*[G_{<\kappa}][H] \to M^*_\tau[G^\tau][H^\tau]$. In order to finish we need to build generic filters over $M^*_{\bar{E}}$. We split the handling into 2 cases:

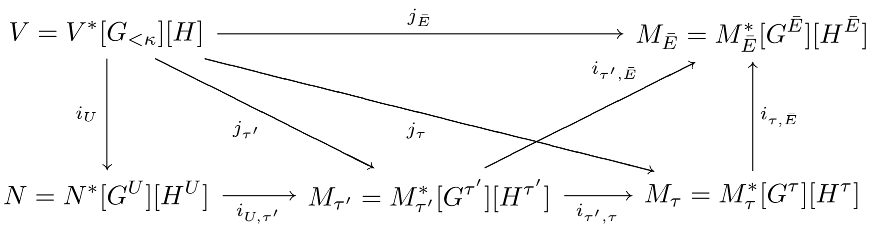
(1) $\mathrm{l}(\bar{E}) = \tau + 1$: In this case we have $M^*_{\bar{E}} = M^*_\tau$, so by setting $G^{\bar{E}} = G^\tau$, $H^{\bar{E}} = H^\tau$ we have the needed filters.
(2) $\mathrm{l}(\bar{E})$ is limit: We let $G^{\bar{E}}$, $H^{\bar{E}}$ be the filters generated by $\bigcup_{\tau < \mathrm{l}(\bar{E})} i''_{\tau,\bar{E}} G^\tau$, $\bigcup_{\tau < \mathrm{l}(\bar{E})} i''_{\tau,\bar{E}} H^\tau$ respectively.

After the forcing the generic extension the power function we have is

$$2^\mu = \begin{cases} \mu^{+3} & \text{if } \mu \in \{\nu^+, \nu^{++}, \nu^{+3}\} \text{ where } \nu \leq \kappa \text{ is inaccessible} \\ \mu^+ & \text{Otherwise} \end{cases},$$

and we still have (3.0.1).

The new diagram we have after all the liftings is




Let $i^2_U$ be the iterate of $i_U$. We choose a function, $R(-,-)$, such that

$$R_U = i^2_U(R)(\kappa, i_U(\kappa)).$$

The function $R(-,-)$ will be used in the definition of the forcing notion $P_{\bar{E}}$ later on.

3.2. **Cardinal structure in $N[I_U]$.** We claim that in $N[I_U]$ there are no cardinals in $[\kappa^{+7}, i_U(\kappa)]$ and all other $N$-cardinals are preserved. The power function differs from the power function of $N$ at the following points: $2^{\kappa^{+4}} = i_U(\kappa)^+$, $2^{\kappa^{+5}} = i_U(\kappa)^{++}$, $2^{\kappa^{+6}} = i_U(\kappa)^{+3}$.

Before continuing we recall that a forcing notion $P$ is called $\lambda$-dense if the intersection of less than $\lambda$ dense open subsets of $P$ is dense open. We recall Easton's lemma:

**Lemma.** *Let $P, Q$ be forcing notions being $\lambda$-closed, $\lambda$-c.c. respectively. Then $\Vdash_Q \ulcorner P$ is $\lambda$-dense$\urcorner$.*

Our construction of $N$ from $N^*$ means that $N[I_U] = N^*[G_{<\kappa}][G^U_\kappa][G^U_{>\kappa}][I_U]$. If we write $N^*[G_{<\kappa}][I_U][G^U_\kappa]$ as $N^*[G_{<\kappa}][I^U_1 \times I^U_2][I^U_3][I^U_4][G^U_\kappa]$ then the usual arguments for product forcing show that the claim is satisfied in this model. However $N^*[G_{<\kappa}][I^U_1 \times I^U_2][I^U_3][I^U_4][G^U_\kappa] \vDash \ulcorner \dot{P}^U_{>\kappa}[G_{<\kappa}][G^U_\kappa]$ is $\kappa^{+4}$-closed$\urcorner$. Hence we might lose $\kappa^{+5}$, $\kappa^{+6}$ in $N^*[G_{<\kappa}][I^U_1 \times I^U_2][I^U_3][I^U_4][G^U_\kappa][G^U_{>\kappa}]$. In order to show that $\kappa^{+5}$ is preserved we work as follows (This argument is due to the referee, our original argument was much more longer and complicated): The usual arguments for Cohen forcing yield

$$N^*[G_{<\kappa}] \vDash \ulcorner \dot{Q}^U_\kappa[G_{<\kappa}] \times R^U_4 \text{ is } \kappa^{+5}\text{-c.c.}\urcorner.$$

Due to the $\kappa^{+5}$-closedness of $R^U_1 \times R^U_2 \times R^U_3$ in $N^*[G_{<\kappa}]$ we still have

$$N^*[G_{<\kappa}][I^U_1 \times I^U_2][I^U_3] \vDash \ulcorner \dot{Q}^U_\kappa[G_{<\kappa}] \times R^U_4 \text{ is } \kappa^{+5}\text{-c.c.}\urcorner.$$

Hence, by general forcing theory, we have

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3] \vDash \ulcorner R^U_4 \text{ is } \kappa^{+5}\text{-c.c.}\urcorner.$$

Since

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3] \vDash \ulcorner \dot{P}^U_{>\kappa}[G_{<\kappa}][G_\kappa] \text{ is } \kappa^{+5}\text{-closed}\urcorner,$$

we get, by Easton's lemma,

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3][I^U_4] \vDash \ulcorner \dot{P}^U_{>\kappa}[G_{<\kappa}][G_\kappa] \text{ is } \kappa^{+5}\text{-dense}\urcorner.$$

Thus $\kappa^{+5}$ remains a cardinal in $N[I_U] = N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3][I^U_4][G_{>\kappa}]$.

Preservation of $\kappa^{+6}$ is proved in a way similar to the above:

$$N^*[G_{<\kappa}] \vDash \ulcorner \dot{Q}^U_\kappa[G_{<\kappa}] \times R^U_4 \times R^U_5 \text{ is } \kappa^{+6}\text{-c.c.}\urcorner.$$

Now we have $\kappa^{+6}$-closedness of $R^U_1 \times R^U_2$ in $N^*[G_{<\kappa}]$ so that

$$N^*[G_{<\kappa}][I^U_1 \times I^U_2] \vDash \ulcorner \dot{Q}^U_\kappa[G_{<\kappa}] \times R^U_4 \times R^U_3 \text{ is } \kappa^{+6}\text{-c.c.}\urcorner.$$

Hence, by general forcing theory, we have

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2] \vDash \ulcorner R^U_4 \times R^U_3 \text{ is } \kappa^{+6}\text{-c.c.}\urcorner.$$

Since

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2] \vDash \ulcorner \dot{P}^U_{>\kappa}[G_{<\kappa}][G_\kappa] \text{ is } \kappa^{+6}\text{-closed}\urcorner,$$

we get, by Easton's lemma,

$$N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3][I^U_4] \vDash \ulcorner \dot{P}^U_{>\kappa}[G_{<\kappa}][G_\kappa] \text{ is } \kappa^{+6}\text{-dense}\urcorner.$$

Thus $\kappa^{+6}$ remains a cardinal in $N[I_U] = N^*[G_{<\kappa}][G_\kappa][I^U_1 \times I^U_2][I^U_3][I^U_4][G_{>\kappa}]$.

By observing that $N^*[G_{<\kappa}][I_U][G^U_\kappa] \vDash \ulcorner|\dot{P}^U_{>\kappa}[G_{<\kappa}][G^U_\kappa]| = \kappa^{+6}\urcorner$ we see that the power function of $N^*[G_{<\kappa}][I_U][G^U_\kappa][G^U_{>\kappa}]$ is the same as the power function of $N^*[G_{<\kappa}][I_U][G^U_\kappa]$.

3.3. **Locating the needed generic filters.** As $U \in M_{\bar{E}}$ and $\forall \tau < \mathrm{l}(\bar{E})\ E(\tau) \in M_{\bar{E}}$ we have the following diagram

$$\begin{array}{ccc} M_{\bar{E}} & & \\ {\scriptstyle i^{\bar{E}}_U}\downarrow & \searrow^{j^{\bar{E}}_\tau} & \\ N^{\bar{E}} & \xrightarrow[i^{\bar{E}}_{U,\tau}]{} & M^{\bar{E}}_\tau \end{array} \qquad \begin{array}{l} U = E_\kappa(0), \\ i^{\bar{E}}_U{:}M_{\bar{E}} \to N^{\bar{E}} \simeq \mathrm{Ult}(M_{\bar{E}}, U), \\ j^{\bar{E}}_\tau{:}M_{\bar{E}} \to M^{\bar{E}}_\tau \simeq \mathrm{Ult}(M_{\bar{E}}, E(\tau)), \\ i^{\bar{E}}_{U,\tau}(i^{\bar{E}}_U(f)(\kappa)) = j^{\bar{E}}_\tau(f)(\kappa). \end{array}$$

Recall that we have $I_U \in M_{\bar{E}}$ which is $R_U$-generic over $N^{\bar{E}}$.

3.3.1. *Generic over $M_\tau$ when $\tau + 1 < \mathrm{l}(\bar{E})$.* Consider the following forcing notion:

$$R_\tau = (\mathrm{Col}(\kappa^{+6}, j_\tau(\kappa)) \times \mathrm{C}(\kappa^{+4}, j_\tau(\kappa)^+) \times \mathrm{C}(\kappa^{+5}, j_\tau(\kappa)^{++}) \times \\ \mathrm{C}(\kappa^{+6}, j_\tau(\kappa)^{+3}))_{M^*_\tau[G_{<\kappa}]}.$$

We show that there is $I_\tau \in M_{\bar{E}}$, an $R_\tau$-generic filter over $M_\tau$. Moreover, whenever $\tau' < \tau < \mathrm{l}(\bar{E})$ we have $i''_{\tau',\tau} I_{\tau'} \subseteq I_\tau$. For this we set

$$R^{\bar{E}}_\tau = (\mathrm{Col}(\kappa^{+6}, j^{\bar{E}}_\tau(\kappa)) \times \mathrm{C}(\kappa^{+4}, j^{\bar{E}}_\tau(\kappa)^+) \times \mathrm{C}(\kappa^{+5}, j^{\bar{E}}_\tau(\kappa)^{++}) \times \\ \mathrm{C}(\kappa^{+6}, j^{\bar{E}}_\tau(\kappa)^{+3}))_{M^{*\bar{E}}_\tau[G_{<\kappa}]}.$$

$R_\tau$, $R^{\bar{E}}_\tau$ are coded in $V^{M^*_\tau[G_{<\kappa}]}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau[G_{<\kappa}]}_{j^{\bar{E}}_\tau(\kappa)+3}$ respectively. $V^{M^*_\tau[G_{<\kappa}]}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau[G_{<\kappa}]}_{j^{\bar{E}}_\tau(\kappa)+3}$ are determined by $V^{V^*}_{\kappa+3}$, $V^{M^*_{\bar{E}}}_{\kappa+3}$ (and $E(\tau)$, of course). As $E(\tau) \in M^*_{\bar{E}}$ and $V^{V^*}_{\kappa+3} = V^{M^*_{\bar{E}}}_{\kappa+3}$ we get that $R_\tau = R^{\bar{E}}_\tau$.

By the same reasoning, each anti-chain of $R_\tau$ appearing in $M_\tau$ is also an anti-chain of $R^{\bar{E}}_\tau$ appearing in $M^{\bar{E}}_\tau$. Hence, if $I_\tau \in M_{\bar{E}}$ is $R^{\bar{E}}_\tau$-generic filter over $M^{\bar{E}}_\tau$ then it is also $R_\tau$-generic filter over $M_\tau$.

Let $I_\tau \in M_{\bar{E}}$ be the filter generated by $i^{\bar{E}\prime\prime}_{U,\tau} I_U$. We have the natural factoring $I_\tau = I^\tau_1 \times \cdots \times I^\tau_4$, We claim that $I_\tau$ is $R_\tau$-generic over $M^{\bar{E}}_\tau$. We start by showing genericity over $M^{*\bar{E}}_\tau[G_{<\kappa}]$. So, let $D \in M^{*\bar{E}}_\tau[G_{<\kappa}]$ be dense open in $R_\tau$.

Let $\dot{D} \in M^{*\bar{E}}_\tau$ be a $P_\kappa$-name for $D$. Choose $f \in M^*_{\bar{E}}$ such that $j^{\bar{E}}_\tau(f)(\bar{E}_\alpha\restriction\tau) = \dot{D}$. So in $M^*_{\bar{E}}$ we have

$$A = \{\bar{\nu} \mid M^*_{\bar{E}} \vDash \ulcorner \Vdash_{P_{\kappa^0(\bar{\nu})}} \ulcorner f(\bar{\nu}) \text{ is dense open in } \mathrm{Col}(\kappa^0(\bar{\nu})^{+6}, \kappa) \times \\ \mathrm{C}(\kappa^0(\bar{\nu})^{+4}, \kappa^+) \times \mathrm{C}(\kappa^0(\bar{\nu})^{+5}, \kappa^{++}) \times \mathrm{C}(\kappa^0(\bar{\nu})^{+6}, \kappa^{+3})\urcorner\urcorner\} \in E_\alpha(\tau).$$

The standard observation $B = \{\mu \mid |\{\bar{\nu} \in A \mid \kappa^0(\bar{\nu}) = \mu\}| \leq \mu^{+3}\} \in E_\kappa(0)$ yields that for each $\mu \in B$ there is a $P_\mu$-name, $f^*(\mu)$, such that for all $\bar{\nu} \in A$ with

$\kappa^0(\bar\nu) = \mu$ we have $M^*_{\bar E} \vDash \ulcorner \Vdash_{P_\mu} \ulcorner f^*(\mu) \subseteq f(\bar\nu)$ is dense open$\urcorner\urcorner$. Hence

$$N^{*\bar E} \vDash \ulcorner \Vdash_{P_\kappa} \ulcorner i^{\bar E}_U(f^*)(\kappa) \text{ is dense open}\urcorner\urcorner,$$
$$M^{*\bar E}_\tau \vDash \ulcorner \Vdash_{P_\kappa} \ulcorner j^{\bar E}_\tau(f^*)(\kappa) \subseteq j^{\bar E}_\tau(f)(\bar E_\alpha\restriction\tau)\urcorner\urcorner.$$

So there is $g \in M^*_{\bar E}$ such that $i^{\bar E}_U(g)(\kappa)[G_{<\kappa}] \in i^{\bar E}_U(f^*)(\kappa)[G_{<\kappa}] \cap I_U$. Noting that we have $i^{\bar E}_{U,\tau}{:}N^{*\bar E}[G_{<\kappa}] \to M^{*\bar E}_\tau[G_{<\kappa}]$, we get $j^{\bar E}_\tau(g)(\kappa)[G_{<\kappa}] \in j^{\bar E}_\tau(f^*)(\kappa)[G_{<\kappa}] \cap i^{\bar E\prime\prime}_{U,\tau} I_U$. That is $j^{\bar E}_\tau(f)(\bar E_\alpha\restriction\tau)[G_{<\kappa}] \cap I_\tau \neq \emptyset$. By this we showed that $I_\tau$ is $R_\tau$-generic over $M^{*\bar E}_\tau[G_{<\kappa}]$. Hence it is $R_\tau$-generic over $M^*_\tau[G_{<\kappa}]$.

So, we can consider $M^*_\tau[G_{<\kappa}][I_\tau]$. As $M^*_\tau \vDash \ulcorner R_\tau$ is $\kappa^{+4}$-closed$\urcorner$ we get that $G^\tau_\kappa$ is $\dot Q^\tau_\kappa[G_{<\kappa}]$-generic over $M^*_\tau[G_{<\kappa}][I_\tau]$. By commutativity of product forcing we get that $I_\tau$ is $R_\tau$-generic over $M^*_\tau[G_{<\kappa}][G^\tau_\kappa]$. Hence we can consider $M^*_\tau[G_{<\kappa}][G^\tau_\kappa][I_\tau]$. We want to show that $G^\tau_{>\kappa}$ is generic over $M^*_\tau[G_{<\kappa}][G^\tau_\kappa][I_\tau]$. For this we lift $i_{U,\tau}$ to $i^*_{U,\tau}{:}N^*[G_{<\kappa}][G^U_\kappa][I_U] \to M^*_\tau[G_{<\kappa}][G^\tau_\kappa][I_\tau]$ which is possible by recalling how we generated $I_\tau$ from $I_U$. Let $D \in M^*_\tau[G_{<\kappa}][G^\tau_\kappa][I_\tau]$ be dense open in $\dot P^\tau_{>\kappa}[G_{<\kappa}][G^\tau_\kappa]$.

Then there is $\dot D \in M^*_\tau[G_{<\kappa}][G^\tau_\kappa]$ such that $\dot D[I_\tau] = D$. Of course $\dot D \in M^*_\tau[G^\tau]$ as $M^*_\tau[G^\tau] = M^*_\tau[G_{<\kappa}][G^\tau_\kappa][G^\tau_{>\kappa}] \supset M^*_\tau[G_{<\kappa}][G^\tau_\kappa]$. Hence there is $f \in V^*[G_{<\kappa}]$ such that $j_{\bar E}(f)(\bar E_\alpha\restriction\tau) = \dot D$. Then in $V^*[G_{<\kappa}][H]$ we have

$$A = \{\bar\nu \mid V^*[G_{<\kappa^0(\bar\nu)}][G_{\kappa^0(\bar\nu)}] \vDash \ulcorner \Vdash_{R(\kappa^0(\bar\nu),\kappa)} \ulcorner f(\bar\nu) \text{ is dense open in } \dot P_{>\kappa^0(\bar\nu)}\urcorner\urcorner\} \in \bar E_\alpha(\tau).$$

We note that for $\mu$ inaccessible we have $f\restriction\mu^{+3} \in V^*[G_{<\mu}][G_\mu]$ and $V^*[G_{<\mu}][G_\mu] \vDash \ulcorner \Vdash_{R(\mu,\kappa)} \ulcorner \dot P_{>\mu}[G_{<\mu}][G_\mu]$ is $\mu^{+4}$-closed$\urcorner\urcorner$. Hence there is $f^* \in V^*[G_{<\kappa}]$ such that $\forall\bar\nu \in A$ $V^*[G_{<\kappa^0(\bar\nu)}][G_{\kappa^0(\bar\nu)}] \vDash \ulcorner \Vdash_{R(\kappa^0(\bar\nu),\kappa)} \ulcorner f^*(\kappa^0(\bar\nu)) \subseteq f(\bar\nu)$ is dense open$\urcorner\urcorner$. This implies

$$N^*[G_{<\kappa}][G^U_\kappa][I_U] \vDash \ulcorner i_U(f^*)(\kappa)[I_U] \text{ is dense open}\urcorner,$$
$$M^*_\tau[G_{<\kappa}][G^\tau_\kappa][I_\tau] \vDash \ulcorner j_\tau(f^*)(\kappa)[I_\tau] \subseteq j_\tau(f)(\bar E_\alpha\restriction\tau)[I_\tau]\urcorner.$$

By genericity of $G^U_{>\kappa}$ over $N^*[G_{<\kappa}][G^U_\kappa][I_U]$ we get that there is $g \in V^*[G_{<\kappa}][H]$ such that $i_U(g)(\kappa) \in i_U(f^*)(\kappa)[I_U] \cap G^U_{>\kappa}$. Hence $j_\tau(g)(\kappa) \in j_\tau(f)(\bar E_\alpha\restriction\tau)[I_\tau] \cap i''_{U,\tau} G^U_{>\kappa}$. That is $D \cap G^\tau_{>\kappa} \neq \emptyset$.

Of course this implies that $I_\tau$ is generic over $M^*_\tau[G^\tau]$. As $H^\tau$ adds no new anti-chains to $R_\tau$ we get that $I_\tau$ is generic over $M^*_\tau[G^\tau][H^\tau]$.

As for the moreover: $V^{N^{*\bar E}}_{i^{\bar E}(\kappa)+3} = V^{N^*}_{i(\kappa)+3}$ and $i^{\bar E}_{U,\tau}\restriction V^{N^{*\bar E}}_{i^{\bar E}(\kappa)+3} = i_{U,\tau}\restriction V^{N^*}_{i(\kappa)+3}$, $i^{\bar E}_{\tau',\tau}\restriction V^{N^{*\bar E}}_{i^{\bar E}(\kappa)+3} = i_{\tau',\tau}\restriction V^{N^*}_{i(\kappa)+3}$. By their definition $i^{\bar E\prime\prime}_{\tau',\tau} \circ i^{\bar E\prime\prime}_{U,\tau'} I^U = i^{\bar E\prime\prime}_{U,\tau} I^U$. Hence $i''_{\tau',\tau} I_{\tau'} \subseteq I_\tau$.

3.3.2. *Generic over $M_{\bar E}$.*

$$R_{\bar E} = (\mathrm{Col}(\kappa^{+6}, j_{\bar E}(\kappa)) \times \mathrm{C}(\kappa^{+4}, j_{\bar E}(\kappa)^+) \times \mathrm{C}(\kappa^{+5}, j_{\bar E}(\kappa)^{++}) \times \mathrm{C}(\kappa^{+6}, j_{\bar E}(\kappa)^{+3}))_{M^*_{\bar E}[G_{<\kappa}]}.$$

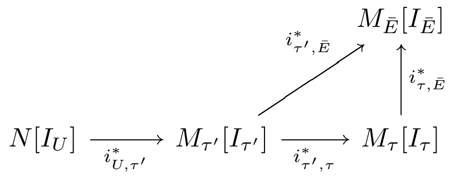
We are going to show that there is $I_{\bar E} \in V$, which is $R_{\bar E}$-generic over $M_{\bar E}$. Moreover, whenever $\tau < \mathrm{l}(\bar E)$ we have $i''_{\tau,\bar E} I_\tau \subseteq I_{\bar E}$. There are 2 different cases

(1) $\mathrm{l}(\bar E)$ is limit: We set $I_{\bar E}$ to be the filter generated by $\bigcup_{\tau<\mathrm{l}(\bar E)} i''_{\tau,\bar E} I_\tau$. Its' genericity is rather straightforward. Let $D \in M_{\bar E}$ be dense open in $R_{\bar E}$.

Then there is $D_\tau \in M_\tau$ such that $i_{\tau,\bar{E}}(D_\tau) = D$. By genericity of $I_\tau$, there is $p \in D_\tau \cap I_\tau$. Hence $i_{\tau,\bar{E}}(p) \in D \cap i''_{\tau,\bar{E}} I_\tau$. That is $D \cap I_{\bar{E}} \neq \emptyset$.

(2) $\mathrm{l}(\bar{E}) = \tau + 1$: Just follow the proof of 3.3.1 with $V^*$, $M^*_\tau$ instead of $M^*_{\bar{E}}$, $M^{*\bar{E}}_\tau$.

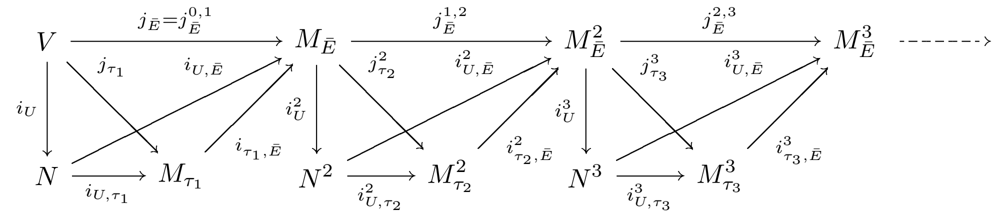
3.4. **Cardinal structure in $M_\tau[I_\tau]$, $M_{\bar{E}}[I_{\bar{E}}]$.** The following lifting says everything which we can possibly say.




We use these embeddings only in this subsection and not carry them on.

The forcing notion we define later, $P_{\bar{E}}$, adds a club to $\kappa$. For each $\nu_1, \nu_2$ successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M_{\bar{E}}[I_{\bar{E}}]$.

3.5. **Generic filters over iterated ultrapowers.** We iterate $j_{\bar{E}}$ and consider the following diagram




$$j^0_{\bar{E}} = \mathrm{id},\ j^n_{\bar{E}} = j^{0,n}_{\bar{E}},$$
$$M^0_{\bar{E}} = V,\ M^1_{\bar{E}} = M_{\bar{E}},$$
$$G^{\bar{E},0} = G_{<\kappa},\ H^{\bar{E},0} = H,$$
$$G^{\bar{E},n} = j^n_{\bar{E}}(G^{\bar{E},0}),\ H^{\bar{E},n} = j^n_{\bar{E}}(H^{\bar{E},0}),$$
$$G^{\bar{E},n} = G^{\bar{E},n}_{<\kappa_{n-1}} \times G^{\bar{E},n}_{\kappa_{n-1}} \times G^{\bar{E},n}_{>\kappa_{n-1}},$$
$$\kappa_0 = \kappa,\ \kappa_n = j^n_{\bar{E}}(\kappa),$$
$$j^{n,n+1}_{\tau_{n+1}}{:}M^n_{\bar{E}} \to M^{n+1}_{\tau_{n+1}} \simeq \mathrm{Ult}(M^n_{\bar{E}}, j^n_{\bar{E}}(E)(\tau_{n+1})),$$
$$M^n_{\bar{E}} = M^{*n}_{\bar{E}}[G^{\bar{E},n}][H^{\bar{E},n}],$$
$$j^{m,n}_{\bar{E}} = j^{n-1,n}_{\bar{E}} \circ \cdots \circ j^{m+1,m+2} \circ j^{m,m+1}_{\bar{E}},$$
$$R^1_{\bar{E}} = R_{\bar{E}},\ R^n_{\bar{E}} = j^{n-1}_{\bar{E}}(R_{\bar{E}}),\ R^1_U = R_U,\ R^n_U = j^{n-1}_U(R_U),$$
$$I^1_{\bar{E}} = I_{\bar{E}},\ I^n_{\bar{E}} = j^{n-1}_{\bar{E}}(I_{\bar{E}}),\ I^1_U = I_U,\ I^n_U = j^{n-1}_U(I_U).$$

We note that

$$R^n_{\bar{E}} = (\mathrm{Col}(j^{n-1}_{\bar{E}}(\kappa)^{+6}, j^n_{\bar{E}}(\kappa)) \times \mathrm{C}(j^{n-1}_{\bar{E}}(\kappa)^{+4}, j^n_{\bar{E}}(\kappa)^{+}) \times \mathrm{C}(j^{n-1}_{\bar{E}}(\kappa)^{+5}, j^n_{\bar{E}}(\kappa)^{++}) \times \mathrm{C}(j^{n-1}_{\bar{E}}(\kappa)^{+6}, j^n_{\bar{E}}(\kappa)^{+3}))_{M^{*n+1}_{\bar{E}}[G^{\bar{E},n}_{<\kappa_{n-1}}]}.$$

We claim that $I_{\bar E}\times I^2_{\bar E}\times\dots\times I^n_{\bar E}$ is $R_{\bar E}\times R^2_{\bar E}\times\dots\times R^n_{\bar E}$-generic over $M^{n+1}_{\bar E}$. Of course genericity over $M^n_{\bar E}$ is more than enough for this. Moreover, if $D\in M^{n+1}_{\bar E}$ is dense open in $R^n_{\bar E}$ then there is $j_{\bar E}(f)(\bar E_\alpha\restriction\tau)\in I_{\bar E}$ such that $j^n_{\bar E}(f)(j^{n-1}_{\bar E}(\bar E_\alpha\restriction\tau))\in D\cap I^n_{\bar E}$.

By construction, $I_{\bar E}$ is $R_{\bar E}$-generic over $M_{\bar E}$. Formally we have

$$V\vDash\ulcorner I_{\bar E}\text{ is }R_{\bar E}\text{-generic over }M_{\bar E}\urcorner.$$

Applying $j^{n-1}_{\bar E}$ we get

$$M^{n-1}_{\bar E}\vDash\ulcorner I^n_{\bar E}\text{ is }R^n_{\bar E}\text{-generic over }M^n_{\bar E}\urcorner.$$

As for the product forcing we note that

$$\begin{aligned}&G^{\bar E,n}_{<\kappa_{n-1}}=G_{<\kappa}\times G^{\bar E,1}_{\kappa}\times G^{\bar E,1}_{>\kappa}\times G^{\bar E,2}_{\kappa_1}\times G^{\bar E,2}_{>\kappa_1}\times\dots\times G^{\bar E,n-1}_{\kappa_{n-2}}\times G^{\bar E,n-1}_{>\kappa_{n-2}},\\ &G^{\bar E,n}=G^{\bar E,n}_{<\kappa_{n-1}}\times G^{\bar E,n}_{\kappa_{n-1}}\times G^{\bar E,n}_{>\kappa_{n-1}}.\end{aligned}$$

Let use assume, by induction, that $I_{\bar E}\times I^2_{\bar E}\times\dots\times I^{n-1}_{\bar E}$ is $R_{\bar E}\times R^2_{\bar E}\times\dots\times R^{n-1}_{\bar E}$-generic over $M^{n-1}_{\bar E}$. Then it is also generic over $M^n_{\bar E}$. As $M^{*n}_{\bar E}[G^{\bar E,n}_{<\kappa_{n-1}}]\vDash\ulcorner(\dot Q^{\bar E,n}_{\kappa_{n-1}}*\dot P^{\bar E,n}_{>\kappa_{n-1}}*\dot Q^{\bar E,n}_{\kappa_n})[G^{\bar E,n}_{<\kappa_{n-1}}]\times R^n_{\bar E}$ is $\kappa^+_{n-1}$-closed$\urcorner$, all anti-chains of $R_{\bar E}\times\dots\times R^{n-1}_{\bar E}$ appearing in $M^{*n}_{\bar E}[G^{\bar E,n}_{<\kappa_{n-1}}][G^{\bar E,n}_{\kappa_{n-1}}][G^{\bar E,n}_{>\kappa_{n-1}}][H^{\bar E,n}][I^n_{\bar E}]$ are already in $M^{*n}_{\bar E}[G^{\bar E,n}_{<\kappa_{n-1}}]$. That is $I^1_{\bar E}\times\dots\times I^{n-1}_{\bar E}$ is $R^1_{\bar E}\times\dots\times R^{n-1}_{\bar E}$-generic over $M^{*n}_{\bar E}[G^{\bar E,n}_{<\kappa_{n-1}}][G^{\bar E,n}_{\kappa_{n-1}}][G^{\bar E,n}_{>\kappa_{n-1}}][H^{\bar E,n}][I^n_{\bar E}]$. So we get what we need: $I^1_{\bar E}\times\dots\times I_{\bar E}$ is $R^1_{\bar E}\times\dots\times R^n_{\bar E}$-generic over $M^n_{\bar E}$. We are left to prove the 'moreover' part.

We start by showing that $j^{n-1}_{\bar E}{}''I_{\bar E}$ is dense in $j^{n-1}_{\bar E}(I_{\bar E})=I^n_{\bar E}$. For this we point out that $i''_{U,\bar E}I_U$ is dense $I_{\bar E}$. By elementarity we get that $i^{n\prime\prime}_{U,\bar E}I^n_U$ is dense in $I^n_{\bar E}$. So, it is enough to show that $j^{n-1}_{\bar E}{}''I_U$ is dense in $j^{n-1}_{\bar E}(I_U)=I^n_U$.

The proof is by induction and we start with $n=2$. Let $p\in I^2_U$. Choose $f$ such that $j_{\bar E}(f)(\bar E_\alpha\restriction\tau)=p$. Then $A=\{\bar\nu\mid f(\bar\nu)\in I_U\}\in E_\alpha(\tau)$. Let $B=\{f(\bar\nu)\mid\bar\nu\in A\}$. As $N\supset N^\kappa$ we get that $B\in N$. As $N\vDash\ulcorner R_U$ is $\kappa^+$-closed$\urcorner$ we get that there is $q\in I_U$ such that $\forall\bar\nu\in A\ q\leq f(\bar\nu)$. Hence $j_{\bar E}(q)\leq j_{\bar E}(f)(\bar E_\alpha\restriction\tau)$. By this we proved $j''_{\bar E}I_U$ is dense in $I^2_U$.

Of course, by elementarity $j^{n-1,n}_{\bar E}{}''I^n_U$ is dense in $j^{n-1,n}_{\bar E}(I^n_U)=I^{n+1}_U$. Our induction hypothesis is that $j^{n-1}_{\bar E}{}''I_U$ is dense in $I^n_U$. Hence $j^{n-1,n}{}''\circ j^{n-1}_{\bar E}{}''I_U$ is dense in $I^{n+1}_U$ as needed.

By the above, if $D\in M^n_{\bar E}$ is dense open in $R^n_{\bar E}$, then $D\cap j^{n-1}_{\bar E}{}''I_{\bar E}\neq\emptyset$. By showing that $j^{n-1}_{\bar E}{}''I_{\bar E}$ is of the required structure we finish the claim. So, let $p\in I_{\bar E}$.

Then $p=j_{\bar E}(f)(\bar E_\alpha\restriction\tau)$. Formally, $V\vDash\ulcorner p=j_{\bar E}(f)(\bar E_\alpha\restriction\tau)\urcorner$. Applying $j^{n-1}_{\bar E}$ we get $M^{n-1}_{\bar E}\vDash\ulcorner j^{n-1}_{\bar E}(p)=j^{n-1,n}_{\bar E}(j^{n-1}_{\bar E}(f))(j^{n-1}_{\bar E}(\bar E_\alpha\restriction\tau))\urcorner$. That is, $j^{n-1}_{\bar E}(p)=j^n_{\bar E}(f)(j^{n-1}_{\bar E}(\bar E_\alpha\restriction\tau))$.

## 4. Redefining extender sequences

Our starting assumption in this section is the models and embeddings constructed in the previous section. The extender sequences we define here are based on the old ones and have the same length. They differ in that we add generic filters into the sequence.

We construct a new extender sequence system, $\bar F$, from $\bar E$. If $\mathrm{l}(\bar E)=0$ then

$$\forall\alpha\in\operatorname{dom}\bar E\ \bar F_\alpha=\langle\alpha\rangle.$$

If $\mathrm{l}(\bar{E}) = 1$ we set $F(0) = E(0)$. According to the previous section construction, there is $I(0) \in V$ which is $R_0$-generic over $M_0$. We set

$$\forall \alpha \in \operatorname{dom} \bar{E}\ \bar{F}_\alpha = \langle \alpha, F(0), I(0) \rangle.$$

By $I(\bar{F})$ we mean $I(0)$.

We continue by induction. Assume we have defined $\langle F(\tau'), I(\tau') \mid \tau' < \tau \rangle$.

If $\tau = \mathrm{l}(\bar{E})$ then we set

$$\forall \alpha \in \operatorname{dom} \bar{E}\ \bar{F}_\alpha = \langle \alpha, F(0), I(0), \dots, F(\tau'), I(\tau'), \dots \mid \tau' < \tau \rangle.$$

We define $I(\bar{F})$ as follows

(1) $\tau$ is limit: By $I(\bar{F})$ we mean the filter generated by $\bigcup_{\tau' < \tau} i''_{\tau', \bar{E}} I(\tau')$. This filter is $R_{\bar{E}}$-generic over $M_{\bar{E}}$.

(2) $\tau = \tau' + 1$: $I(\bar{F})$ is $I(\tau')$. Note that in this case $M_{\bar{E}} = M_{\tau'}$, so $I(\bar{F})$ is $R_{\bar{E}}$-generic over $M_{\bar{E}}$.

If $\tau < \mathrm{l}(\bar{E})$ then we define

$$A \in F_\alpha(\tau) \iff \langle \alpha, F(0), I(0), \dots, F(\tau'), I(\tau'), \dots \mid \tau' < \tau \rangle \in j_{\bar{E}}(A).$$

If $\tau + 1 < \mathrm{l}(\bar{E})$ then there is $I(\tau) \in M_{\bar{E}}$ which is $R_\tau$-generic over $M_\tau$. If $\tau + 1 = \mathrm{l}(\bar{E})$ then there is $I(\tau) \in V$ which is $R_\tau$-generic over $M_\tau$.

By this we finished the definition of the new extender sequence. We point out that $\mathrm{Ult}(V, E(\tau)) = \mathrm{Ult}(V, F(\tau))$. This is due to $I(\tau') \in M_\tau$ when $\tau' < \tau$. Hence, the $F(\tau)$'s, do not 'pull' into $M_\tau$ sets which were not already pulled in by the $E(\tau)$.

From now on we continue with this new definition of extender sequence and we use $\bar{E}$ for the new extender sequence system constructed.

**Definition 4.1.** We say $T \in \bar{E}_\alpha$ if $\forall \xi < \mathrm{l}(\bar{E}_\alpha)\ T \in E_\alpha(\xi)$.

*Note* 4.2. In [19] we defined here what is an $\bar{E}_\alpha$-tree. In this work we use just sets which are in $\bar{E}_\alpha$. We use the letters $S$, $T$, $R$, etc. (used for trees in [19]) for these sets.

The operations defined next are the substitute for $T_{\langle \bar{\nu} \rangle}$, and $T(\bar{\nu})$ from [19].

**Definition 4.3.**

$$T \setminus \bar{\nu} = T \setminus V_{\kappa^0(\bar{\nu})},$$
$$T \restriction \bar{\nu} = T \cap V_{\kappa^0(\bar{\nu})}.$$

We define next a form of diagonal intersection which works well also for the non-normal measures.

**Definition 4.4.** Assume $\forall \xi < \kappa\ T^\xi \subseteq V_\kappa$ such that the elements of $T^\xi$ are extender sequences. Then ${\triangle^0}_{\xi < \kappa} T^\xi = \{ \bar{\nu} \in V_\kappa \mid \forall \xi < \kappa^0(\bar{\nu})\ \bar{\nu} \in T^\xi \}$.

Obviously if $\forall \xi < \kappa\ T^\xi \in \bar{E}_\alpha$ then ${\triangle^0}_{\xi < \kappa} T^\xi \in \bar{E}_\alpha$.

**Definition 4.5.** $S \subseteq (V_\kappa)^m$ is called an $\bar{E}_\alpha$-fat tree if

(1) $\exists \xi < \mathrm{l}(\bar{E}_\alpha)\ \mathrm{Lev}_0(S) \in \bar{E}_\alpha(\xi)$,

(2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S\ \exists \xi < \mathrm{l}(\bar{E}_\alpha)\ \mathrm{Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle) \in \bar{E}_\alpha(\xi)$.

As an example, suppose $S$ is $\bar{E}_\alpha$-fat tree of height two. Then the first level of $S$ is composed of elements of the form $\langle \bar{\nu} \rangle$. The second level of $S$ is composed of elements of the form $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle$. Now, the first level of $S$ belongs to *one* of the measures composing $\bar{E}_\alpha$. As for the second level, we look how we got there. So, if $\langle \bar{\nu}_1 \rangle \in S$ then there are many $\langle \bar{\nu}_2 \rangle$ such that $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in S$. Namely, the set of such $\langle \bar{\nu}_2 \rangle$'s is also in *one* of the measures composing $\bar{E}_\alpha$.

## 5. $P_{\bar{E}}$-Forcing

At the suggestion of the referee we add some verbal explanation to the rather technical definition of the forcing notion.

Our aim is to add many Radin sequences to a cardinal $\kappa$. The amount added depends on the size of the extender sequence system we use, and the order type of the sequences depends on the length of the extender sequence system. A condition gives information about initial part of up to $\kappa$ Radin sequences. The Radin sequences added are indexed by the extender sequences. Hence for each $\bar{E}_\alpha \in \bar{E}$ we have the Radin sequence $M^{\bar{E}_\alpha}$. A condition, $p$, gives information regarding $M^{\bar{E}_\alpha}$ if $\bar{E}_\alpha$ is in the *support* of $p$. Namely, one can view $p$ as containing a function from $\kappa$ extender sequences into $\kappa$ initial-segments of a Radin sequence. If $\bar{E}_\alpha$ is in the support of $p$ we name the start-segment information given by $p$ regarding the Radin sequence $M^{\bar{E}_\alpha}$ by $p^{\bar{E}_\alpha}$. Hence, if $G$ is a generic filter then $M^{\bar{E}_\alpha} = \bigcup\{p^{\bar{E}_\alpha} \mid p \in G,\ \bar{E}_\alpha \in \operatorname{supp}(p)\}$. In order to have the Prikry property we need a union of measure one sets from which to choose an extender sequence to enlarge the initial segments. As in the Gitik-Magidor forcing we do not carry an independent set for each $\bar{E}_\alpha$ in $\operatorname{supp} p$. Instead we have a distinguished sequence, $\operatorname{mc}(p)$, maximal in the $<_{\bar{E}}$ ordering among all elements of *suppp*, and the set from which we choose a point to add, $\bar{\mu}$, belongs to $\operatorname{mc}(p)$. We do not extend too many initial-segments at once, due to some technical reasons. When we do decide to extend an initial sequence, $p^{\bar{E}_\alpha}$, we do it with the point $\pi_{\operatorname{mc}(p),\bar{E}_\alpha}(\bar{\mu})$.

Like in Radin forcing, there are in fact two types of initial sequence extensions. When a singleton is added, we just put it on top of the sequence. If an extender sequence is added, we remove the initial segment and leave only the added extender sequence. All the sequences which were removed are collected into a new block and grow independently using the extender sequence just added.

Alongside the sequence $M^{\bar{E}_\kappa}$ we attach a sequence of cardinal collapsing and Cohen extensions forcing notions which decide what cardinals are successors in the model, and what their power size is.

We return to the technical side. The following definition is for the degenerate case. It is the equivalent of adding a singleton in Radin forcing. Note that extension in this forcing means only extending the functions.

**Definition 5.1.** Assume $\bar{E}$ is an extender sequence system such that $\operatorname{l}(\bar{E}) = 0$. A condition $p$ in $P^*_{\bar{E}}$ is of the form

$$\{\langle \bar{E}_\kappa, p^{\bar{E}_\kappa} \rangle\} \cup \{\langle \bar{E}_\kappa, f \rangle\}$$

where

(1) $p^{\bar{E}_\kappa} \in V_{\kappa^0(\bar{E})}$ is an extender sequence (We allow $p^{\bar{E}_\kappa} = \emptyset$.). We write $p^0$ for $p^{\min s}$ (i.e. $p^{\bar{E}_\kappa}$),

(2) $f \in R(\kappa(p^0), \kappa^0(\bar{E}))$. If $p^0 = \emptyset$ then $f = \emptyset$.

We write $f^p$, $\operatorname{mc}(p)$, $\operatorname{supp} p$, for $f$, $\bar{E}_\kappa$, $\{\bar{E}_\kappa\}$, respectively.

**Definition 5.2.** Assume $\operatorname{l}(\bar{E}) = 0$. Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a Prikry extension of $q$ ($p \leq^* q$) if

(1) $p^0 = q^0$,
(2) $f^p \leq f^q$.

When $\mathrm{l}(\bar{E}) = 0$ the order $\leq^{**}$ disappears:

**Definition 5.3.** Assume $\mathrm{l}(\bar{E}) = 0$. Let $p, q \in P^*_{\bar{E}}$. We say that $p \leq^{**} q$ if $p = q$.

Clearly $\langle P^*_{\bar{E}}/p, \leq^* \rangle \simeq R(\kappa(p^0), \kappa^0(\bar{E}))/f^p$.

**Definition 5.4.** Assume $\mathrm{l}(\bar{E}) > 0$. A condition $p$ in $P^*_{\bar{E}}$ is of the form

$$\{\langle \bar{\gamma}, p^{\bar{\gamma}} \rangle \mid \bar{\gamma} \in s\} \cup \{\langle \bar{E}_\alpha, T, f, F \rangle\}$$

where

(1) $s \subseteq \bar{E}$, $|s| \leq \kappa$, $\min \bar{E} \in s$,
(2) $p^{\bar{E}_\kappa} \in V_{\kappa^0(\bar{E})}$ is an extender sequence (We allow $p^{\bar{E}_\kappa} = \emptyset$.). We write $p^0$ for $p^{\min s}$ (i.e. $p^{\bar{E}_\kappa}$),
(3) $\forall \bar{\gamma} \in s \setminus \{\min s\}$ $p^{\bar{\gamma}} \in (V_{\kappa^0(\bar{E})})^{<\omega}$ is a sequence of extender sequences where $\kappa^0(p^{\bar{\gamma}})$ is increasing. (We allow $p^{\bar{\gamma}} = \emptyset$.),
(4) $\forall \bar{\gamma} \in s$ $\kappa(p^0) \leq \max \kappa^0(p^{\bar{\gamma}})$,
(5) $T \in \bar{E}_\alpha$,
(6) $\forall \bar{\nu} \in T$ $|\{\bar{\gamma} \in s \mid \max \kappa^0(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu})\}| \leq \kappa^0(\bar{\nu})$,
(7) $\forall \bar{\gamma} \in s$ $\bar{E}_\alpha \geq_{\bar{E}} \bar{\gamma}$,
(8) $\forall \bar{\beta}, \bar{\gamma} \in s$ $\forall \bar{\nu} \in T$ $\max \kappa^0(p^{\bar{\beta}}), \max \kappa^0(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \implies \pi_{\bar{E}_\alpha, \bar{\beta}}(\bar{\nu}) \neq \pi_{\bar{E}_\alpha, \bar{\gamma}}(\bar{\nu})$,
(9) $f$ is a function such that
  (9.1) $\operatorname{dom} f = \{\bar{\nu} \in T \mid \mathrm{l}(\bar{\nu}) = 0\}$,
  (9.2) $f(\nu_1) \in R(\kappa(p^0), \nu_1^0)$. If $p^0 = \emptyset$ then $f(\nu_1) = \emptyset$.
(10) $F$ is a function such that
  (10.1) $\operatorname{dom} F = (\{\bar{\nu} \in T \mid \mathrm{l}(\bar{\nu}) = 0\})^2$,
  (10.2) $F(\nu_1, \nu_2) \in R(\nu_1^0, \nu_2^0)$,
  (10.3) $j^2_{\bar{E}}(F)(\alpha, j_{\bar{E}}(\alpha)) \in I(\bar{E})$.

As usual we write $f^p$, $F^p$, $T^p$, $\mathrm{mc}(p)$, $\operatorname{supp} p$ for $f$, $F$, $T$, $\bar{E}_\alpha$, $s$ respectively. Note that we do not require $\bar{E}_\alpha \in s$. That is, we do not consider $\mathrm{mc}(p)$ a part of the support. We note that the properties of $R_{\bar{E}} = j^2_{\bar{E}}(R)(\kappa, j_{\bar{E}}(\kappa))$ we use are the $\kappa^+$-

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | $\bar{\mu}_{4,3}$ | |
| | | $\bar{\mu}_{1,2}$ | | $\bar{\mu}_{3,2}$ | $\bar{\mu}_{4,2}$ | |
| | | $\bar{\mu}_{1,1}$ | $\bar{\mu}_{2,1}$ | $\bar{\mu}_{3,1}$ | $\bar{\mu}_{4,1}$ | |
| | $\bar{\mu}_0$ | $\bar{\mu}_{1,0}$ | $\bar{\mu}_{2,0}$ | $\bar{\mu}_{3,0}$ | $\bar{\mu}_{4,0}$ | $T, f, F$ |
| Support | $\bar{E}_\kappa$ | $\bar{E}_{\alpha_1}$ | $\bar{E}_{\alpha_2}$ | $\bar{E}_{\alpha_3}$ | $\bar{E}_{\alpha_4}$ | $\bar{E}_{\alpha_5} = \mathrm{mc}$ |

FIGURE 1. An example of $p_0 \in P^*_{\bar{E}}$.

closedness and $j_{\bar{E}}(\kappa)^+$-c.c. Any forcing notion satisfying these requirements can be used instead.

**Definition 5.5.** Assume $\mathrm{l}(\bar{E}) > 0$. Let $p, q \in P^*_{\bar{E}}$. We say that $p$ is a Prikry extension of $q$ ($p \leq^* q$) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$,

(2) $\mathrm{mc}(p) \geq_{\bar{E}} \mathrm{mc}(q)$,
(3) If $\mathrm{mc}(p) >_{\bar{E}} \mathrm{mc}(q)$ then $\mathrm{mc}(q) \in \operatorname{supp} p$,
(4) $\forall \bar{\gamma} \in \operatorname{supp} q\ p^{\bar{\gamma}} = q^{\bar{\gamma}}$,
(5) $\forall \bar{\gamma} \in \operatorname{supp} p \setminus \operatorname{supp} q\ \max \kappa^0(p^{\bar{\gamma}}) > \cup \cup j_{\bar{E}}(f'^q)(\kappa(\mathrm{mc}(q)))$ where $f'^q$ is the collapsing part of $f^q$,
(6) $T^p \subseteq \pi^{-1}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$,
(7) $\forall \bar{\gamma} \in \operatorname{supp} q\ \forall \bar{\nu} \in T^p$

$$\max \kappa^0(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \implies \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) = \pi_{\mathrm{mc}(q),\bar{\gamma}}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\bar{\nu})),$$

(8) $\forall \nu_1 \in \operatorname{dom} f^p\ f^p(\nu_1) \leq f^q \circ \pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu_1)$,
(9) $\forall \langle \nu_1, \nu_2 \rangle \in \operatorname{dom} F^p\ F^p(\nu_1, \nu_2) \leq F^q \circ \pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu_1, \nu_2)$.

The requirement 5 is essential for the proof of the homogeneity of dense open subsets and hence to the proof of Prikry's condition.

$$\begin{array}{ccccccccc}
 & \bar{\nu}_{0,3} & & & & & \bar{\mu}_{4,3} & & \\
 & \bar{\nu}_{0,2} & \bar{\mu}_{1,2} & & & \bar{\mu}_{3,2} & \bar{\mu}_{4,2} & & \bar{\nu}_{2,2} \\
 & \bar{\nu}_{0,1} & \bar{\mu}_{1,1} & \bar{\nu}_{1,1} & \bar{\mu}_{2,1} & \bar{\mu}_{3,1} & \bar{\mu}_{4,1} & \bar{\mu}_{5,1} & \bar{\nu}_{2,1} \\
\bar{\mu}_0 & \bar{\nu}_{0,0} & \bar{\mu}_{1,0} & \bar{\nu}_{1,0} & \bar{\mu}_{2,0} & \bar{\mu}_{3,0} & \bar{\mu}_{4,0} & \bar{\mu}_{5,0} & \bar{\nu}_{2,0} \\
\hline
\bar{E}_\kappa & \bar{E}_{\beta_0} & \bar{E}_{\alpha_1} & \bar{E}_{\beta_1} & \bar{E}_{\alpha_2} & \bar{E}_{\alpha_3} & \bar{E}_{\alpha_4} & \bar{E}_{\alpha_5} & \bar{E}_{\beta_2}
\end{array}$$

$$\begin{array}{c}
\pi^{-1}_{\beta_3,\alpha_5} T, f \circ \pi_{\beta_3,\alpha_5}, F \circ \pi_{\beta_3,\alpha_5} \\
\hline
\bar{E}_{\beta_3} = \mathrm{mc}
\end{array}$$

FIGURE 2. A direct extension of $p_0$ from figure 1.

The order $\leq^{**}$ we define now allows only shrinkage of the measure 1 set. Everything else is the same.

**Definition 5.6.** Assume $\mathrm{l}(\bar{E}) > 0$. Let $p, q \in P^*_{\bar{E}}$. We say $p \leq^{**} q$ if

(1) $\operatorname{supp} p = \operatorname{supp} q$,
(2) $\mathrm{mc}(p) = \mathrm{mc}(q)$,
(3) $\forall \gamma \in \operatorname{supp} q\ p^{\gamma} = q^{\gamma}$,
(4) $T^p \subseteq T^q$,
(5) $\forall \nu_1 \in \operatorname{dom} f^p\ f^p(\nu_1) = f^q(\nu_1)$,
(6) $\forall \langle \nu_1, \nu_2 \rangle \in \operatorname{dom} F^p\ F^p(\nu_1, \nu_2) = F^q(\nu_1, \nu_2)$.

**Definition 5.7.** A condition in $P_{\bar{E}}$ is of the form

$$p_n {}^\frown \cdots {}^\frown p_0$$

where

- $p_0 \in P^*_{\bar{E}}$, $\kappa^0(p^0_0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in P^*_{\bar{\mu}_1}$, $\kappa^0(p^0_1) \geq \kappa^0(\bar{\mu}_2)$,
- $\vdots$
- $p_n \in P^*_{\bar{\mu}_n}$,

where $\bar{E}, \bar{\mu}_1, \dots, \bar{\mu}_n$ are extender sequence systems satisfying

$$\kappa^0(\bar{\mu}_n) < \cdots < \kappa^0(\bar{\mu}_1) < \kappa^0(\bar{E}).$$

When $p = p_n {}^\frown \cdots {}^\frown p_0$ we use the short cut $p_{k..l}$ for $p_k {}^\frown \cdots {}^\frown p_l$.

$$\begin{array}{cc} \bar\tau_0 & g \\ \hline \bar\mu_{0,0} & \bar\mu_{0,0} = \mathrm{mc} \end{array}$$

$$\begin{array}{cccccc} & & \bar\mu_{2,3} & & & \\ & \bar\mu_{1,2} & \bar\mu_{2,2} & & \bar\mu_{4,2} & \\ & \bar\mu_{1,1} & \bar\mu_{2,1} & \bar\mu_{3,1} & \bar\mu_{4,1} & \\ \bar\mu_{0,0} & \bar\mu_{1,0} & \bar\mu_{2,0} & \bar\mu_{3,0} & \bar\mu_{4,0} & S, h, H \\ \hline \bar\nu_{0,0} & \bar\nu_{1,0} & \bar\nu_2 & \bar\nu_3 & \bar\nu_{4,0} & \bar\nu_5 = \mathrm{mc} \end{array}$$

$$\begin{array}{cccccc} & & & \bar\nu_{6,3} & & \\ & & \bar\nu_{5,2} & \bar\nu_{6,2} & & \\ & & \bar\nu_{5,1} & \bar\nu_{6,1} & & \\ \bar\nu_{0,0} & \bar\nu_{1,0} & \bar\nu_{5,0} & \bar\nu_{6,0} & \bar\nu_{4,0} & T, f, F \\ \hline \bar E_\kappa & \bar E_{\alpha_1} & \bar E_{\alpha_2} & \bar E_{\alpha_3} & \bar E_{\alpha_4} & \bar E_{\alpha_5} = \mathrm{mc} \end{array}$$

FIGURE 3. An Example of a Condition in $P_{\bar E}$.

**Definition 5.8.** Let $p, q \in P_{\bar E}$. We say that $p$ is a Prikry extension of $q$ ($p \leq^* q$) if $p, q$ are of the form

$$p = p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and

- $p_0, q_0 \in P^*_{\bar E}$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in P^*_{\bar\mu_1}$, $p_1 \leq^* q_1$,
- $\vdots$
- $p_n, q_n \in P^*_{\bar\mu_n}$, $p_n \leq^* q_n$.

**Definition 5.9.** Let $p, q \in P_{\bar E}$. We say $p \leq^{**} q$ if $p, q$ are of the form

$$p = p_n {}^\frown \cdots {}^\frown p_0,$$
$$q = q_n {}^\frown \cdots {}^\frown q_0,$$

and

- $p_0, q_0 \in P^*_{\bar E}$, $p_0 \leq^{**} q_0$,
- $p_1, q_1 \in P^*_{\bar\mu_1}$, $p_1 \leq^{**} q_1$,
- $\vdots$
- $p_n, q_n \in P^*_{\bar\mu_n}$, $p_n \leq^{**} q_n$.

$p_{0\langle\bar\nu\rangle}$, defined now, is the basic non-direct extension in $P_{\bar E}$ of the condition $p_0$, which adds the extender sequence $\bar\nu \in T^{p_0}$ (and hence a condition $p'_1 \in P_{\bar\nu}$) to the finite sequence.

**Definition 5.10.** Let $p_0 \in P^*_{\bar E}$, $\bar\nu \in T^{p_0}$, $\cup\cup j_{\bar E}(f'^{p_0})(\kappa(\mathrm{mc}(p_0))) < \kappa^0(\bar\nu)$, where $f'^{p_0}$ is the collapse part of $f^{p_0}$. We define $p_{0\langle\bar\nu\rangle}$ to be $p'_1 {}^\frown p'_0$ where

(1) $\operatorname{supp} p'_0 = \operatorname{supp} p_0$,

(2) $\forall \bar\gamma \in \operatorname{supp} p'_0\ p'^{\bar\gamma}_0 = \begin{cases} \pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu) & \max \kappa^0(p^{\bar\gamma}_0) < \kappa^0(\bar\nu),\ \operatorname{l}(\bar\nu) > 0. \\ \pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu) & \max \kappa^0(p^{\bar\gamma}_0) < \kappa^0(\bar\nu),\ \operatorname{l}(\bar\nu) = 0, \\ & \bar\gamma = \bar E_\kappa. \\ p^{\bar\gamma} \frown \pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu) & \max \kappa^0(p^{\bar\gamma}_0) < \kappa^0(\bar\nu),\ \operatorname{l}(\bar\nu) = 0, \\ & \bar\gamma \neq \bar E_\kappa. \\ p^{\bar\gamma}_0 & \text{Otherwise.} \end{cases}$,

(3) $\operatorname{mc}(p'_0) = \operatorname{mc}(p_0)$,

(4) $T^{p'_0} = T^{p_0} \setminus \bar\nu$,

(5) $\forall \nu_2 \in T^{p'_0}\ f^{p'_0}(\nu_2) = F^{p_0}(\kappa(\bar\nu), \nu_2)$,

(6) $F^{p'_0} = F^{p_0}$,

(7) $\operatorname{supp} p'_1 = \begin{cases} \{\pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu) \mid \bar\gamma \in \operatorname{supp} p_0,\ \max \kappa^0(p^{\bar\gamma}_0) < \kappa^0(\bar\nu)\} & \operatorname{l}(\bar\nu) > 0 \\ \{\pi_{\operatorname{mc}(p_0),0}(\bar\nu)\} & \operatorname{l}(\bar\nu) = 0 \end{cases}$,

(8) $\forall \bar\gamma \in \operatorname{supp} p_0\ \forall \pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu) \in \operatorname{supp} p'_1\ p'^{\pi_{\operatorname{mc}(p_0),\bar\gamma}(\bar\nu)}_1 = p^{\bar\gamma}_0$,

(9) If $\operatorname{l}(\bar\nu) = 0$ then
  (9.1) $\operatorname{mc}(p'_1) = \bar\nu^0$,
  (9.2) $T^{p'_1} = \emptyset$,
  (9.3) $f^{p'_1} = f^{p_0}(\kappa(\bar\nu))$,
  (9.4) $F^{p'_1} = \emptyset$.

(10) If $\operatorname{l}(\bar\nu) > 0$ then
  (10.1) $\operatorname{mc}(p'_1) = \bar\nu$,
  (10.2) $T^{p'_1} = T^{p_0} \restriction \bar\nu$,
  (10.3) $f^{p'_1} = f^{p_0} \restriction \bar\nu$,
  (10.4) $F^{p'_1} = F^{p_0} \restriction \bar\nu$.

$$\begin{array}{ccccc} & \bar\mu_{1,2} & \bar\mu_{3,2} & \bar\mu_{4,3} \\ & \bar\mu_{1,1} & \bar\mu_{3,1} & \bar\mu_{4,2} \\ & & & \bar\mu_{4,1} \\ \bar\mu_0 & \bar\mu_{1,0} & \bar\mu_{3,0} & \bar\mu_{4,0} & T\restriction\bar\nu, f\restriction\bar\nu, F\restriction\bar\nu \\ \hline \bar\nu^0 & \pi_{\alpha_5,\alpha_1}(\bar\nu) & \pi_{\alpha_5,\alpha_3}(\bar\nu) & \pi_{\alpha_5,\alpha_4}(\bar\nu) & \bar\nu \end{array}$$

$$\begin{array}{cccccc} & & \bar\mu_{2,1} \\ \bar\nu^0 & \pi_{\alpha_5,\alpha_1}(\bar\nu) & \bar\mu_{2,0} & \pi_{\alpha_5,\alpha_3}(\bar\nu) & \pi_{\alpha_5,\alpha_4}(\bar\nu) & T\setminus\bar\nu, f, F \\ \hline \bar E_\kappa & \bar E_{\alpha_1} & \bar E_{\alpha_2} & \bar E_{\alpha_3} & \bar E_{\alpha_4} & \bar E_{\alpha_5} = \operatorname{mc} \end{array}$$

FIGURE 4. $p_{0\langle\bar\nu\rangle}$ for $p_0$ of figure 1 for $\operatorname{l}(\bar\nu) > 0$.

We use the following notation: $(p_{0\langle\bar\nu\rangle})_0 = p'_0$, $(p_{0\langle\bar\nu\rangle})_1 = p'_1$. If $\langle\bar\nu_1,\bar\nu_2\rangle \in (T^{p_0})^2$ then $p_{0\langle\bar\nu_1,\bar\nu_2\rangle} = (p_{0\langle\bar\nu_1\rangle})_1 \frown (p_{0\langle\bar\nu_1\rangle})_{0\langle\bar\nu_2\rangle}$ and so on.

**Definition 5.11.** Let $p, q \in P_{\bar E}$. We say that $p$ is a 1-point extension of $q$ ($p \leq^1 q$) if $p, q$ are of the form

$$p = p_{n+1} \frown p_n \frown \cdots \frown p_0,$$
$$q = q_n \frown \cdots \frown q_0,$$

and there is $0 \leq k \leq n$ such that

| $\bar{\mu}_0$ | $f(\bar{\nu})$ |
|---|---|
| $\bar{\nu}^0$ | $\bar{\nu}^0$ |

| | | | | | |
|---|---|---|---|---|---|
| | | | | $\pi_{\alpha_5,\alpha_4}(\bar{\nu})$ | |
| | $\pi_{\alpha_5,\alpha_1}(\bar{\nu})$ | | $\pi_{\alpha_5,\alpha_3}(\bar{\nu})$ | $\bar{\mu}_{4,3}$ | |
| | $\bar{\mu}_{1,2}$ | | $\bar{\mu}_{3,2}$ | $\bar{\mu}_{4,2}$ | |
| | $\bar{\mu}_{1,1}$ | $\bar{\mu}_{2,1}$ | $\bar{\mu}_{3,1}$ | $\bar{\mu}_{4,1}$ | |
| $\bar{\nu}^0$ | $\bar{\mu}_{1,0}$ | $\bar{\mu}_{2,0}$ | $\bar{\mu}_{3,0}$ | $\bar{\mu}_{4,0}$ | $T \setminus \bar{\nu}, f, F$ |
| $\bar{E}_\kappa$ | $\bar{E}_{\alpha_1}$ | $\bar{E}_{\alpha_2}$ | $\bar{E}_{\alpha_3}$ | $\bar{E}_{\alpha_4}$ | $\bar{E}_{\alpha_5} = \text{mc}$ |

FIGURE 5. $p_{0\langle\bar{\nu}\rangle}$ for $p_0$ of figure 1 for $\text{l}(\bar{\nu}) = 0$.

- $p_i, q_i \in P^*_{\bar{\mu}_i}$, $p_i \leq^* q_i$ for $i = 0, \ldots, k-1$,
- $p_{i+1}, q_i \in P^*_{\bar{\mu}_i}$, $p_{i+1} \leq^* q_i$ for $i = k+1, \ldots, n$,
- There is $\langle\bar{\nu}\rangle \in T^{q_k}$ such that $p_{k+1} {}^\frown p_k \leq^* q_{k\langle\bar{\nu}\rangle}$.

**Definition 5.12.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is an $n$-point extension of $q$ ($p \leq^n q$) if there are $p^n, \ldots, p^0$ such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

We consider Prikry extension to be a 0-point extension. That is

**Definition 5.13.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is a 0-point extension of $q$ ($p \leq^0 q$) if $p \leq^* q$.

**Definition 5.14.** Let $p, q \in P_{\bar{E}}$. We say that $p$ is an extension of $q$ ($p \leq q$) if there is an $n$ such that $p \leq^n q$.

Later on by $P_{\bar{E}}$ we mean $\langle P_{\bar{E}}, \leq\rangle$.

*Note* 5.15. When $\text{l}(\bar{E}) = 1$ the forcing $P_{\bar{E}}$ is as the forcing defined in [13].

**Definition 5.16.** Let $\bar{\epsilon}$ be an extender sequence such that $\kappa^0(\bar{\epsilon}) < \kappa^0(\bar{E})$.

$$P_{\bar{E}}/P_{\bar{\epsilon}} = \{p \mid \exists q \in P_{\bar{\epsilon}}\ q {}^\frown p \in P_{\bar{E}}\}.$$

When $\tau_1 < \tau_2$ we have $A \in \bar{E}_\alpha\restriction\tau_2 \implies A \in \bar{E}_\alpha\restriction\tau_1$. So it is our convention that $P_{\bar{E}\restriction\tau_2} \subseteq P_{\bar{E}\restriction\tau_1}$.

We conclude this section with an example as to why any kind of extender sequence can appear on new coordinates. Drop the interleaved functions and trees etc. Begin with

$$p = \{\langle\langle\kappa, E(0), E(1)\rangle, \langle\rangle\rangle\},$$

Assuming we can add the points $\langle\nu, e(0)\rangle$ and $\langle\mu, f(0)\rangle$ we get

$$q = q_2 {}^\frown q_1 {}^\frown q_0 = p_{\langle\langle\nu,e(0)\rangle,\langle\mu,f(0)\rangle\rangle}$$

where

$$\begin{aligned} q_2 &= \{\langle\langle\nu, e(0)\rangle, \langle\rangle\rangle\} \\ q_1 &= \{\langle\langle\mu, f(0)\rangle, \langle\nu, e(0)\rangle\rangle\} \\ q_0 &= \{\langle\langle\kappa, E(0), E(1)\rangle, \langle\langle\mu, f(0)\rangle\rangle\rangle\} \end{aligned}$$

Let us start with another condition

$$r = \{\langle\langle\kappa, E(0), E(1)\rangle, \langle\rangle\rangle, \langle\langle\alpha, E(0), E(1)\rangle, \langle\rangle\rangle\}.$$

Taking $\mu'$ and $\nu'$ such that $\pi_{\alpha,\kappa}(\mu') = \mu$ and $\pi_{\alpha,\kappa}(\nu') = \nu$, we get that

$$s = s_2 {}^\frown s_1 {}^\frown s_0 = r_{\langle\langle\nu', e(0)\rangle, \langle\mu', f(0)\rangle\rangle}$$

where

$$\begin{aligned} s_2 &= \{\langle\langle\nu, e(0)\rangle, \langle\rangle\rangle, \langle\langle\nu', e(0)\rangle, \langle\rangle\rangle\} \\ s_1 &= \{\langle\langle\mu, f(0)\rangle, \langle\nu, e(0)\rangle\rangle, \langle\langle\mu', f(0)\rangle, \langle\nu', e(0)\rangle\rangle\} \\ s_0 &= \{\langle\langle\kappa, E(0), E(1)\rangle, \langle\langle\mu, f(0)\rangle\rangle\rangle, \langle\langle\alpha, E(0), E(1)\rangle, \langle\langle\mu', f(0)\rangle\rangle\rangle\} \end{aligned}$$

We would certainly expect at this point to have $s \leq^* q$. For this we have to allow extender sequences on new coordinates.

Of course this might cause "accidental pointing". Hence we might have some bad behavior in the beginning of a sequence. This is to expected with Radin/Prikry sequences, there is "noise" in the beginning. It just might be a little longer than the noise we are used to.

## 6. Basic properties of $P_{\bar{E}}$

**Claim 6.1.** *$P_{\bar{E}}$ satisfies $\kappa^{++}$-c.c.*

*Proof.* Let $\{p^\xi \mid \xi < \kappa^{++}\} \subset P_{\bar{E}}$. As $|\{p^\xi_{n_\xi} {}^\frown \cdots {}^\frown p^\xi_1 \mid \xi < \kappa^{++}\}| = \kappa$ we can assume without loss of generality that there is $p_n {}^\frown \cdots {}^\frown p_1$ such that $\forall\xi < \kappa^{++}$ $p_n {}^\frown \cdots {}^\frown p_1 = p^\xi_{n_\xi} {}^\frown \cdots {}^\frown p^\xi_1$. Hence we can ignore these lower parts and assume that the set of conditions we start with is $\{p^\xi_0 \mid \xi < \kappa^{++}\}$.

Let $d_\xi = \operatorname{supp} p^\xi_0 \cup \{\operatorname{mc}(p^\xi_0)\}$. As $|d_\xi| < \kappa^+ < \kappa^{++}$, $(\kappa^+)^\kappa = \kappa^+ < \kappa^{++}$ we can invoke the $\Delta$-lemma. Hence, without loss of generality, there is $d$ such that $\forall\xi_1 \neq \xi_2$ $d_{\xi_1} \cap d_{\xi_2} = d$. As $\kappa^\kappa = \kappa^+ < \kappa^{++}$ we can assume, without loss of generality, $\forall\bar{\alpha} \in d$ $\forall\xi_1, \xi_2$ $(p^{\xi_1}_0)^{\bar{\alpha}} = (p^{\xi_2}_0)^{\bar{\alpha}}$.

The $T^{p^\xi_0}$'s, $F^{p^\xi_0}$'s are always compatible. Hence we are left with handling of $\{f^{p^\xi_0} \mid \xi < \kappa^{++}\}$. $\{j_{\bar{E}}(f^{p^\xi_0})(\kappa(\operatorname{mc}(p^\xi_0))) \mid \xi < \kappa^{++}\} \subset j_{\bar{E}}(R)(\kappa((p^\xi_0)^0), \kappa)$ and $j_{\bar{E}}(R)(\kappa((p^\xi_0)^0), \kappa)$ satisfies $\kappa^+$-c.c. Hence there are $\xi_1, \xi_2$ such that

$$j_{\bar{E}}(f^{p^{\xi_1}_0})(\kappa(\operatorname{mc}(p^{\xi_1}_0))) \parallel j_{\bar{E}}(f^{p^{\xi_2}_0})(\kappa(\operatorname{mc}(p^{\xi_2}_0))).$$

Hence $p^{\xi_1}_0 \parallel p^{\xi_2}_0$. □

**Lemma 6.2.** *Let $p = p_1 {}^\frown p_0 \in P_{\bar{E}}$. Assume that we have $S^0$, $t'_1(\bar{\nu}_1)$ such that*

(1) $S^0 \subseteq T^{p_0}$,
(2) $S^0 \in \operatorname{mc}(p_0)(\xi_0)$,
(3) $\forall\langle\bar{\nu}_1\rangle \in S^0$ $t'_1(\bar{\nu}_1) \leq^{**} (p_{0\langle\bar{\nu}_1\rangle})_1$.

*Then there are $p^*_0 \leq^{**} p_0$, $S^{0*} \subseteq S^0$, such that $S^{0*} \in \operatorname{mc}(p_0)(\xi_0)$, and*

$$\{p_1 {}^\frown t'_1(\bar{\nu}_1) {}^\frown (p^*_{0\langle\bar{\nu}_1\rangle})_0 \mid \langle\bar{\nu}_1\rangle \in S^{0*}\}$$

*is pre-dense below $p_1 {}^\frown p^*_0$.*

*Proof.* Let $\bar{E}_\alpha = \mathrm{mc}(p_0)$. Set

$$T_{<\xi_0} = j_{\bar{E}}(T^{t'_1})^{(\bar{E}_\alpha\restriction\xi_0)},$$
$$T_{\xi_0} = \{\bar{\nu}_1 \in S^0 \mid T_{<\xi_0}\restriction\bar{\nu}_1 = T^{t'_1(\bar{\nu}_1)}\},$$
$$T_{>\xi_0} = \{\bar{\nu} \in T^{p_0} \mid \exists\tau < \mathrm{l}(\bar{\nu})\ T_{\xi_0}\restriction\bar{\nu} \in \bar{\nu}(\tau)\}.$$

We set $S^{0*} = T_{\xi_0}$. It is clear that $T_{<\xi_0} \in \bar{E}_\alpha\restriction\xi_0$, $T_{\xi_0} \in E_\alpha(\xi_0)$, $\forall\xi_0 < \xi < \mathrm{l}(\bar{E})\ T_{>\xi_0} \in E_\alpha(\xi)$. Let $T^* = T_{<\xi_0} \cup T_{\xi_0} \cup T_{>\xi_0}$. We define the condition $p^*_0$ to be $p_0$ with $T^*$ substituted for $T^{p_0}$. Let $r \frown q \leq p_1 \frown p^*_0$ such that $r \leq p_1$, $q \leq p^*_0$. Of course the part below $p_1$ poses no problem. So we are left to show that there is $\bar{\nu} \in S^{0*}$ such that $q \parallel t'_1(\bar{\nu}_1) \frown (p^*_{0\langle\bar{\nu}_1\rangle})_0$. By the definition of $\leq$ there is $\langle\bar{\nu}_1, \dots, \bar{\nu}_n\rangle \in (T^{p^*_0})^n$ such that $q = q_n \frown \dots \frown q_0 \leq^* p^*_{0\langle\bar{\nu}_1,\dots,\bar{\nu}_n\rangle}$. Let $k$ be the last such that $\langle\bar{\nu}_1, \dots, \bar{\nu}_k\rangle \in (T_{<\xi_0})^k$. ($k$ can be $0, 1, \dots, n$.) If $k = n$ we choose $\bar{\nu}_{k+1} \in T_{\xi_0} \setminus \bar{\nu}_n$. We split the handling according to where $\bar{\nu}_{k+1}$ is.

(1) $\bar{\nu}_{k+1} \in T_{\xi_0}$: $q_n \frown \dots \frown q_{n-k+1} \leq (p^*_{0\langle\bar{\nu}_{k+1}\rangle})_1$, $q_{n-k} \frown \dots \frown q_0 \leq (p^*_{0\langle\bar{\nu}_{k+1}\rangle})_0$. As $T^{t'_1(\bar{\nu}_{k+1})} = T_{<\xi_0}\restriction\bar{\nu}_{k+1}$ we have $\langle\bar{\nu}_1, \dots, \bar{\nu}_k\rangle \in (T^{t'_1(\bar{\nu}_{k+1})})^k$. Hence $q_n \frown \dots \frown q_{n-k+1} \parallel t'_1(\bar{\nu}_{k+1})$.
(2) $\bar{\nu}_{k+1} \in T_{>\xi_0}$: So $\exists\tau < \mathrm{l}(\bar{\nu}_{k+1})\ S^{0*}\restriction\bar{\nu}_{k+1} \in \bar{\nu}_{k+1}(\tau)$. In particular there is $\bar{\nu} \in (S^{0*} \cap T^*)\restriction\bar{\nu}_{k+1}\setminus\bar{\nu}_k$. Let $s = (p^*_{0\langle\bar{\nu}_1,\dots,\bar{\nu}_k,\bar{\nu}\rangle})_1$. Then $q_n \frown \dots \frown q_{n-k+1} \frown s \frown q_{n-k} \frown \dots \frown q_0 \leq p^*_{0\langle\bar{\nu}\rangle}$. Once more, as $\bar{\nu} \in T_{\xi_0}$ we have $T^{t'_1(\bar{\nu})} = T_{<\xi_0}\restriction\bar{\nu}$, hence $\langle\bar{\nu}_1, \dots, \bar{\nu}_k\rangle \in T^{t'_1(\bar{\nu})}$. That is $q_n \frown \dots \frown q_{n-k+1} \frown s \frown q_{n-k} \frown \dots \frown q_0 \parallel t'_1(\bar{\nu}) \frown (p^*_{0\langle\bar{\nu}\rangle})_0$.

□

**Lemma 6.3.** *Let $p = p_1 \frown p_0 \in P_{\bar{E}}$. Assume that we have $S^0$, $t'_2(\bar{\nu}_1)$, $t'_1(\bar{\nu}_1, \bar{\nu}_2)$ such that*

(1) $S^0 \subseteq (T^{p_0})^2$,
(2) $S^0$ *is* $\mathrm{mc}(p_0)$*-fat tree,*
(3) $\forall\langle\bar{\nu}_1, \bar{\nu}_2\rangle \in S^0\ t'_2(\bar{\nu}_1) \frown t'_1(\bar{\nu}_1, \bar{\nu}_2) \leq^{**} (p_{0\langle\bar{\nu}_1,\bar{\nu}_2\rangle})_{2..1}$.

*Then there are $p^*_0 \leq^{**} p_0$, $S^{0*} \subseteq S^0$, such that $S^{0*}$ is $\mathrm{mc}(p_0)$-fat tree, and*

$$\{p_1 \frown t'_2(\bar{\nu}_1) \frown t'_1(\bar{\nu}_1, \bar{\nu}_2) \frown (p^*_{0\langle\bar{\nu}_1,\bar{\nu}_2\rangle})_0 \mid \langle\bar{\nu}_1, \bar{\nu}_2\rangle \in S^{0*}\}$$

*is pre-dense below $p_1 \frown p^*_0$.*

*Proof.* Fix $\bar{\nu}_1 \in \mathrm{Lev}_0(S^0)$. Then we can invoke 6.2 on $t'_2(\bar{\nu}_1) \frown (p_{0\langle\bar{\nu}_1\rangle})_0$, $t'_1(\bar{\nu}_1, \bar{\nu}_2)$, $\mathrm{Suc}_{S^0}(\bar{\nu}_1)$ to get $r_0(\bar{\nu}_1) \leq^{**} (p_{0\langle\bar{\nu}_1\rangle})_0$, $R^0(\bar{\nu}_1) \subseteq \mathrm{Suc}_{S^0}(\bar{\nu}_1)$ such that

$$\{p_1 \frown t'_2(\bar{\nu}_1) \frown t'_1(\bar{\nu}_1, \bar{\nu}_2) \frown (r_0(\bar{\nu}_1)_{\langle\bar{\nu}_2\rangle})_0 \mid \bar{\nu}_2 \in R^0(\bar{\nu}_1)\}$$

is pre-dense below $p_1 \frown t'_2(\bar{\nu}_1) \frown r_0(\bar{\nu}_1)$.

We do the above for all $\bar{\nu}_1 \in \mathrm{Lev}_0(S^0)$. We let $T^{r_0} = \triangle^0{}_{\bar{\nu}_1\in\mathrm{Lev}_0(S^0)} T^{r_0(\bar{\nu}_1)}$. Of course, $r_0$ is $p_0$ with $T^{p_0}$ substituted by $T^{r_0}$. We point out that $\forall\bar{\nu}_1 \in \mathrm{Lev}_0(S^0)$ $(r_{0\langle\bar{\nu}_1\rangle})_0 \leq^{**} r_0(\bar{\nu}_1)$.

We invoke 6.2 with $p_1 \frown r_0$, $t'_2(\bar{\nu}_1)$, $\mathrm{Lev}_0(S^0)$ to get $p^*_0 \leq^{**} r_0$, $R^{0*} \subseteq \mathrm{Lev}_0(S^0)$ such that

$$\{p_1 \frown t'_2(\bar{\nu}_1) \frown (p^*_{0\langle\bar{\nu}_1\rangle})_0 \mid \bar{\nu}_1 \in R^{0*}\}$$

is pre-dense below $p_1 {}^\frown p^*_0$. We set $\mathrm{Lev}_0(S^{0*}) = R^{0*}$, $\mathrm{Suc}_{S^{0*}}(\bar\nu_1) = R^0(\bar\nu_1)$. We claim that

$$\{p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown t'_1(\bar\nu_1,\bar\nu_2) {}^\frown (p^*_{0\langle\bar\nu_1,\bar\nu_2\rangle})_0 \mid \langle\bar\nu_1,\bar\nu_2\rangle \in S^{0*}\}$$

is pre-dense below $p_1 {}^\frown p^*_0$. Let $q \leq p_1 {}^\frown p^*_0$.

Then there is $\bar\nu_1 \in R^{0*}$ such that $q \parallel p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown (p^*_{0\langle\bar\nu_1\rangle})_0$. As $(p^*_{0\langle\bar\nu_1\rangle})_0 \leq^{**} r_0(\bar\nu_1)$ we have $q \parallel p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown r_0(\bar\nu_1)$. Choose $s \leq q, p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown r_0(\bar\nu_1)$. There is $\bar\nu_2 \in R(\bar\nu_1)$ such that $s \parallel p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown t'_1(\bar\nu_1,\bar\nu_2) {}^\frown (r_0(\bar\nu_1)_{\langle\bar\nu_2\rangle})_0$. As $s \leq p_1 {}^\frown p^*_0$, $(p^*_{0\langle\bar\nu_1,\bar\nu_2\rangle})_0 \parallel (r_0(\bar\nu_1)_{\langle\bar\nu_2\rangle})_0$ we get $s \parallel p_1 {}^\frown t'_2(\bar\nu_1) {}^\frown t'_1(\bar\nu_1,\bar\nu_2) {}^\frown (p^*_{0\langle\bar\nu_1,\bar\nu_2\rangle})_0$. We complete the proof by noting that $S^{0*}$ was constructed such that $\langle\bar\nu_1,\bar\nu_2\rangle \in S^{0*}$. □

Repeat invocation of the above proof yields

**Claim 6.4.** *Let $p = p_1 {}^\frown p_0 \in P_{\bar E}$. Assume we have $S^0$, $t'_n(\bar\nu_1)$, …, $t'_1(\bar\nu_1,\dots,\bar\nu_n)$ such that*

(1) $S^0 \subseteq (T^{p_0})^n$,
(2) $S^0$ *is* $\mathrm{mc}(p_0)$*-fat tree,*
(3) $\forall\langle\bar\nu_1,\dots,\bar\nu_n\rangle \in S^0\ t'_n(\bar\nu_1) {}^\frown \cdots {}^\frown t'_1(\bar\nu_1,\dots,\bar\nu_n) \leq^{**} (p_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$.

*Then there are $p^*_0 \leq^{**} p_0$, $S^{0*} \subseteq S^0$, such that $S^{0*}$ is $\mathrm{mc}(p_0)$-fat tree, and*

$$\{p_1 {}^\frown t'_n(\bar\nu_1) {}^\frown \ldots {}^\frown t'_1(\bar\nu_1,\dots,\bar\nu_n) {}^\frown (p^*_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0 \mid \langle\bar\nu_1,\dots,\bar\nu_n\rangle \in S^{0*}\}$$

*is pre-dense below $p_1 {}^\frown p^*_0$.*

## 7. Homogeneity in dense open subsets

Our aim in this section is to prove the following theorem. Unlike [19] we do not carry exact information on the measure 1 set of new blocks. This made us use $\leq^{**}$ in this theorem.

**Theorem 7.1.** *Let $D$ be dense open in $P_{\bar E}$, $p = p_{l..0} \in P_{\bar E}$. Then there is $p^* \leq^* p$ such that*

$$\begin{aligned}
&\exists S^l\ \forall\langle\bar\nu_{l,1},\dots,\bar\nu_{l,n_l}\rangle \in S^l\ \exists t'_{l,n_l} {}^\frown \cdots {}^\frown t'_{l,1} \leq^{**} (p^*_{l\langle\bar\nu_{l,1},\dots,\bar\nu_{l,n_l}\rangle})_{n_l..1} \\
&\qquad\qquad\vdots \\
&\quad\exists S^0\ \forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle \in S^0\ \exists t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} \leq^{**} (p^*_{0\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_{n_0..1} \\
&\qquad\qquad t'_{l,n_l} {}^\frown \cdots {}^\frown t'_{l,1} {}^\frown (p^*_{l\langle\bar\nu_{l,1},\dots,\bar\nu_{l,n_l}\rangle})_0 {}^\frown \\
&\qquad\qquad\vdots \\
&\qquad\qquad t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} {}^\frown (p^*_{0\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_0 \in D
\end{aligned}$$

*where for each $k = 0,\dots,l$ $S^k \subseteq (V_{\kappa^0(\mathrm{mc}(p^*_k))})^{n_k}$ is an $\mathrm{mc}(p^*_k)$-fat tree of height $n_k$.*

A word of caution is in order. If one of the elements of $p^*$ does not contain a tree then we mean in the above formula just a direct extension of it. For example, let $p = p_{2..0} \in P_{\bar E}$ and $p_2 \in P_{\bar\epsilon_2}$, $p_1 \in P_{\bar\epsilon}$ where $\mathrm{l}(\bar\epsilon_2) > 0$, $\mathrm{l}(\bar\epsilon) = 0$. Then the above formula should be read as

$$\begin{aligned}
&\exists S^2\ \forall\langle\bar\nu_{2,1},\dots,\bar\nu_{2,n_2}\rangle \in S^2\ \exists t'_{2,n_2} {}^\frown \cdots {}^\frown t'_{2,1} \leq^{**} (p^*_{2\langle\bar\nu_{2,1},\dots,\bar\nu_{2,n_2}\rangle})_{n_2..1} \\
&\quad\exists S^0\ \forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle \in S^0\ \exists t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} \leq^{**} (p^*_{0\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_{n_0..1}
\end{aligned}$$

$$t'_{2,n_2} {}^\frown \cdots {}^\frown t'_{2,1} {}^\frown (p^*_{2\langle \bar\nu_{2,1},\dots,\bar\nu_{2,n_2}\rangle})_0 {}^\frown$$
$$p^*_1 {}^\frown$$
$$t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} {}^\frown (p^*_{0\langle \bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_0 \in D.$$

We prove the theorem by induction on $l$, the number of blocks in $p$. We give the proof in a series of lemmas. 7.3, 7.4 are the case $l = 0$ of the theorem.

**Lemma 7.2.** *Let $D$ be dense open in $P_{\bar E}/P_{\bar\epsilon}$, $p = p_0 \in P_{\bar E}/P_{\bar\epsilon}$, $n < \omega$. Then there is $p^*_0 \leq^* p_0$ such that one and only one of the following is satisfied*

(1) *There is $S \subseteq (T^{p^*_0})^n$, an $\mathrm{mc}(p^*_0)$-fat tree, such that*

$$\forall \langle \bar\nu_1,\dots,\bar\nu_n\rangle \in S\ \exists t'_n {}^\frown \cdots {}^\frown t'_1 \leq^{**} (p^*_{0\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$$
$$t'_n {}^\frown \cdots {}^\frown t'_1 {}^\frown (p^*_{0\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_0 \in D.$$

(2) $\forall \langle \bar\nu_1,\dots,\bar\nu_n\rangle \in (T^{p^*_0})^n\ \forall q \leq^* p^*_{0\langle \bar\nu_1,\dots,\bar\nu_n\rangle}\ q \notin D.$

*Proof.* We give the proof for $n = 1$. It is essentially the same for all $n$.

Let $\chi$ be large enough so that $H_\chi$ catch 'what interests us' and let

$$N \prec H_\chi,$$
$$|N| = \kappa,$$
$$N \supset \kappa,$$
$$N \supseteq N^{<\kappa},$$
$$P_{\bar E}, P_{\bar\epsilon}, D, p_0 \in N.$$

Choose $\alpha \in \operatorname{dom} \bar E$ such that

$$\forall \gamma \in \operatorname{dom} \bar E \cap N\ \gamma <_{\bar E} \alpha,$$

and set

$$A = \pi^{-1}_{\alpha,\beta} T^{p_0}$$

where $\bar E_\beta = \mathrm{mc}(p_0)$.

Let $\preceq$ be a well ordering of $A$ such that

$$\forall \bar\nu_1, \bar\nu_2 \in A\ \bar\nu_1 \preceq \bar\nu_2 \implies \kappa^0(\bar\nu_1) \leq \kappa^0(\bar\nu_2).$$

We shrink $A$ a bit so that the following is satisfied

$$\forall \bar\nu \in A\ |\{\bar\mu \in A \mid \kappa^0(\bar\mu) < \kappa^0(\bar\nu)\}| \leq \kappa^0(\bar\nu).$$

We start an induction on $\bar\nu$ in which we build

$$\langle \alpha^{\bar\nu}_0, u^{\bar\nu}_0, T^{\bar\nu}_0, F^{\bar\nu}_0 \mid \bar\nu \in A\rangle,$$

where $(u^{\bar\nu}_0)_{\langle \pi_{\alpha,\alpha^{\bar\nu}_0}(\bar\nu)\rangle} \cup \{\langle \bar E_{\alpha^{\bar\nu}_0}, F^{p_0}(\pi_{\alpha,\beta}(\bar\nu), \pi_{\alpha^{\bar\nu}_0,\beta}(-)), T^{\bar\nu}_0, F^{\bar\nu}_0\rangle\} \in P_{\bar E}$.

Assume that we have constructed

$$\langle \alpha^{\bar\nu}_0, u^{\bar\nu}_0, T^{\bar\nu}_0, F^{\bar\nu}_0 \mid \bar\nu \prec \bar\nu_0\rangle.$$

Set the following:

- $\bar\nu_0$ is $\prec$-minimal:

$$q' = p_0 \setminus \{\langle \bar E_\beta, T^{p_0}, f^{p_0}, F^{p_0}\rangle\},$$
$$\alpha' = \beta.$$

- $\bar\nu_0$ is the immediate $\prec$-successor of $\bar\nu$:

$$q' = u_0^{\bar\nu},$$
$$\alpha' = \alpha^{\bar\nu}.$$

- $\bar\nu_0$ is $\prec$-limit: Choose $\alpha' \in N$ such that $\forall \bar\nu \prec \bar\nu_0\ \alpha' >_{\bar E} \alpha^{\bar\nu}$ and set

$$q' = \bigcup_{\bar\nu \prec \bar\nu_0} u_0^{\bar\nu}.$$

We start an induction on $i$. We construct in it

$$\langle \alpha_0^{\bar\nu_0,i}, u_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, f_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i}, \bar\alpha_1^{\bar\nu_0,i}, u_1^{\bar\nu_0,i}, T_1^{\bar\nu_0,i}, f_1^{\bar\nu_0,i}, F_1^{\bar\nu_0,i} \mid i < \kappa \rangle,$$

where

$$u_1 \cup \{\langle \bar\alpha_1^{\bar\nu_0,i}, f_1^{\bar\nu_0,i}, T_1^{\bar\nu_0,i}, F_1^{\bar\nu_0,i}\rangle\} {}^\frown \\ (u_0^{\bar\nu_0,i})_{\langle \pi_{\alpha,\alpha_0^{\bar\nu_0,i}}(\bar\nu_0)\rangle} \cup \{\langle \bar E_{\alpha_0^{\bar\nu_0,i}}, f_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i}\rangle\} \in P_{\bar E},$$

and $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}) \mid i < \kappa\rangle$ is a maximal anti-chain in $j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa)$ below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)), \beta)$.

The construction of this anti-chain is done carefully in order to have the maximal anti-chain after $\kappa$ steps (and not at some ordinal of cardinality $\kappa$): If $f, \alpha_0$ are such that $j_{\bar E}(f)(\alpha_0) \in j_{\bar E}(R)(\kappa^0(\bar\nu), \kappa)$ then let $j_{\bar E}(f)(\alpha_0) \wedge \mu$ be the same function with the collapsing part not mentioning ordinals above $\mu$. In the same way we define $j_{\bar E}(R)(\kappa^0(\bar\nu), \kappa) \wedge \mu$. The 'carefulness' of the construction is the added fact that for each inaccessible $\mu < \kappa$ the set $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \wedge \mu \mid i < \mu^+\rangle$ is pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu), \kappa) \wedge \mu$ below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)), \beta) \wedge \mu$.

Assume we have constructed

$$\langle \alpha_0^{\bar\nu_0,i}, u_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, f_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i}, \bar\alpha_1^{\bar\nu_0,i}, u_1^{\bar\nu_0,i}, T_1^{\bar\nu_0,i}, f_1^{\bar\nu_0,i}, F_1^{\bar\nu_0,i} \mid i < i_0 \rangle,$$

and we do step $i_0$.

- $i_0 = 0$:

$$q'' = q',$$
$$\alpha'' = \alpha',$$
$$f'' = F^{p_0}(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)), \pi_{\alpha'',\beta}(-)).$$

- $i_0 = i + 1$: If $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \mid i < i_0\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)), \beta)$ then we finish the induction on $i$.

  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_0$, $i_0 < \mu^+$, and $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \wedge \mu \mid i < i_0\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that

$$j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu,$$
$$\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}).$$

  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$

  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i})$, and for each inaccessible $\mu < i_0$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$.

Then we set

$$q'' = u_0^{\bar\nu_0,i},$$
$$\alpha'' >_{\bar E} \alpha^{\bar\nu_0,i}, \beta''.$$

- $i_0$ is limit: If $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \mid i < i_0\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)), \beta)$ then we finish the induction on $i$.

  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_0$, $i_0 < \mu^+$, and $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \wedge \mu \mid i < i_0\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that

  $$j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu,$$
  $$\forall i < i_0 \; j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}).$$

  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$

  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_0 \; j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i})$, and for each inaccessible $\mu < i_0$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$.

  Choose $\alpha'' \in N$ such that $\forall i < i_0 \; \bar\alpha'' >_{\bar E} \bar\alpha^{\bar\nu_0,i}, \beta''$ and set

  $$q'' = \bigcup_{i<i_0} u_0^{\bar\nu_0,i}.$$

Set

$$T_0'' = \pi^{-1}_{\alpha'',\beta} T^{p_0} \setminus \pi_{\alpha,\alpha''}(\bar\nu_0),$$
$$F_0'' = F^{p_0} \circ \pi_{\alpha'',\beta},$$
$$u_0'' = (q''_{\langle \pi_{\alpha,\alpha''}(\bar\nu_0)\rangle})_0 \cup \{\langle \alpha'', T_0'', f'', F_0''\rangle\}),$$

$$u_1'' = (q''_{\langle \pi_{\alpha,\alpha''}(\bar\nu_0)\rangle})_1 \cup \{\langle \pi_{\alpha,\alpha''}(\bar\nu_0), T_0''\restriction\pi_{\alpha,\kappa}(\bar\nu_0), f^{p_0} \circ \pi_{\alpha'',\beta}\restriction\pi_{\alpha,\kappa}(\bar\nu_0), F_0''\restriction\pi_{\alpha,\kappa}(\bar\nu_0)\rangle\}.$$

If there is $q_1 \frown q_0 \in D \cap N$ such that

$$q_1 \frown q_0 \leq^* u_1'' \frown u_0''$$

then set

$$\alpha_0^{\bar\nu_0,i_0} = \kappa(\mathrm{mc}(q_0)),$$
$$u_0^{\bar\nu_0,i_0} = q'' \cup \{\langle \bar\gamma, q_0^{\bar\gamma}\rangle \mid \bar\gamma \in \operatorname{supp} q_0 \setminus \operatorname{supp} q''\},$$
$$T_0^{\bar\nu_0,i_0} = T^{q_0},$$
$$f_0^{\bar\nu_0,i_0} = f^{q_0},$$
$$F_0^{\bar\nu_0,i_0} = F^{q_0},$$
$$\bar\alpha_1^{\bar\nu_0,i_0} = \mathrm{mc}(q_1),$$
$$u_1^{\bar\nu_0,i_0} = \{\langle \bar\gamma, q_1^{\bar\gamma}\rangle \mid \bar\gamma \in \operatorname{supp} q_1\},$$
$$T_1^{\bar\nu_0,i_0} = T^{q_1},$$
$$f_1^{\bar\nu_0,i_0} = f^{q_1},$$
$$F_1^{\bar\nu_0,i_0} = F^{q_1},$$

otherwise we set

$$\begin{aligned}
\alpha_0^{\bar\nu_0,i_0} &= \alpha'',\\
u_0^{\bar\nu_0,i_0} &= q'',\\
T_0^{\bar\nu_0,i_0} &= T_0'',\\
f_0^{\bar\nu_0,i_0} &= f_0'',\\
F_0^{\bar\nu_0,i_0} &= F_0'',\\
\bar\alpha_1^{\bar\nu_0,i_0} &= \operatorname{mc}(u_1''),\\
u_1^{\bar\nu_0,i_0} &= \{\langle\bar\gamma, u_1''^{\bar\gamma}\rangle \mid \bar\gamma\in\operatorname{supp} u_1''\},\\
T_1^{\bar\nu_0,i_0} &= T^{u_1''},\\
f_1^{\bar\nu_0,i_0} &= f^{u_1''},\\
F_1^{\bar\nu_0,i_0} &= F^{u_1''}.
\end{aligned}$$

When the induction on $i$ terminates we have

$$\langle \alpha_0^{\bar\nu_0,i}, u_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, f_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i}, \bar\alpha_1^{\bar\nu_0,i}, u_1^{\bar\nu_0,i}, T_1^{\bar\nu_0,i}, f_1^{\bar\nu_0,i}, F_1^{\bar\nu_0,i} \mid i<\kappa\rangle.$$

We point out that indeed $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i}) \mid i<\kappa\rangle$ is a maximal anti-chain: Assume $f\in j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa)$, $f\leq j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)),\beta)$. Then there is an inaccessible $\mu<\kappa$ such that $f\in j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa)\wedge\mu$. By the construction $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i})\wedge\mu \mid i<\mu^+\rangle$ is pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa)\wedge\mu$ below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)),\beta)\wedge\mu$. Hence there is $i<\mu^+$ such that $f \parallel j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i})\wedge\mu$. Hence $f\parallel j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha_0^{\bar\nu_0,i})$.

We complete step $\bar\nu_0$ by setting

$$\begin{aligned}
&\alpha_0^{\bar\nu_0}\in N,\\
&\forall i<i^{\bar\nu_0}\ \alpha_0^{\bar\nu_0} >_{\bar E} \alpha_0^{\bar\nu_0,i},\\
&u_0^{\bar\nu_0} = \bigcup_{i<i^{\bar\nu_0}} u_0^{\bar\nu_0,i},\\
&T_0^{\bar\nu_0} = \mathop{\triangle^0}_{i<i^{\bar\nu_0}} \pi^{-1}_{\alpha_0^{\bar\nu_0},\alpha_0^{\bar\nu_0,i}} T_0^{\bar\nu_0,i},\\
&\forall i<i^{\bar\nu_0}\ F_0^{\bar\nu_0}\leq F_0^{\bar\nu_0,i}\circ\pi_{\alpha^{\bar\nu_0},\alpha^{\bar\nu_0,i}}.
\end{aligned}$$

When the induction on $\bar\nu$ terminates we have

$$\langle \alpha_0^{\bar\nu}, u_0^{\bar\nu}, T_0^{\bar\nu}, F_0^{\bar\nu} \mid \bar\nu\in A\rangle.$$

We define the following function with domain $A$:

$$g(\bar\nu) = \langle j_{\bar E}(f_0^{\bar\nu,i})(\alpha_0^{\bar\nu,i}) \mid i<i^{\bar\nu}\rangle.$$

By the construction $g(\bar\nu)\in M_{\bar E}$ is a maximal anti-chain in $j_{\bar E}(R)(\kappa^0(\bar\nu),\kappa)$ below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu)),\beta)$. Hence, $\forall\tau<\mathrm{l}(\bar E)$ $j_{\bar E}(g)(\bar E_\alpha\restriction\tau)\in M^2_{\bar E}$ is a maximal anti-chain below $j^2_{\bar E}(F^{p_0})(\beta, j_{\bar E}(\beta))$. By genericity of $I(\bar E)$ over $M^2_{\bar E}$, there are $g^\tau$'s such that $j^2_{\bar E}(g^\tau)(\alpha, j_{\bar E}(\alpha))\in I(\bar E)$ and $j^2_{\bar E}(g^\tau)(\alpha, j_{\bar E}(\alpha))$ is stronger than a condition in $j_{\bar E}(g)(\bar E_\alpha\restriction\tau)$. Let $h^\tau$ be such that $j^2_{\bar E}(g^\tau)(\alpha, j_{\bar E}(\alpha))\leq j_{\bar E}(g)(\bar E_\alpha\restriction\tau)(j_{\bar E}(h^\tau)(\bar E_\alpha\restriction\tau))$. Note that we can use $\alpha$ here because the generic was built through the normal ultrafilter. If we would not have had this property we would have enlarged $\alpha$ to accommodate the intersection and we might have needed different $\alpha$ for each $\tau$.

We are ready to combine the information gathered into $\mathrm{l}(\bar{E})+1$ conditions as follows. We begin with $p'_0$:

$$\begin{aligned}
&p'_0 = \bigcup_{\bar{\nu}\in A} u_0^{\bar{\nu}},\\
&\forall \nu \in A\ \mathrm{l}(\nu)=0 \implies f^{p'_0}(\nu) = f^{p_0}\circ \pi_{\alpha,\beta}(\nu),\\
&\forall \bar{\nu}\in A\ F^{p'_0} \leq F_0^{\bar{\nu}} \circ \pi_{\alpha,\alpha^{\bar{\nu}}}.
\end{aligned}$$

Now for each $\tau < \mathrm{l}(\bar{E})$ we construct $p_0^\tau = p'_0 \cup u_1^\tau \cup \{\langle \bar{E}_{\beta^\tau}, T_0^\tau, f_0^\tau, F_0^\tau\rangle\}$ as follows:

$$\begin{aligned}
&T_0'^\tau = \mathop{\triangle^0}_{\bar{\nu}\in A} \pi^{-1}_{\alpha,\alpha^{\bar{\nu},h^\tau(\bar{\nu})}} T_0^{\bar{\nu},h^\tau(\bar{\nu})},\\
&\forall \bar{\nu}\in A\ F_0'^\tau \leq F_0^{\bar{\nu},h^\tau(\bar{\nu})},\\
&\bar{\alpha}_1^\tau = j_{\bar{E}}(\bar{\alpha}_1)^{\bar{E}_\alpha\restriction\tau, j_{\bar{E}}(h^\tau)(\bar{E}_\alpha\restriction\tau)},\\
&u_1'^\tau = j_{\bar{E}}(u_1)^{\bar{E}_\alpha\restriction\tau, j_{\bar{E}}(h^\tau)(\bar{E}_\alpha\restriction\tau)},\\
&u_1^\tau = \{\langle \bar{E}_\gamma, (u_1'^\tau)^{\bar{E}_\gamma\restriction\tau}\rangle \mid \bar{E}_\gamma\restriction\tau \in \operatorname{supp} u_1'^\tau\},\\
&T_1^\tau = j_{\bar{E}}(T_1)^{\bar{E}_\alpha\restriction\tau, j_{\bar{E}}(h^\tau)(\bar{E}_\alpha\restriction\tau)},\\
&\text{If } \tau>0 \text{ then } f_1^\tau = j_{\bar{E}}(f_1)^{\bar{E}_\alpha\restriction\tau, j_{\bar{E}}(h^\tau)(\bar{E}_\alpha\restriction\tau)},\\
&\text{If } \tau=0 \text{ then } \forall\bar{\nu}\in A\ \mathrm{l}(\bar{\nu})=0 \implies f_1^0(\bar{\nu}) = f^{\bar{\nu},h^0(\bar{\nu})},\\
&F_1^\tau = j_{\bar{E}}(F_1)^{\bar{E}_\alpha\restriction\tau, j_{\bar{E}}(h^\tau)(\bar{E}_\alpha\restriction\tau)}.
\end{aligned}$$

We note that the construction ensures us that $f_1^\tau \leq f^{p_0}$, $F_1^\tau \leq F^{p_0}$ when $\tau > 0$, and $u_1^\tau$ and $p'_0$ do not contain contradictory information. Hence we can define

$$\begin{aligned}
&\beta^\tau >_{\bar{E}} \alpha_1^\tau, \alpha,\\
&T_0^\tau = \pi^{-1}_{\beta^\tau,\alpha} T^{p'_0},\\
&f_0^\tau = f_1^\tau \circ \pi_{\beta^\tau,\alpha_1^\tau},\\
&F_0^\tau \leq F_1^\tau \circ \pi_{\beta^\tau,\alpha_1^\tau}, g^\tau\circ\pi_{\beta^\tau,\alpha}.
\end{aligned}$$

With this we have constructed for each $\tau < \mathrm{l}(\bar{E})$ the condition $p_0^\tau$.

We consider the sets

$$A^\tau = \{\bar{\nu} \mid \exists t'_1 \leq^{**} (p^\tau_{0\langle\bar{\nu}\rangle})_1\ t'_1{}^\frown (p^\tau_{0\langle\bar{\nu}\rangle})_0 \in D\}$$

for each $\tau < \mathrm{l}(\bar{E})$. There are two options at this point:

(1) There is $\tau < \mathrm{l}(\bar{E})$ such that $A^\tau \in E_{\beta^\tau}(\tau)$: Of course, $p_0^\tau$ satisfies the requested conclusion. Hence we set $p_0^* = p_0^\tau$ and the theorem is proved.

(2) $\forall \tau < \mathrm{l}(\bar{E})\ A^\tau \notin \bar{E}_{\beta^\tau}(\tau)$: We claim that some shrinkage of $T^{p'_0}$ is enough to get us into clause 2 of the theorem. Let us assume, by contradiction, that a small shrinkage can not bring us to $p_0^*$. This means that there is $\tau < \mathrm{l}(\bar{E}_\alpha)$ such that

$$\{\bar{\nu}\in T^{p'_0} \mid \exists q_1{}^\frown q_0 \leq^* p'_{0\langle\bar{\nu}\rangle}\ q_1{}^\frown q_0 \in D\} \in E_\alpha(\tau).$$

Let $F'' \leq F^{p'_0}, g^\tau$ and $p''_0$ be $p'_0$ with $F''$ substituted for $F^{p'_0}$. Due to openness of $D$ we still have

$$\{\bar{\nu}\in T^{p'_0} \mid \exists q_1{}^\frown q_0 \leq^* p''_{0\langle\bar{\nu}\rangle}\ q_1{}^\frown q_0 \in D\} \in E_\alpha(\tau).$$

Hence, by the construction, we have

$$\{\bar\nu \in T^{p'_0} \mid u_1^{\bar\nu,h^\tau(\bar\nu)} \cup \{\langle \bar\alpha_1^{\bar\nu,h^\tau(\bar\nu)}, T_1^{\bar\nu,h^\tau(\bar\nu)}, f_1^{\bar\nu,h^\tau(\bar\nu)}, F_1^{\bar\nu,h^\tau(\bar\nu)}\rangle\} {}^\frown (u^{\bar\nu,h^\tau(\bar\nu)}_{0\langle \pi_{\alpha,\alpha_0^{\bar\nu,h^\tau(\bar\nu)}}(\bar\nu)\rangle})_0 \cup \{\langle \bar E_{\alpha_0^{\bar\nu,h^\tau(\bar\nu)}}, T_0^{\bar\nu,h^\tau(\bar\nu)}, f_0^{\bar\nu,h^\tau(\bar\nu)}, F_0^{\bar\nu,h^\tau(\bar\nu)}\rangle\} \in D\} \in E_\alpha(\tau).$$

Invoking $\pi^{-1}_{\beta^\tau,\alpha}$ on the above set yields

$$\{\bar\nu \in T_0^\tau \mid u_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))} \cup \{\langle \bar\alpha_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, T_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, f_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, F_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}\rangle\} {}^\frown (u^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}_{0\langle \pi_{\beta^\tau,\alpha_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}}(\bar\nu)\rangle})_0 \cup \{\langle \bar E_{\alpha_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}}, T_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, f_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, F_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}\rangle\} \in D\} \in E_{\beta^\tau}(\tau).$$

Now, from the construction of $p^\tau$ we see that

$$\{\bar\nu \in T_0^\tau \mid \exists t'_1 \leq^{**} (p^\tau_{\langle\bar\nu\rangle})_1\ t'_1 {}^\frown (p^\tau_{\langle\bar\nu\rangle})_0 \leq^* u_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))} \cup \{\langle \bar\alpha_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, T_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, f_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, F_1^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}\rangle\} {}^\frown (u^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}_{0\langle \pi_{\alpha,\alpha_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}}\rangle})_0 \cup \{\langle \bar E_{\alpha_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}}, T_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, f_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}, F_0^{\pi_{\beta^\tau,\alpha}(\bar\nu),h^\tau(\pi_{\beta^\tau,\alpha}(\bar\nu))}\rangle\}\} \in E_{\beta^\tau}(\tau).$$

Combining the above 2 formulas and recalling that $D$ is open we get

$$\{\bar\nu \in T_0^\tau \mid \exists t'_1 \leq^{**} (p^\tau_{\langle\bar\nu\rangle})_1\ t'_1 {}^\frown (p^\tau_{\langle\bar\nu\rangle})_0 \in D\} \in E_{\beta^\tau}(\tau).$$

That is $A^\tau \in E_{\beta^\tau}(\tau)$. Contradiction. So, we have shown that

$$T^{p^*_0} = \{\bar\nu \in T^{p'_0} \mid \forall q_1 {}^\frown q_0 \leq^* p'_{0\langle\bar\nu\rangle}\ q_1 {}^\frown q_0 \notin D\} \in \bar E_\alpha.$$

By letting $p^*_0$ be $p'_0$ with $T^{p'_0}$ substituted by $T^{p^*_0}$ we get clause 2.

□

**Lemma 7.3.** *Let $D$ be dense open in $P_{\bar E}/P_{\bar\epsilon}$, $p = p_0 \in P_{\bar E}/P_{\bar\epsilon}$. Then there are $n < \omega$, $p^*_0 \leq^* p_0$ and $S \subseteq (T^{p^*_0})^n$, an* $\mathrm{mc}(p^*_0)$*-fat tree, such that*

$$\forall \langle \bar\nu_1, \ldots, \bar\nu_n\rangle \in S\ \exists t'_n {}^\frown \cdots {}^\frown t'_1 \leq^{**} (p^*_{0\langle \bar\nu_1,\ldots,\bar\nu_n\rangle})_{n..1}\ t'_n {}^\frown \cdots {}^\frown t'_1 {}^\frown (p^*_{0\langle \bar\nu_1,\ldots,\bar\nu_n\rangle})_0 \in D.$$

*Proof.* Construct, by repeat invocation of 7.2 for each $n < \omega$, a $\leq^*$-decreasing sequence $\langle p^n_0 \mid n < \omega\rangle$. Let $p^*_0 \leq^* p^n_0$ for all $n < \omega$. Choose $q \in D$ such that $q \leq p^*_0$. There is $\langle \bar\nu_1, \ldots, \bar\nu_n\rangle \in (T^{p^*_0})^n$ such that $q \leq^* p^*_{0\langle \bar\nu_1,\ldots,\bar\nu_n\rangle}$. By this we eliminated clause 2 of claim 7.2 for $n$. □

**Claim 7.4.** *Assume* $\mathrm{l}(\bar\epsilon) = 0$ *and let $D$ be dense open in $P_{\bar\epsilon}/P_{\bar\epsilon_2}$, $p = p_0 \in P_{\bar\epsilon}/P_{\bar\epsilon_2}$. Then there is $p^*_0 \leq^* p_0$ such that $p^*_0 \in D$.*

*Proof.* Of course this is completely trivial as $(P_{\bar\epsilon}/P_{\bar\epsilon_2})/p_0 \simeq R(\kappa(p^0_0), \kappa^0(\bar\epsilon))/p_0$. In this case $\leq^*$ and $\leq$ are the same. □

**Lemma 7.5.** *Assume* $\mathrm{l}(\bar\epsilon) = 0$ *and let* $G$ *be* $P_{\bar\epsilon}/P_{\bar\epsilon_2}$*-generic with* $p = p_0 \in G$*. Then* $V_{\kappa^0(\bar\epsilon_2)+1} = V^{V[G]}_{\kappa^0(\bar\epsilon_2)+1}$ *and* $P_{\bar\epsilon_2} = P^{V[G]}_{\bar\epsilon_2}$.

*Proof.* Let $\nu = \kappa(p^0_0)$. As $P_{\bar\epsilon}/P_{\bar\epsilon_2}$ is $\nu^+$-closed we get immediately that $V_{\kappa^0(\bar\epsilon_2)+1} = V^{V[G]}_{\kappa^0(\bar\epsilon_2)+1}$.

We have much more than that. Namely, $\nu^{++}, \dots, \nu^{+6}$ are not collapsed. For this we remind the reader that the $V$ we work with is a generic extension of $V^*$ for a reverse Easton forcing. Let $Q_1$ be the reverse Easton forcing up to $\kappa^0(\bar\epsilon_2)$ and $H_1$ be its generic. Let $Q_2$ be the forcing at stage $\kappa^0(\bar\epsilon_2)$ and $H_2$ be its generic over $V^*[H_1]$. Let $Q_3$ be the rest of the reverse Easton forcing up to $\kappa^0(\bar\epsilon)$ and $H_3$ be its generic over $V^*[H_1][H_2]$. Then we have

$$\mathcal{P}^{V[G]}(\kappa^0(\bar\epsilon)) = \mathcal{P}^{V^*[H_1][H_2][H_3][G]}(\kappa^0(\bar\epsilon)).$$

Note that $V^*[H_1][H_2][H_3][G]$ is a reflection of the situation at 3.4 and by this we see that $\nu^{++}, \dots, \nu^{+6}$ are not collapsed.

In fact we see that nothing has changed as far as the definition of $P_{\bar\epsilon_2}$ in $V[G]$ is concerned. (We might have new anti-chains which is no obstacle to us). □

**Lemma 7.6.** *Assume* $\mathrm{l}(\bar\epsilon) = 0$ *and let* $p = p_{l..0} \in P_{\bar\epsilon}$*. Assume that 7.1 is true for* $p_{l..1} \in P_{\bar\epsilon_2}$ *and dense open subsets of* $P_{\bar\epsilon_2}$*. Then it is true for* $p_{l..0}$ *and dense open subsets of* $P_{\bar\epsilon}$.

*Proof.* In order to avoid excess of indices we give the proof of the case $p = p_{1..0}$.

Let $G$ be $P_{\bar\epsilon}/P_{\bar\epsilon_2}$-generic with $p_0 \in G$ and let $D$ be dense open in $P_{\bar\epsilon}$. Then $D_{\bar\epsilon_2} = \{q \leq p_1 \mid q \frown r \in D,\ r \in G\} \in V[G]$ is a dense open subset of $P_{\bar\epsilon_2}$. By 7.5 and 7.1 for $p_1$ there are $p^*_1 \in V$ and $S^1 \in V$, an $\mathrm{mc}(p^*_1)$-fat tree, such that $p^*_1 \leq^* p_1$ and

$$\forall \langle \bar\nu_1, \dots, \bar\nu_n\rangle \in S^1\ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$$
$$t'_n \frown \cdots \frown t'_1 \frown (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_0 \in D_{\bar\epsilon_2}.$$

Hence there is $p'_0 \leq^* p_0$ which forces the above. That is for each $\langle \bar\nu_1, \dots, \bar\nu_n\rangle \in S^1$ there is a maximal anti-chain, $A(\bar\nu_1, \dots, \bar\nu_n)$, of $P_{\bar\epsilon}/P_{\bar\epsilon_2}$ below $p'_0$ such that

$$\forall \langle \bar\nu_1, \dots, \bar\nu_n\rangle \in S^1\ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$$
$$\forall q_0 \in A(\bar\nu_1, \dots, \bar\nu_n)$$
$$t'_n \frown \cdots \frown t'_1 \frown (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_0 \frown q_0 \in D.$$

By noting that $\langle P_{\bar\epsilon}/P_{\bar\epsilon_2}, \leq^*\rangle$ is $\kappa^0(\bar\epsilon_2)^+$-closed and that $|S^1| = \kappa^0(\bar\epsilon_2)$ we see that there is $p^*_0 \leq^* p'_0$ such that

$$\forall \langle \bar\nu_1, \dots, \bar\nu_n\rangle \in S^1\ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$$
$$t'_n \frown \cdots \frown t'_1 \frown (p^*_{1\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_0 \frown p^*_0 \in D.$$

□

Our main obstacle in proving 7.1 is that in general $P_{\bar E}/P_{\bar\epsilon}$ is $\kappa^0(\bar\epsilon)^+$-closed while $P_{\bar\epsilon}$ is $\kappa^0(\bar\epsilon)^{++}$-c.c. However, when $\mathrm{l}(\bar\epsilon) = 0$ we have $P_{\bar\epsilon}$ is $\kappa^0(\bar\epsilon)^+$-c.c. The following 2 lemmas give us facts in this case which help us to overcome the obstacle.

**Lemma 7.7.** *Let $D$ be dense open in $P_{\bar{E}}$, $p = p_l \frown \cdots \frown p_0 \in P_{\bar{E}}$. Then there are $n < \omega$, $p^*_0 \leq^* p_0$, $q \leq p_{l..1}$ and $S \subseteq (T^{p^*_0})^n$, an $\text{mc}(p^*_0)$-fat tree, such that*

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad q \frown t'_n \frown \cdots \frown t'_1 \frown (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D.$$

*Proof.* Let $D_{\bar{E}} = \{r \leq p_0 \mid \exists s \leq p_{l..1} \ s \frown r \in D\}$. Then $D_{\bar{E}}$ is dense open in $P_{\bar{E}}/P_{\bar{\epsilon}}$ below $p_0$ where $p_{l..1} \in P_{\bar{\epsilon}}$. By 7.3 there are $n < \omega$, $p^*_0 \leq^* p_0$, $S \subseteq (T^{p^*_0})^n$ such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad t'_n \frown \cdots \frown t'_1 \frown (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D_{\bar{E}}.$$

By the definition of $D_{\bar{E}}$ we see that there is a function $q(\bar{\nu}_1, \ldots, \bar{\nu}_n)$ with domain $S$ such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad q(\bar{\nu}_1, \ldots, \bar{\nu}_n) \frown t'_n \frown \cdots \frown t'_1 \frown (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D.$$

As $q(\bar{\nu}_1, \ldots, \bar{\nu}_n) \leq p_{l..1}$ there is $q \leq p_{l..1}$ such that except on a measure 0 set we have $q(\bar{\nu}_1, \ldots, \bar{\nu}_n) = q$. By removing this measure 0 set from $S$ we get

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad q \frown t'_n \frown \cdots \frown t'_1 \frown (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D.$$

□

**Lemma 7.8.** *Let $D$ be dense open in $P_{\bar{E}}$, $p = p_{l..0} \in P_{\bar{E}}$. Let $\bar{\epsilon}$ be such that $p_{l..1} \in P_{\bar{\epsilon}}$ where $\text{l}(\bar{\epsilon}) = 0$. Then there are $p^*_0 \leq^* p_0$ and $\langle q^\xi \mid \xi < \kappa^0(\bar{\epsilon}) \rangle$ a maximal anti-chain below $p_{l..1}$ such that for each $\xi < \kappa^0(\bar{\epsilon})$ there are $p'_0 \geq^* p^*_0$, $n < \omega$ and $S \subseteq (T^{p'_0})^n$, an $\text{mc}(p'_0)$-fat tree, such that*

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p'_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad q^\xi \frown t'_n \frown \cdots \frown t'_1 \frown (p'_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D$$

*and*

$$\{q^\xi \frown t'_n \frown \cdots \frown t'_1 \frown (p'_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \mid \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S\}$$

*is pre-dense below $q^\xi \frown p'_0$.*

*Of course an immediate corollary is for each $\xi < \kappa^0(\bar{\epsilon})$ there are $n < \omega$ and $S \subseteq (T^{p^*_0})^n$, an $\text{mc}(p^*_0)$-fat tree, such that*

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1} \quad q^\xi \frown t'_n \frown \cdots \frown t'_1 \frown (p^*_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n\rangle})_0 \in D.$$

*Proof.* Our first observation is that $P_{\bar{\epsilon}}$ is $\kappa^0(\bar{\epsilon})^+$-c.c. (As opposed to the usual $\kappa^0(\bar{\epsilon})^{++}$-c.c. we have when $\text{l}(\bar{\epsilon}) > 0$). We construct, by induction, the sequence $\langle q^\xi \mid \xi < \xi_0 \rangle$ where $\xi_0 < \kappa^0(\bar{\epsilon})^+$. Together with it we construct an auxiliary $\leq^*$-decreasing sequence $\langle p^\xi_0 \mid \xi < \xi_0 \rangle$.

- $\xi_0 = 0$: By 7.7, 6.4 and openness of $D$ there are $q^0 \leq p_{l..1}$, $p^0_0 \leq^* p_0$, $S \subseteq (T^{p^0_0})^n$ such that

$$\forall \langle \bar{\nu}_1, \ldots, \bar{\nu}_n \rangle \in S \ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^0_{0\langle \bar{\nu}_1,\ldots,\bar{\nu}_n \rangle})_{n..1}$$

$$q^0 \frown t'_n \frown \cdots \frown t'_1 \frown (p^0_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0 \in D$$

and $\{q^0 \frown t'_n \frown \cdots \frown t'_1 \frown (p^0_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0 \mid \langle\bar\nu_1,\dots,\bar\nu_n\rangle \in S\}$ is pre-dense below $q^0 \frown p^0_0$.

- $\xi_0 > 0$: If $\langle q^\xi \mid \xi < \xi_0\rangle$ is a maximal anti-chain below $p_{l..1}$ then the induction is finished. If it is not a maximal anti-chain we observe that $\xi_0 < \kappa^0(\bar\epsilon)^+$ as $P_{\bar\epsilon}$ is $\kappa^0(\bar\epsilon)^+$-c.c. Let $q' < p_{l..1}$ be such that $\forall \xi < \xi_0\ q' \perp q^\xi$. As $\langle P_{\bar E}, \leq^*\rangle$ is $\kappa^0(\bar\epsilon)^+$-closed there is $p'_0$ such that $p'_0 \leq^* p^\xi_0$ for all $\xi < \xi_0$. By 7.7, 6.4 starting from $q' \frown p'_0$ there are $q^{\xi_0} \leq q'$, $p^{\xi_0}_0 \leq^* p'_0$, $S \subseteq (T^{p^{\xi_0}_0})^n$ such that

$$\forall\langle\bar\nu_1,\dots,\bar\nu_n\rangle \in S\ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (p^{\xi_0}_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}$$
$$q^{\xi_0} \frown t'_n \frown \cdots \frown t'_1 \frown (p^{\xi_0}_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0 \in D$$

and $\{q^{\xi_0} \frown t'_n \frown \cdots \frown t'_1 \frown (p^{\xi_0}_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0 \mid \langle\bar\nu_1,\dots,\bar\nu_n\rangle \in S\}$ is pre-dense below $q^{\xi_0} \frown p^{\xi_0}_0$.

When the induction terminates we have a $\leq^*$-decreasing sequence $\langle p^\xi_0 \mid \xi < \xi_0\rangle$ where $\xi_0 < \kappa^0(\bar\epsilon)^+$. By choosing $p^*_0 \leq^* p^\xi_0$ for all $\xi < \xi_0$ we finish the proof. □

**Lemma 7.9.** *Let $p = p_{l..0} \in P_{\bar E}$. Assume that 7.1 is true for $p_{l..1} \in P_{\bar\epsilon}$ and dense open subsets of $P_{\bar\epsilon}$. Then it is true for $p_{l..0}$ and dense open subsets of $P_{\bar E}$.*

*Proof.* In order to avoid too many indices we prove the lemma for the case $p = p_{1..0}$.

Choose $\chi$ large enough so that $H_\chi$ contains everything we are interested in. Let $N \prec H_\chi$ be such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $p \in N$, $P_{\bar E} \in N$.

Let $\bar E_\beta = \text{mc}(p_0)$. Choose $\alpha \in \operatorname{dom} \bar E$ such that $\alpha >_{\bar E} \gamma$ for all $\gamma \in \operatorname{dom} \bar E \cap N$. Let $A = \{\bar\nu \in \pi^{-1}_{\alpha,\text{mc}(p_0)} T^{p_0} \mid \text{l}(\bar\nu) = 0\}$. Note that $A \in E_\alpha(0)$. Let $\preceq$ be a well ordering of $A$ such that $\forall \bar\nu_1, \bar\nu_2 \in A\ \bar\nu_1 \preceq \bar\nu_2 \implies \kappa^0(\bar\nu_1) \leq \kappa^0(\bar\nu_2)$. We shrink $A$ a bit so that the following is satisfied: $\forall \bar\nu \in A\ |\{\bar\mu \in A \mid \kappa^0(\bar\mu) < \kappa^0(\bar\nu)\}| \leq \kappa^0(\bar\nu)$. We start an induction on $\bar\nu$ in which we build

$$\langle \alpha^{\bar\nu}, u^{\bar\nu}_0, T^{\bar\nu}_0, F^{\bar\nu}_0 \mid \bar\nu \in A\rangle,$$

where $(u^{\bar\nu}_0)_{\langle\pi_{\alpha,\alpha^{\bar\nu}}(\bar\nu)\rangle} \cup \{\langle \bar E_{\alpha^{\bar\nu}}, F^{p_0}(\pi_{\alpha,\beta}(\bar\nu), \pi_{\alpha^{\bar\nu},\beta}(-), T^{\bar\nu}_0, F^{\bar\nu}_0\rangle\} \in P_{\bar E}$. Assume that we have constructed $\langle \alpha^{\bar\nu}, u^{\bar\nu}_0, F^{\bar\nu}_0, T^{\bar\nu}_0 \mid \bar\nu \prec \bar\nu_0\rangle$. We start working in $N$. Set the following:

- $\bar\nu_0$ is $\prec$-minimal:
$$q' = p_0 \setminus \{\langle \bar E_\beta, T^{p_0}, f^{p_0}, F^{p_0}\rangle\},$$
$$\alpha' = \beta.$$
- $\bar\nu_0$ is the immediate $\prec$-successor of $\bar\nu$:
$$q' = u^{\bar\nu}_0,$$
$$\alpha' = \alpha^{\bar\nu}.$$
- $\bar\nu_0$ is $\prec$-limit: Choose $\alpha' \in N$ such that $\forall \bar\nu \prec \bar\nu_0\ \alpha' >_{\bar E} \alpha^{\bar\nu}$ and set
$$q' = \bigcup_{\bar\nu \prec \bar\nu_0} u^{\bar\nu}_0.$$

We make an induction on $i$ which builds $\langle \alpha^{\bar\nu_0,i}, u_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, f_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i} \mid i < \kappa\rangle$. Assume we have constructed

$$\langle \alpha^{\bar\nu_0,i}, u_0^{\bar\nu_0,i}, T_0^{\bar\nu_0,i}, f_0^{\bar\nu_0,i}, F_0^{\bar\nu_0,i} \mid i < i_0\rangle,$$

and we do step $i_0$.

- $i_0 = 0$:
$$\begin{aligned} q'' &= q', \\ \alpha'' &= \alpha', \\ f'' &= F^{p_0}(\kappa(\pi_{\alpha,\beta}(\bar\nu_0)), \pi_{\alpha'',\beta}(-)). \end{aligned}$$
- $i_0 = i+1$: If $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}) \mid i < i_0\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\kappa(\pi_{\alpha,\beta}(\bar\nu_0)), \beta)$ we terminate the induction on $i$.
  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_0$, $i_0 < \mu^+$, and $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}) \wedge \mu \mid i < i_0\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that
$$\begin{aligned} &j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu, \\ &\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}). \end{aligned}$$
  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$
  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i})$, and for each inaccessible $\mu < i_0$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$.
  We set
$$\begin{aligned} q'' &= u_0^{\bar\nu_0,i}, \\ \alpha'' &\geq_{\bar E} \alpha^{\bar\nu_0,i}, \beta''. \end{aligned}$$
- $i_0$ is limit: If $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}) \mid i < i_0\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\kappa(\pi_{\alpha,\beta}(\bar\nu_0)), \beta)$ we terminate the induction on $i$.
  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_0$, $i_0 < \mu^+$, and $\langle j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}) \wedge \mu \mid i < i_0\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that
$$\begin{aligned} &j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu, \\ &\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i}). \end{aligned}$$
  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$
  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_0\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\bar\nu_0,i})(\alpha^{\bar\nu_0,i})$, and for each inaccessible $\mu < i_0$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\bar\nu_0), \kappa) \wedge \mu$.
  Choose $\alpha'' \in N$ such that $\forall i < i_0\ \alpha'' >_{\bar E} \alpha^{\bar\nu_0,i}, \beta''$. We set
$$q'' = \bigcup_{i<i_0} u_0^{\bar\nu_0,i}.$$

Set

$$u''_1 = \big(q''_{\langle \pi_{\alpha,\alpha''}(\bar\nu_0)\rangle}\big)_1,$$

$$
\begin{aligned}
&u''_0 = \big(q''_{\langle \pi_{\alpha,\alpha''}(\bar\nu_0)\rangle}\big)_0,\\
&T''_0 = \pi^{-1}_{\alpha'',\beta} T^{p_0} \setminus \pi_{\alpha,\alpha''}(\bar\nu_0),\\
&F''_0 = F^{p_0} \circ \pi_{\alpha'',\beta},\\
&T''_1 = \emptyset,\\
&f''_1 = f^{p_0} \circ \pi_{\alpha,\beta}(\kappa(\bar\nu_0)),\\
&F''_1 = \emptyset.
\end{aligned}
$$

Using the corollary of 7.8 construct $q'''_0 \leq^* u''_0 \cup \{\langle \bar E_{\alpha''}, T''_0, f'' \circ \pi_{\alpha'',\beta''}, F''_0\rangle\}$ and $B^{\bar\nu_0,i_0}$ a maximal anti-chain below $p_{l..1} {}^\frown u''_1 \cup \{f''_1\}$. So for each $b \in B^{\bar\nu_0,i_0}$ there is $S \subseteq (T^{q'''_0})^n$, an $\mathrm{mc}(q'''_0)$-tree, such that

$$
\forall \langle \bar\nu_1,\dots,\bar\nu_n\rangle \in S\ \exists t'_n {}^\frown \cdots {}^\frown t'_1 \leq^{**} (q'''_{0\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}
\quad b {}^\frown t'_n {}^\frown \cdots {}^\frown t'_1 {}^\frown (q'''_{0\langle \bar\nu_1,\dots,\bar\nu_n\rangle})_0 \in D.
$$

We set

$$
\begin{aligned}
&\alpha^{\bar\nu_0,i_0} = \kappa(\mathrm{mc}(q'''_0)),\\
&u^{\bar\nu_0,i_0}_0 = q'' \cup \{\langle \bar E_\gamma, q'''^{\bar E_\gamma}_0\rangle \mid \bar E_\gamma \in \operatorname{supp} q'''_0 \setminus \operatorname{supp} q''\},\\
&T^{\bar\nu_0,i_0}_0 = T^{q'''_0},\\
&f^{\bar\nu_0,i_0}_0 = f^{q'''_0},\\
&F^{\bar\nu_0,i_0}_0 = F^{q'''_0}.
\end{aligned}
$$

When the induction on $i$ terminates we have

$$
\langle \alpha^{\bar\nu_0,i}, u^{\bar\nu_0,i}_0, T^{\bar\nu_0,i}_0, f^{\bar\nu_0,i}_0, F^{\bar\nu_0,i}_0 \mid i < \kappa\rangle.
$$

We point out that $\langle j_{\bar E}(f^{\bar\nu_0,i}_0)(\alpha^{\bar\nu_0,i}) \mid i < \kappa\rangle$ is a maximal anti-chain: Assume $f \in j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa)$, $f \leq j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)),\beta)$. Then there is an inaccessible $\mu < \kappa$ such that $f \in j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa) \wedge \mu$. By the construction $\langle j_{\bar E}(f^{\bar\nu_0,i}_0)(\alpha^{\bar\nu_0,i}) \wedge \mu \mid i < \mu^+\rangle$ is pre-dense in $j_{\bar E}(R)(\kappa^0(\bar\nu_0),\kappa) \wedge \mu$ below $j_{\bar E}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\bar\nu_0)),\beta) \wedge \mu$. Hence there is $i < \mu^+$ such that $f \parallel j_{\bar E}(f^{\bar\nu_0,i}_0)(\alpha^{\bar\nu_0,i}) \wedge \mu$. Hence $f \parallel j_{\bar E}(f^{\bar\nu_0,i}_0)(\alpha^{\bar\nu_0,i})$.

We complete step $\bar\nu_0$ by setting

$$
\begin{aligned}
&\alpha^{\bar\nu_0} \in N,\\
&\forall i < \kappa\ \alpha^{\bar\nu_0} >_{\bar E} \alpha^{\bar\nu_0,i},\\
&u^{\bar\nu_0}_0 = \bigcup_{i<\kappa} u^{\bar\nu_0,i}_0,\\
&T^{\bar\nu_0}_0 = \mathop{\triangle^0}_{i<\kappa} \pi^{-1}_{\alpha^{\bar\nu_0},\alpha^{\bar\nu_0,i}} T^{\bar\nu_0,i}_0,\\
&\forall i < \kappa\ F^{\bar\nu_0}_0 \leq F^{\bar\nu_0,i}_0 \circ \pi_{\alpha^{\bar\nu_0},\alpha^{\bar\nu_0,i}}.
\end{aligned}
$$

When the induction on $\bar\nu$ terminates we return to work in $V$ and we have

$$
\langle \alpha^{\bar\nu}, u^{\bar\nu}_0, T^{\bar\nu}_0, F^{\bar\nu}_0 \mid \bar\nu \in A\rangle.
$$

We define the following function with domain $A$:

$$
g(\bar\nu) = \langle j_{\bar E}(f^{\bar\nu,i}_0)(\alpha^{\bar\nu,i}) \mid i < \kappa\rangle.
$$

By the construction, $g(\bar\nu)$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\kappa(\pi_{\alpha,\beta}(\bar\nu)),\beta)$. We note that $\langle j_0(f_0^{\bar\nu,i})(\alpha^{\bar\nu,i}) \mid i<\kappa\rangle \in M_0$ as $M_0$ is closed under $\kappa$-sequences. Hence $g(\bar\nu)\in M_{\bar E}$ as $g(\bar\nu) = i_{0,\bar E}(\langle j_0(f_0^{\bar\nu,i})(\alpha^{\bar\nu,i}) \mid i<\kappa\rangle)$. So $j_{\bar E}(g)(\alpha)\in M^2_{\bar E}$ is a maximal anti-chain below $j^2_{\bar E}(F^{p_0})(\beta, j_{\bar E}(\beta))$. As $I(\bar E)$ is $j^2_{\bar E}(R)(\kappa, j_{\bar E}(\kappa))$-generic over $M^2_{\bar E}$, there is $f_0$ such that $j^2_{\bar E}(f_0)(\alpha, j_{\bar E}(\alpha))\in I(\bar E)$ and $j^2_{\bar E}(f_0)(\alpha, j_{\bar E}(\alpha))$ is stronger than a condition in $j_{\bar E}(g)(\alpha)$. Note that we can use $\alpha$ here because the generic was build through the normal measure. If we would not have had this property we would have enlarged $\alpha$ to accommodate the intersection. We combine everything into one condition, $p'^*_0$, as follows:

$$
\begin{aligned}
&p'^*_0 = \bigcup_{\bar\nu\in A} u_0^{\bar\nu},\\
&T^{p'^*_0} = \mathop{\triangle^0}_{\bar\nu\in A} \pi^{-1}_{\alpha,\alpha^{\bar\nu}} T_0^{\bar\nu},\\
&f^{p'^*_0}(\nu_1) = f^{p_0}\circ\pi_{\alpha,\beta}(\nu_1),\\
&\forall\bar\nu\in A\ F^{p'^*_0} \le F_0^{\bar\nu}\circ\pi_{\alpha,\alpha^{\bar\nu}}, f_0.
\end{aligned}
$$

We write what we have gained so far: For each $\bar\nu\in A$ there is $B_{\bar\nu}$, a maximal anti-chain below $p_1{}^\frown(p'^*_{0\langle\bar\nu\rangle})_1$, such that for each $b\in B_{\bar\nu}$ there is $S\subseteq (T^{p'^*_0})^n$, an mc($p'^*_0$)-tree, such that

$$
\begin{aligned}
\forall\langle\bar\nu_1,\dots,\bar\nu_n\rangle\in S\ \exists t'_n{}^\frown\cdots{}^\frown t'_1 \le^{**} (p'^*_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}&\\
b^\frown t'_n{}^\frown\cdots{}^\frown t'_1{}^\frown(p'^*_{0\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0\in D.&
\end{aligned}
$$

We set $D_{\bar\nu}=\{r\in P_{\bar\nu}\mid r\le q,\ q\in B_{\bar\nu}\}$. Then $D_{\bar\nu}$ is a dense open subset of $P_{\bar\nu}$. By invoking 7.6 for $D_{\bar\nu}$, $p_1{}^\frown(p'^*_{0\langle\bar\nu\rangle})_1$ we find $p_1(\bar\nu){}^\frown h(\bar\nu)\le^* p_1{}^\frown(p'^*_{0\langle\bar\nu\rangle})_1$, $S^1(\bar\nu)$ such that

$$
\begin{aligned}
\forall\langle\bar\nu_1,\dots,\bar\nu_n\rangle\in S^1(\bar\nu)\ \exists t'_n{}^\frown\cdots{}^\frown t'_1 \le^{**} (p_1(\bar\nu)_{\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}&\\
t'_n{}^\frown\cdots{}^\frown t'_1{}^\frown(p_1(\bar\nu)_{\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0{}^\frown h(\bar\nu)\in D_{\bar\nu}.&
\end{aligned}
$$

Immediately we see that there are $p^*_1$, $S^1$ such that by removing a measure 0 set from $A$ we get $\forall\bar\nu\in A\ p^*_1=p_1(\bar\nu)$, $S^1=S^1(\bar\nu)$. So after the shrinkage of $A$ we have for each $\bar\nu\in A$

$$
\begin{aligned}
\forall\langle\bar\nu_1,\dots,\bar\nu_n\rangle\in S^1\ \exists t'_n{}^\frown\cdots{}^\frown t'_1 \le^{**} (p^*_{1\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_{n..1}&\\
t'_n{}^\frown\cdots{}^\frown t'_1{}^\frown(p^*_{1\langle\bar\nu_1,\dots,\bar\nu_n\rangle})_0{}^\frown h(\bar\nu)\in D_{\bar\nu}.&
\end{aligned}
$$

We gather the additional information we have by setting $f^{p^*_0}(\bar\nu)=h(\bar\nu)$ and letting the condition $p^*_0$ be $p'^*_0$ with $f^{p^*_0}$ substituted for $f^{p'^*_0}$. So at this point we have the following

$$
\begin{aligned}
&\forall\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle\in S^1\ \exists t'_{1,n_1}{}^\frown\cdots{}^\frown t'_{1,1}\le^{**}(p^*_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_{n_1..1}\\
&\qquad\forall\bar\nu\in A\ \exists S^0\subseteq (T^{p^*_0})^{n_0}\\
&\forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle\in S^0\ \exists t'_{0,n_0+1}{}^\frown\cdots{}^\frown t'_{0,1}\le^{**}(p^*_{0\langle\bar\nu,\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_{n_0+1..1}\\
&\qquad t'_{1,n_1}{}^\frown\cdots{}^\frown t'_{1,1}{}^\frown(p^*_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_0{}^\frown\\
&\qquad\qquad t'_{0,n_0+1}{}^\frown\cdots{}^\frown t'_{0,1}{}^\frown(p^*_{0\langle\bar\nu,\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_0\in D.
\end{aligned}
$$

Of course as $A \in E_\alpha(0)$ the above is just a convoluted form of

$$\forall \langle \bar\nu_{1,1}, \dots, \bar\nu_{1,n_1}\rangle \in S^1 \ \exists t'_{1,n_1} {}^\frown \cdots {}^\frown t'_{1,1} \leq^{**} (p^*_{1\langle \bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_{n_1..1}$$
$$\exists S^0 \subseteq (T^{p^*_0})^{n_0}$$
$$\forall \langle \bar\nu_{0,1}, \dots, \bar\nu_{0,n_0}\rangle \in S^0 \ \exists t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} \leq^{**} (p^*_{0\langle \bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_{n_0..1}$$
$$t'_{1,n_1} {}^\frown \cdots {}^\frown t'_{1,1} {}^\frown (p^*_{1\langle \bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_0 {}^\frown$$
$$t'_{0,n_0} {}^\frown \cdots {}^\frown t'_{0,1} {}^\frown (p^*_{0\langle \bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_0 \in D.$$

□

*proof of 7.1.* The proof is done by induction on $l$.

The case $l = 0$ is done in 7.4 for $\mathrm{l}(\bar E) = 0$ and 7.3 for $\mathrm{l}(\bar E) > 0$.

The case $l + 1$ is done in 7.6 for $\mathrm{l}(\bar E) = 0$ and 7.9 for $\mathrm{l}(\bar E) > 0$. □

## 8. Prikry's condition

**Lemma 8.1.** *Let $\sigma$ be a formula in the forcing language, $q {}^\frown p_k {}^\frown r \in P_{\bar E}$ and $S \subseteq (T^{p_k})^m$ an $\mathrm{mc}(p_k)$-fat tree such that*

$$\forall \langle \bar\nu_1, \dots, \bar\nu_m\rangle \in S \ \exists t'_m {}^\frown \cdots {}^\frown t'_1 \leq^{**} (p_{k\langle \bar\nu_1,\dots,\bar\nu_m\rangle})_{m..1}$$
$$q {}^\frown t'_m {}^\frown \cdots {}^\frown t'_1 {}^\frown (p_{k\langle \bar\nu_1,\dots,\bar\nu_m\rangle})_0 {}^\frown r \parallel \sigma.$$

*Then there is $p^*_k \leq^{**} p_k$ such that $q {}^\frown p^*_k {}^\frown r \parallel \sigma$.*

*Proof.* Let

$$A^1_{\bar\nu_1,\dots,\bar\nu_{m-1}} = \{\bar\nu_m \in \mathrm{Suc}_S(\bar\nu_1, \dots, \bar\nu_{m-1}) \mid$$
$$q {}^\frown t'_m {}^\frown \cdots {}^\frown t'_1 {}^\frown (p_{k\langle \bar\nu_1,\dots,\bar\nu_m\rangle})_0 {}^\frown r \Vdash \sigma\},$$

$$A^2_{\bar\nu_1,\dots,\bar\nu_{m-1}} = \{\bar\nu_m \in \mathrm{Suc}_S(\bar\nu_1, \dots, \bar\nu_{m-1}) \mid$$
$$q {}^\frown t'_m {}^\frown \cdots {}^\frown t'_1 {}^\frown (p_{k\langle \bar\nu_1,\dots,\bar\nu_m\rangle})_0 {}^\frown r \Vdash \neg\sigma\}.$$

Then

$$\mathrm{Suc}_S(\bar\nu_1, \dots, \bar\nu_{m-1}) = A^1_{\bar\nu_1,\dots,\bar\nu_{m-1}} \cup A^2_{\bar\nu_1,\dots,\bar\nu_{m-1}},$$
$$A^1_{\bar\nu_1,\dots,\bar\nu_{m-1}} \cap A^2_{\bar\nu_1,\dots,\bar\nu_{m-1}} = \emptyset.$$

Let $A_{\bar\nu_1,\dots,\bar\nu_{m-1}}$ be the $A^i_{\bar\nu_1,\dots,\bar\nu_{m-1}}$, $i \in \{1, 2\}$ such that

$$\exists \xi < \mathrm{l}(\bar E) \ A_{\bar\nu_1,\dots,\bar\nu_{m-1}} \in \mathrm{mc}(p_k)(\xi).$$

We choose $t'_m(\bar\nu_1) {}^\frown \cdots {}^\frown t'_1(\bar\nu_1, \dots, \bar\nu_m)$ such that for all $\langle \bar\nu_1, \dots, \bar\nu_{m-1}\rangle$, for all $\langle \bar\nu_m\rangle \in A_{\bar\nu_1,\dots,\bar\nu_{m-1}}$

$$q {}^\frown t'_m(\bar\nu_1) {}^\frown \cdots {}^\frown t'_1(\bar\nu_1, \dots, \bar\nu_m) {}^\frown (p_{k\langle \bar\nu_1,\dots,\bar\nu_m\rangle})_0 {}^\frown r \Vdash \sigma^i$$

where $\sigma^i \in \{\sigma, \neg\sigma\}$ according to the selection of $i$. By 6.3, for each $\langle \bar\nu_1, \dots, \bar\nu_{m-1}\rangle \in S$ there are $B_{\bar\nu_1,\dots,\bar\nu_{m-1}} \subseteq A_{\bar\nu_1,\dots,\bar\nu_{m-1}}$, $p^{m-1}_k(\bar\nu_1, \dots, \bar\nu_{m-1}) \leq^{**} (p_{k\langle \bar\nu_1,\dots,\bar\nu_{m-1}\rangle})_0$ such that below $q {}^\frown t'_m(\bar\nu_1) {}^\frown \cdots {}^\frown t'_2(\bar\nu_1, \dots, \bar\nu_{m-1}) {}^\frown p^{m-1}_k(\bar\nu_1, \dots, \bar\nu_{m-1}) {}^\frown r$

$$\{q {}^\frown t'_m(\bar\nu_1) {}^\frown \cdots {}^\frown t'_1(\bar\nu_1, \dots, \bar\nu_m) {}^\frown (p^{m-1}_k(\bar\nu_1, \dots, \bar\nu_{m-1})_{\langle \bar\nu_m\rangle})_0 {}^\frown r \mid$$
$$\bar\nu_m \in B_{\bar\nu_1,\dots,\bar\nu_{m-1}}\}$$

is pre-dense. Hence

$$q {}^\frown t'_m(\bar\nu_1) {}^\frown \dots {}^\frown t'_2(\bar\nu_1,\dots,\bar\nu_{m-1}) {}^\frown p^{m-1}_k(\bar\nu_1,\dots,\bar\nu_{m-1}) {}^\frown r \parallel \sigma.$$

Let $T^{m-1} = T^{p_k} \cap \triangle^0_{\bar\nu_1,\dots,\bar\nu_{m-1}} T^{p^{m-1}_k(\bar\nu_1,\dots,\bar\nu_{m-1})}$. Let $p^{m-1}_k$ be the condition $p_k$ with its measure 1 set substituted by $T^{m-1}$. We get that for all $\langle\bar\nu_1,\dots,\bar\nu_{m-1}\rangle \in S$

$$q {}^\frown t'_m(\bar\nu_1) {}^\frown \dots {}^\frown t'_2(\bar\nu_1,\dots,\bar\nu_{m-1}) {}^\frown (p^{m-1}_{k\langle\bar\nu_1,\dots,\bar\nu_{m-1}\rangle})_0 {}^\frown r \parallel \sigma.$$

Letting $S^{m-1}$ be $S$ restricted to $m-1$ levels bring us to the beginning of the proof but with $m-1$ instead of $m$.

Hence, repeating another $m-1$ steps as the above build $p^0_k \leq^{**} \dots \leq^{**} p^{m-1}_k \leq^{**} p_k$ and $q {}^\frown p^0_k {}^\frown r \parallel \sigma$. ☐

**Theorem 8.2.** *Let $p \in P_{\bar E}$, $\sigma$ a formula in the forcing language. Then there is $p^* \leq^* p$ such that $p^* \parallel \sigma$.*

*Proof.* As usual we give the proof for the case $p = p_1 {}^\frown p_0$.

Let $D = \{r \leq p \mid r \parallel \sigma\}$. $D$ is a dense open subset in $P_{\bar E}$. By 7.1 there is $p' = p'_{1..0} \leq^* p$ such that

$$\begin{aligned}&\exists S^1\ \forall\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle \in S^1\ \exists t'_{1,n_1} {}^\frown \dots {}^\frown t'_{1,1} \leq^{**} (p'_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_{n_1..1}\\ &\quad\exists S^0\ \forall\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle \in S^0\ \exists t'_{0,n_0} {}^\frown \dots {}^\frown t'_{0,1} \leq^{**} (p'_{0\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_{n_0..1}\\ &\qquad t'_{1,n_1} {}^\frown \dots {}^\frown t'_{1,1} {}^\frown (p'_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_0 {}^\frown\\ &\qquad\quad t'_{0,n_0} {}^\frown \dots {}^\frown t'_{0,1} {}^\frown (p'_{0\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle})_0 \parallel \sigma.\end{aligned}$$

We use the above formula to fix $S^1$. Then for each $\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle \in S^1$ we fix $t'_1(\bar\nu_{1,1})$, …, $t'_{1,n_1}(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1})$, $S^0(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1})$. In the same way we fix $t'_{0,1}$, …, $t'_{0,n_0}$ for each $\langle\bar\nu_{0,1},\dots,\bar\nu_{0,n_0}\rangle \in S^0(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1})$.

By 8.1 for each $\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle \in S^1$ there is $p'_0(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}) \leq^{**} p'_0$ such that

$$t'_{1,n_1} {}^\frown \dots {}^\frown t'_{1,1} {}^\frown (p'_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_0 {}^\frown p'_0(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}) \parallel \sigma.$$

We choose $p^*_0 \leq^{**} p'_0(\bar\nu_{1,1},\dots,\bar\nu_{1,n_1})$ for all $\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle \in S^1$. Hence we get

$$\forall\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle \in S^1\ t'_{1,n_1} {}^\frown \dots {}^\frown t'_{1,1} {}^\frown (p'_{1\langle\bar\nu_{1,1},\dots,\bar\nu_{1,n_1}\rangle})_0 {}^\frown p^*_0 \parallel \sigma.$$

Invoking 8.1 again we get $p^*_1 \leq^{**} p'_1$ such that $p^*_1 {}^\frown p^*_0 \parallel \sigma$. ☐

With Prikry condition at our hand and $\langle P_{\bar E}/P_{\bar\epsilon}, \leq^*\rangle$ being $\kappa^0(\bar\epsilon)^+$-closed we get

**Theorem 8.3.** *Let $G$ be $P_{\bar E}/P_{\bar\epsilon}$-generic. Then $\mathcal{P}^{V[G]}(\kappa^0(\bar\epsilon)) = \mathcal{P}(\kappa^0(\bar\epsilon))$.*

## 9. Properness

The following definitions (which are *not* used in this work) are due to Saharon Shelah [26].

**Definition.** Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \aleph_0$, $P_{\bar E} \in N$. $p \in P_{\bar E}$ is called $\langle N, P_{\bar E}\rangle$-generic if

$$p \Vdash_{P_{\bar E}} \ulcorner \tilde G \text{ is } \widehat P_{\bar E}\text{-generic over } \widehat N \urcorner.$$

**Definition.** The forcing $P_{\bar E}$ is called proper if given $N \prec H_\chi$, $|N| = \aleph_0$, $P_{\bar E} \in N$, $p \in P_{\bar E} \cap N$ there is $q \leq p$ which is $\langle N, P_{\bar E}\rangle$-generic.

We adapt the above definitions to handle elementary submodels of size $\kappa$. We keep the names from the original definitions.

**Definition 9.1.** Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $P_{\bar{E}} \in N$. $p \in P_{\bar{E}}$ is called $\langle N, P_{\bar{E}}\rangle$-generic if

$$p \Vdash_{P_{\bar{E}}} \ulcorner \tilde{G} \text{ is } \widehat{P}_{\bar{E}}\text{-generic over } \widehat{N}\urcorner.$$

**Definition 9.2.** The forcing $P_{\bar{E}}$ is called proper if given $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $P_{\bar{E}} \in N$, $p \in P_{\bar{E}} \cap N$ there is $q \leq p$ which is $\langle N, P_{\bar{E}}\rangle$-generic.

**Theorem 9.3.** *$P_{\bar{E}}$ is proper.*

*Proof.* Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supseteq N^{<\kappa}$, $P_{\bar{E}} \in N$, $p = p_{l..0} \in P_{\bar{E}} \cap N$. We find $p_0^* \leq^* p_0$ such that $p_{l..1} \frown p_0^*$ is $\langle N, P_{\bar{E}}\rangle$-generic.

Let $\{D_\xi \mid \xi < \kappa\}$ be enumeration of all dense open subsets of $P_{\bar{E}}$ appearing in $N$. Note that for $\xi_0 < \kappa$ we have $\{D_\xi \mid \xi < \xi_0\} \in N$.

Let $\bar{E}_\beta = \mathrm{mc}(p_0)$. Choose $\alpha \in \operatorname{dom} \bar{E}$ such that $\alpha >_{\bar{E}} \gamma$ for all $\gamma \in \operatorname{dom} \bar{E} \cap N$. Let $A' = T^{p_0}$. We shrink $A'$ a bit so that the following is satisfied: $\forall \bar{\nu} \in A'$ $|\{\bar{\mu} \in A \mid \kappa^0(\bar{\mu}) < \kappa^0(\bar{\nu})\}| \leq \kappa^0(\bar{\nu})$. Let $A = \{\langle \bar{\nu}_1, \dots, \bar{\nu}_n\rangle \in (A')^{<\omega} \mid \mathrm{l}(\bar{\nu}_n) = 0,\ \kappa^0(\bar{\nu}_1) < \dots < \kappa^0(\bar{\nu}_n)\}$. Elements of $A$ are written in the form $\vec{\nu}$. That is $\vec{\nu} = \langle \bar{\nu}_1, \dots, \bar{\nu}_n\rangle$. By $\max^0 \vec{\nu}$ we mean $\bar{\nu}_n^0$. Let $\preceq$ be well ordering of $A$ such that $\forall \vec{\nu}, \vec{\mu} \in A$ $\vec{\nu} \preceq \vec{\mu} \implies \max^0 \vec{\nu} \leq \max^0 \vec{\mu}$. We start an induction on $\vec{\nu}$ in which we build

$$\langle \alpha^{\vec{\nu}}, u_0^{\vec{\nu}}, T_0^{\vec{\nu}}, F_0^{\vec{\nu}} \mid \vec{\nu} \in A\rangle,$$

where $(u_0^{\vec{\nu}})_{\langle \pi_{\alpha,\alpha^{\vec{\nu}}}(\vec{\nu})\rangle} \cup \{\langle \bar{E}_{\alpha^{\vec{\nu}}}, F^{p_0}_{\langle \pi_{\alpha,\beta}(\vec{\nu})\rangle}(\pi_{\alpha^{\vec{\nu}},\beta}(-)), T_0^{\vec{\nu}}, F_0^{\vec{\nu}}\rangle\} \in P_{\bar{E}}$. Assume that we have constructed $\langle \alpha^{\vec{\nu}}, u_0^{\vec{\nu}}, F_0^{\vec{\nu}}, T_0^{\vec{\nu}} \mid \vec{\nu} \prec \vec{\nu}_*\rangle$. Recall our convention: $\vec{\nu}_* = \langle \bar{\nu}_{*1}, \dots, \bar{\nu}_{*k}\rangle$. We start working in $N$.

Set the following:

- $\vec{\nu}_*$ is $\prec$-minimal:
$$q' = p_0 \setminus \{\langle \bar{E}_\beta, T^{p_0}, f^{p_0}, F^{p_0}\rangle\},$$
$$\alpha' = \beta.$$
- $\vec{\nu}_*$ is the immediate $\prec$-successor of $\vec{\nu}$:
$$q' = u_0^{\vec{\nu}},$$
$$\alpha' = \alpha^{\vec{\nu}}.$$
- $\vec{\nu}_*$ is $\prec$-limit: Choose $\alpha' \in N$ such that $\forall \vec{\nu} \prec \vec{\nu}_*$ $\alpha' >_{\bar{E}} \alpha^{\vec{\nu}}$ and set
$$q' = \bigcup_{\vec{\nu} \prec \vec{\nu}_*} u_0^{\vec{\nu}}.$$

We begin an induction on $i$ which builds

$$\langle \alpha^{\vec{\nu}_*,i}, u_0^{\vec{\nu}_*,i}, T_0^{\vec{\nu}_*,i}, f_0^{\vec{\nu}_*,i}, F_0^{\vec{\nu}_*,i} \mid i < \kappa\rangle,$$

where $(u_0^{\vec{\nu}_*,i})_{\langle \pi_{\alpha,\alpha^{\vec{\nu}_*,i}}(\vec{\nu}_*)\rangle} \cup \{\langle \bar{E}_{\alpha^{\vec{\nu}_*,i}}, f_0^{\vec{\nu}_*,i}, T_0^{\vec{\nu}_*,i}, F_0^{\vec{\nu}_*,i}\rangle\} \in P_{\bar{E}}$, and

$$\langle j_{\bar{E}}(f_0^{\vec{\nu}_*,i})(\alpha^{\vec{\nu}_*,i}) \mid i < \kappa\rangle$$

is a maximal anti-chain in $j_{\bar{E}}(R)(\kappa^0(\vec{\nu}_*), \kappa)$ below $j_{\bar{E}}(F^{p_0})_{\langle \pi_{\alpha,\beta}(\kappa(\vec{\nu}_*))\rangle}(\beta)$.

Assume we have constructed

$$\langle \alpha^{\vec{\nu}_*,i}, u_0^{\vec{\nu}_*,i}, T_0^{\vec{\nu}_*,i}, f_0^{\vec{\nu}_*,i}, F_0^{\vec{\nu}_*,i} \mid i < i_*\rangle,$$

and we do step $i_*$.

- $i_* = 0$:
$$q'' = q',$$
$$\alpha'' = \alpha',$$
$$f'' = F^{p_0}(\kappa(\pi_{\alpha,\beta}(\bar\nu_{*k})), \pi_{\alpha'',\beta}(-)).$$

- $i_* = i+1$: If $\langle j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}) \mid i < i_*\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\kappa(\pi_{\alpha,\beta}(\bar\nu_{*k})), \beta)$ we terminate the induction on $i$.

  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_*$, $i_* < \mu^+$, and $\langle j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}) \wedge \mu \mid i < i_*\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that
$$j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu,$$
$$\forall i < i_*\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}).$$
  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$.

  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_*\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i})$, and for each inaccessible $\mu < i_*$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$.

  We set
$$q'' = u_0^{\vec\nu_*,i},$$
$$\alpha'' \geq_{\bar E} \alpha^{\vec\nu_*,i}, \beta''.$$

- $i_*$ is limit: If $\langle j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}) \mid i < i_*\rangle$ is a maximal anti-chain below $j_{\bar E}(F^{p_0})(\kappa(\pi_{\alpha,\beta}(\bar\nu_{*k})), \beta)$ we terminate the induction on $i$.

  Otherwise we work as follows: If there is $\mu$ which is maximal inaccessible $< i_*$, $i_* < \mu^+$, and $\langle j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}) \wedge \mu \mid i < i_*\rangle$ is not pre-dense in $j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$ then we choose $f''$, $\beta''$ such that
$$j_{\bar E}(f'')(\beta'') \in j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu,$$
$$\forall i < i_*\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i}).$$
  *Then* we enlarge $f''$ slightly so as to ensure $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$

  If one of the above conditions is not met we just choose $f'', \beta''$ such that $\forall i < i_*\ j_{\bar E}(f'')(\beta'') \perp j_{\bar E}(f_0^{\vec\nu_*,i})(\alpha^{\vec\nu_*,i})$, and for each inaccessible $\mu < i_*$ $j_{\bar E}(f'')(\beta'') \notin j_{\bar E}(R)(\kappa^0(\vec\nu_{*k}), \kappa) \wedge \mu$.

  Choose $\alpha'' \in N$ such that $\forall i < i_*\ \alpha'' >_{\bar E} \alpha^{\vec\nu_*,i}, \beta''$ . We set
$$q'' = \bigcup_{i<i_*} u_0^{\vec\nu_*,i}.$$

Set
$$u''_{k+1..0} = (q'' \cup \{\bar E_{\alpha''}, \pi^{-1}_{\alpha'',\beta} T^{p_0}, f'' \circ \pi_{\alpha'',\beta''}, F^{p_0} \circ \pi_{\alpha'',\beta}\})_{\langle \pi_{\alpha,\alpha''}(\vec\nu_*)\rangle}.$$

We construct $q_0''' \leq^* u_0''$ by invoking 7.8 repeatedly for each $\{D_\xi \mid \xi < \max^0 \vec\nu\}$ starting from $p_{l..1} {}^\frown u''$. We write explicitly what we have here: For each $\xi <$

$\max^0 \vec{\nu}$ there is $B^{\vec{\nu}_*, i_*, \xi}$, a maximal anti-chain below $p_{l..1} {}^\frown u''_{k+1..1}$ such that for each $b \in B^{\vec{\nu}_*, i_*, \xi}$ there are $r_0 \geq^* q'''_0$, $S \subseteq (T^{r_0})^n$, an mc$(r_0)$-fat tree such that

$$\forall \langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle \in S \ \exists t'_n {}^\frown \cdots {}^\frown t'_1 \leq^{**} (r_{0\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle})_{n..1} \\ b {}^\frown t'_n {}^\frown \cdots {}^\frown t'_1 {}^\frown (r_{0\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle})_0 \in D,$$

and

$$\{ b {}^\frown t'_n {}^\frown \cdots {}^\frown t'_1 {}^\frown (r_{0\langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle})_0 \mid \langle \bar{\mu}_1, \ldots, \bar{\mu}_n \rangle \in S \}$$

is pre-dense below $b {}^\frown r_0$. We set

$$\begin{aligned} \alpha^{\vec{\nu}_*, i_*} &= \kappa(\mathrm{mc}(q'''_0)), \\ u_0^{\vec{\nu}_*, i_*} &= q'' \cup \{ \langle \bar{E}_\gamma, q_0'''^{\bar{E}_\gamma} \rangle \mid \bar{E}_\gamma \in \operatorname{supp} q'''_0 \setminus \operatorname{supp} q'' \}, \\ T_0^{\vec{\nu}_*, i_*} &= T^{q'''_0}, \\ f_0^{\vec{\nu}_*, i_*} &= f^{q'''_0}, \\ F_0^{\vec{\nu}_*, i_*} &= F^{q'''_0}. \end{aligned}$$

When the induction on $i$ terminates we have

$$\langle \alpha^{\vec{\nu}_*, i}, u_0^{\vec{\nu}_*, i}, T_0^{\vec{\nu}_*, i}, f_0^{\vec{\nu}_*, i}, F_0^{\vec{\nu}_*, i} \mid i < \kappa \rangle.$$

We point out that $\langle j_{\bar{E}}(f_0^{\vec{\nu}_*, i})(\alpha^{\vec{\nu}_*, i}) \mid i < \kappa \rangle$ is a maximal anti-chain: Assume $f \in j_{\bar{E}}(R)(\kappa^0(\vec{\nu}_{*k}), \kappa)$, $f \leq j_{\bar{E}}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\vec{\nu}_{*k})), \beta)$. Then there is an inaccessible $\mu < \kappa$ such that $f \in j_{\bar{E}}(R)(\kappa^0(\vec{\nu}_{*k}), \kappa) \wedge \mu$. By the construction $\langle j_{\bar{E}}(f_0^{\vec{\nu}_*, i})(\alpha^{\vec{\nu}_*, i}) \wedge \mu \mid i < \mu^+ \rangle$ is pre-dense in $j_{\bar{E}}(R)(\kappa^0(\vec{\nu}_{*k}), \kappa) \wedge \mu$ below $j_{\bar{E}}(F^{p_0})(\pi_{\alpha,\beta}(\kappa(\vec{\nu}_{*k})), \beta) \wedge \mu$. Hence there is $i < \mu^+$ such that $f \parallel j_{\bar{E}}(f_0^{\vec{\nu}_*, i})(\alpha^{\vec{\nu}_{*k}, i}) \wedge \mu$. Hence $f \parallel j_{\bar{E}}(f_0^{\vec{\nu}_*, i})(\alpha^{\vec{\nu}_*, i})$.

We complete step $\vec{\nu}_*$ by setting

$$\begin{aligned} & \alpha^{\vec{\nu}_*} \in N, \\ & \forall i < \kappa \ \alpha^{\vec{\nu}_*} >_{\bar{E}} \alpha^{\vec{\nu}_*, i}, \\ & u_0^{\vec{\nu}_*} = \bigcup_{i<\kappa} u_0^{\vec{\nu}_*, i}, \\ & T_0^{\vec{\nu}_*} = \mathop{\triangle^0}_{i<\kappa} \pi^{-1}_{\alpha^{\vec{\nu}_*}, \alpha^{\vec{\nu}_*, i}} T_0^{\vec{\nu}_*, i}, \\ & \forall i < \kappa \ F_0^{\vec{\nu}_*} \leq F_0^{\vec{\nu}_*, i} \circ \pi_{\alpha^{\vec{\nu}_*}, \alpha^{\vec{\nu}_*, i}}. \end{aligned}$$

When the induction on $\vec{\nu}$ terminates we return to work in $V$ and we have

$$\langle \alpha^{\vec{\nu}}, u_0^{\vec{\nu}}, T_0^{\vec{\nu}}, F_0^{\vec{\nu}} \mid \vec{\nu} \in A \rangle.$$

For each $\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle \in (T^{p_0^*})^{<\omega}$ we define the following function with domain $\{ \bar{\mu} \in A' \mid \mathrm{l}(\bar{\mu}) = 0 \}$

$$g^{\bar{\nu}_1, \ldots, \bar{\nu}_k}(\bar{\mu}) = \langle j_{\bar{E}}(f_0^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i})(\alpha^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i}) \mid i < \kappa \rangle.$$

So $g^{\bar{\nu}_1, \ldots, \bar{\nu}_k}(\bar{\mu})$ is a maximal anti-chain below $j_{\bar{E}}(F^{p_0}_{\langle \kappa(\pi_{\alpha,\beta}(\bar{\nu}_1, \ldots, \bar{\nu}_k)) \rangle})(\kappa(\pi_{\alpha,\beta}(\bar{\mu})), \beta)$.

We note that $\langle j_0(f_0^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i})(\alpha^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i}) \mid i < \kappa \rangle \in M_0$ as $M_0$ is closed under $\kappa$-sequence. As $g^{\bar{\nu}_1, \ldots, \bar{\nu}_k}(\bar{\mu}) = i_{0,\bar{E}}(\langle j_0(f_0^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i})(\alpha^{\langle \bar{\nu}_1, \ldots, \bar{\nu}_k, \bar{\mu} \rangle, i}) \mid i < \kappa \rangle)$ we get $g^{\bar{\nu}_1, \ldots, \bar{\nu}_k}(\bar{\mu}) \in M_{\bar{E}}$. So $j_{\bar{E}}(g^{\bar{\nu}_1, \ldots, \bar{\nu}_k})(\alpha) \in M^2_{\bar{E}}$ is a maximal anti-chain below $j^2_{\bar{E}}(F^{p_0})_{\kappa(\pi_{\alpha,\beta}(\langle \bar{\nu}_1, \ldots, \bar{\nu}_k \rangle))}(\beta, j_{\bar{E}}(\beta))$. As $I(\bar{E})$ is $j^2_{\bar{E}}(R)(\kappa, j_{\bar{E}}(\kappa))$-generic over $M^2_{\bar{E}}$, there is $f^{\bar{\nu}_1, \ldots, \bar{\nu}_k}$ such that $j^2_{\bar{E}}(f^{\bar{\nu}_1, \ldots, \bar{\nu}_k})(\alpha, j_{\bar{E}}(\alpha)) \in I(\bar{E})$ and in $j_{\bar{E}}(g^{\bar{\nu}_1, \ldots, \bar{\nu}_k})(\alpha)$

there is a condition weaker than $j^2_{\bar{E}}(f^{\bar{\nu}_1,\dots,\bar{\nu}_k})(\alpha, j_{\bar{E}}(\alpha))$. Note that we can use $\alpha$ here because the generic was built through the normal measure. If we would not have had this property we would have enlarged $\alpha$ to accommodate the intersection. We combine everything into one condition, $p^*_0$, as follows:

$$p^*_0 = \bigcup_{\vec{\nu}\in A} u^{\vec{\nu}}_0,$$
$$T^{p^*_0} = \mathop{\triangle^0}_{\vec{\nu}\in A} \pi^{-1}_{\alpha,\alpha^{\vec{\nu}}} T^{\vec{\nu}}_0,$$
$$f^{p^*_0}(\nu_1) = f^{p_0} \circ \pi_{\alpha,\beta}(\nu_1),$$
$$\forall \langle \bar{\nu}_1,\dots,\bar{\nu}_k,\bar{\mu}\rangle \in A\ F^{p^*_0} \leq F_0^{\langle \bar{\nu}_1,\dots,\bar{\nu}_k,\bar{\mu}\rangle} \circ \pi_{\alpha,\alpha^{\langle \bar{\nu}_1,\dots,\bar{\nu}_k,\bar{\mu}\rangle}}, f^{\bar{\nu}_1,\dots,\bar{\nu}_k}.$$

We claim that $p_{l..1} \frown p^*_0$ is $\langle N, P_{\bar{E}}\rangle$-generic.

So, let $G$ be $P_{\bar{E}}$-generic with $p_{l..1} \frown p^*_0 \in G$. Let $D \in N$ be a dense open subset of $P_{\bar{E}}$. There is $\xi < \kappa$ such that $D = D_\xi$. Let $\vec{\nu} = \langle \bar{\nu}_1,\dots,\bar{\nu}_k,\bar{\mu}\rangle \in A$ be such that $p_{l..1} \frown p^*_{0\langle\vec{\nu}\rangle} \in G$ and $\kappa^0(\bar{\mu}) > \xi$. For convenience let us set $u_{k+2..0} = p^*_{0\langle\vec{\nu}\rangle}$. By the construction there is $B \in N$, a maximal anti-chain below $p_{l..1} \frown u_{k+2..1}$, such that for each $b \in B$ there are $r_0 \geq^* u_0$, $r_0 \in N$ and $S \in N$, an $\mathrm{mc}(r_0)$-fat tree, such that

$$\forall \langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle \in S\ \exists t'_n \frown \cdots \frown t'_1 \leq^{**} (r_{0\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle})_{n..1}$$
$$b \frown t'_n \frown \cdots \frown t'_1 \frown (r_{0\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle})_0 \in D_\xi$$

and

$$\{b \frown t'_n \frown \cdots \frown t'_1 \frown (r_{0\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle})_0 \mid \langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle \in S\}$$

is pre-dense below $b \frown r_0$. Moreover, this pre-dense set is contained in $N$.

We do the natural factoring $G/p_{l..1} \frown u_{k+2..0} = G_{\bar{\mu}}/p_{l..1} \frown u_{k+2..1} \times G_{\bar{E}}/u_0$. By genericity there is $b \in B \cap G_{\bar{\mu}}/p_{l..1} \frown u_{k+2..1}$. Necessarily $b \frown u_0 \in G$. Hence $b \frown r_0 \in G$. So there is $\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle \in S$ such that $b \frown t'_n \frown \cdots \frown t'_1 \frown (r_{0\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle})_0 \in G$.

So we finally got $b \frown t'_n \frown \cdots \frown t'_1 \frown (r_{0\langle \bar{\mu}_1,\dots,\bar{\mu}_n\rangle})_0 \in D \cap G \cap N$. □

We remind the reader of our convention that when $\tau_1 < \tau_2$ we have $P_{\bar{E}\restriction\tau_2} \subseteq P_{\bar{E}\restriction\tau_1}$. For the sake of completeness we mention the following rather obvious propositions.

**Proposition 9.4.** *Assume that we have $p = p_l \frown \cdots \frown p_0 \in P_{\bar{E}}$, $S \subseteq (T^{p_0})^n$ an $\mathrm{mc}(p_0)$-fat tree and $t'_n(\bar{\nu}_1) \frown \cdots \frown t'_1(\bar{\nu}_1,\dots,\bar{\nu}_n) \leq^{**} (p_{0\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle})_{n..1}$ such that $D = \{p_{l..1} \frown t'_n(\bar{\nu}_1) \frown \cdots \frown t'_1(\bar{\nu}_1,\dots,\bar{\nu}_n) \frown (p_{0\langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle})_0 \mid \langle \bar{\nu}_1,\dots,\bar{\nu}_n\rangle \in S\}$ is pre-dense below $p$. Let $\tau \leq \mathrm{l}(\bar{E})$ be such that $S$ is $\mathrm{mc}(p_0)\restriction\tau$-fat tree, then $D$ is pre-dense in $P_{\bar{E}\restriction\tau}$ below $p$.*

**Proposition 9.5.** *Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $P_{\bar{E}} \in N$, $p = p_{l..0} \in P_{\bar{E}} \cap N$. Assume $S \in N$, $S \subseteq (T^{p_0})^n$ is an $\mathrm{mc}(p_0)$-fat tree. Then $S$ is $mc(p_0)\restriction\tau$-fat tree for each $\sup\{\tau'+1 \mid \tau' \in \mathrm{l}(\bar{E}) \cap N\} \leq \tau \leq \mathrm{l}(\bar{E})$.*

With these propositions in mind we see that the properness proof actually gave us more:

**Theorem 9.6.** *Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $P_{\bar{E}} \in N$, $p = p_{l..0} \in P_{\bar{E}} \cap N$. Then there is $p^*_0 \leq^* p_0$ such that for all $\sup\{\tau'+1 \mid \tau' \in \mathrm{l}(\bar{E}) \cap N\} \leq \tau \leq \mathrm{l}(\bar{E})$*

$$p_{l..1} \frown p^*_0 \Vdash_{P_{\bar{E}\restriction\tau}} \ulcorner \tilde{G} \text{ is } \widehat{P}_{\bar{E}}\text{-generic over } \widehat{N} \urcorner.$$

The following is a method to get a generic over elementary submodel from a 'small' forcing. It is given here, even though it does not belong to this section, as it has the same proof as 9.3.

**Definition 9.7.** Let $s \subseteq \bar{E}$. We define

$$P_{\bar{E}}\restriction s = \{p \in P_{\bar{E}} \mid \operatorname{supp} p_0 \cup \operatorname{mc}(p_0) \subseteq s\}$$

with $\leq$, $\leq^*$ inherited from $P_{\bar{E}}$.

**Definition 9.8.** Let $s \subseteq \bar{E}$. If $G$ is $P_{\bar{E}}$-generic then

$$G\restriction s = G \cap P_{\bar{E}}\restriction s.$$

When $|s| \leq \kappa$ this forcing is a somewhat convoluted Radin forcing. Simple analysis reveals that $\Vdash_{P_{\bar{E}}\restriction s} \ulcorner 2^\kappa = \kappa^+ \urcorner$. We also point out that $P_{\bar{E}}\restriction s$ is completely embedded in $P_{\bar{E}}$. Hence, if $G$ is $P_{\bar{E}}$-generic then $G\restriction s$ is $P_{\bar{E}}\restriction s$-generic.

**Theorem 9.9.** *Let $\chi$ be large enough, $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $P_{\bar{E}} \in N$, $p = p_{l..0} \in N$. Then there is $p^*_0 \leq^* p_0$ such that*

$$p_{l..1} {}^\frown p^*_0 \Vdash_{P_{\bar{E}}\restriction s} \ulcorner \tilde{G} \text{ is } \widehat{P}_{\bar{E}}\text{-generic over } \widehat{N} \urcorner$$

*where $s = \operatorname{supp} p^*_0 \cup \operatorname{mc}(p^*_0)$.*

*Moreover, if $G$ is $P_{\bar{E}}$-generic with $p_{l..1} {}^\frown p^*_0 \in G$ then $H = G \cap P_{\bar{E}}\restriction s$ is $P_{\bar{E}}\restriction s$-generic and, obviously, $V[G] = V[H]$.*

## 10. Cardinals in $V^{P_{\bar{E}}}$

The following claim is just an exercise in properness.

**Claim 10.1.** $\Vdash_{P_{\bar{E}}} \ulcorner \widehat{\kappa^+}$ *is cardinal*$\urcorner$.

*Proof.* If $\mathrm{l}(\bar{E}) = 0$ then the claim is trivial. Hence we assume that $\mathrm{l}(\bar{E}) > 0$. Let $p \Vdash \ulcorner \dot{f}{:}\hat{\kappa} \to \widehat{\kappa^+} \urcorner$. Choose $\chi$ large enough so that $H_\chi$ contains everything we are interested in. By 9.3 there are $p^* \leq^* p$, $N \prec H_\chi$ such that

(1) $p, P_{\bar{E}}, \dot{f} \in N$,
(2) $|N| = \kappa$,
(3) $N \supset \kappa$,
(4) $N \supset N^{<\kappa}$,
(5) $p^*$ is $\langle N, P_{\bar{E}}\rangle$-generic.

Let us set $\lambda = N \cap \kappa^+$. Note that $\lambda < \kappa^+$.

Let $G$ be $P_{\bar{E}}$-generic with $p^* \in G$. The $\langle N, P_{\bar{E}}\rangle$-genericity ensures us that for all $\xi < \kappa$ $\dot{f}(\xi)^{N[G]} \in N$, $\dot{f}(\xi)^{N[G]} = \dot{f}(\xi)^{V[G]}$. Hence $\operatorname{ran} \dot{f}^{V[G]} \subseteq \lambda$. That is $p^* \Vdash \ulcorner \dot{f}$ is bounded in $\kappa^+ \urcorner$. ☐

**Claim 10.2.** *No cardinals $> \kappa$ are collapsed by $P_{\bar{E}}$.*

*Proof.* $\kappa^+$ is not collapsed by 10.1. No cardinals $\geq \kappa^{++}$ are collapsed as $P_{\bar{E}}$ satisfies $\kappa^{++}$-c.c. ☐

**Claim 10.3.** *Assume* $\mathrm{l}(\bar{E}) > 0$. $\Vdash_{P_{\bar{E}}} \ulcorner 2^\kappa = \kappa^{+3} \urcorner$.

*Proof.* Let $G$ be $P_{\bar{E}}$-generic. For each $\alpha \in \operatorname{dom} \bar{E}$ define $M^\alpha = \bigcup\{p_0^{\bar{E}_\alpha} \mid p \in G\}$. It is routine to check that for $\alpha \neq \beta$ we have $M^\alpha \neq M^\beta$. Hence $\Vdash_{P_{\bar{E}}} \ulcorner 2^{\kappa^0(\bar{E})} \geq \kappa^0(\bar{E})^{+3} \urcorner$.

For the other direction let $\Vdash_{P_{\bar{E}}} \ulcorner \dot{A} \subseteq \widehat{\kappa} \urcorner$. By 9.9 there are $\chi$ large enough, $N \prec H_\chi$, $|N| = \kappa$, $P_{\bar{E}}, \dot{A} \in N$ and $p \in G$ such that $G \restriction s$ is $P_{\bar{E}}$-generic over N where $s = \operatorname{supp} p_0 \cup \operatorname{mc}(p_0)$. Hence, $\dot{A}[G] \in V[G\restriction s]$. That is $\mathcal{P}^{V[G]}(\kappa) = \bigcup_{s \in ([\bar{E}]^\kappa)_V} \mathcal{P}^{V[G\restriction s]}(\kappa)$. As $V[G] \vDash \ulcorner |\mathcal{P}^{V[G\restriction s]}(\kappa)| = \kappa^+ \urcorner$ when $|s| \leq \kappa$ and $|([\bar{E}]^\kappa)_V| = \kappa^{+3}$ the proof is completed. $\square$

Simple counting of anti-chains shows that we did not destroy the behavior of the power function on $\kappa^+$, $\kappa^{++}$, $\kappa^{+3}$:

**Claim 10.4.** *Assume* $\mathrm{l}(\bar{E}) > 0$. $\Vdash_{P_{\bar{E}}} \ulcorner 2^{(\kappa^+)} = \kappa^{+4},\ 2^{(\kappa^{++})} = \kappa^{+5},\ 2^{(\kappa^{+3})} = \kappa^{+6} \urcorner$.

**Lemma 10.5.** *Let* $p = p_l {}^\frown \cdots {}^\frown p_k {}^\frown \cdots {}^\frown p_0 \in G$ *and* $\bar{\epsilon}$ *be such that* $p_{l..k} \in P_{\bar{\epsilon}}$. *Let* $G/p = G_{\bar{\epsilon}} \times G_{\bar{E}}$ *be the obvious factoring. Then* $\mathcal{P}(\kappa^0(\bar{\epsilon}))^{V[G]} = \mathcal{P}(\kappa^0(\bar{\epsilon}))^{V[G_{\bar{\epsilon}}]}$.

*Proof.* This is immediate due to $\mathcal{P}(\kappa^0(\bar{\epsilon}))^{V[G]} = \mathcal{P}(\kappa^0(\bar{\epsilon}))^{V[G/p]}$ and 8.3. $\square$

**Claim 10.6.** *Let* $G$ *be* $P_{\bar{E}}$*-generic with* $p = p_l {}^\frown \cdots {}^\frown p_k {}^\frown \cdots {}^\frown p_0 \in G$ *and* $\bar{\epsilon}$ *be such that* $p_{l..k} \in P_{\bar{\epsilon}}$ *and* $\mathrm{l}(\bar{\epsilon}) = 0$. *Let* $\nu = \kappa(p_k^0)$. *Then, in V[G],* $\nu^+, \ldots, \nu^{+6}$ *remain cardinals, all cardinals in* $[\nu^{+7}, \kappa^0(\bar{\epsilon})]$ *are collapsed and* $2^{\nu^+} = \nu^{+4}$, $2^{\nu^{++}} = \nu^{+5}$, $2^{\nu^{+3}} = \nu^{+6}$, $2^{\nu^{+4}} = \kappa^0(\bar{\epsilon})^+$, $2^{\nu^{+5}} = \kappa^0(\bar{\epsilon})^{++}$, $2^{\nu^{+6}} = \kappa^0(\bar{\epsilon})^{+3}$.

*Proof.* Let $G/p = G_{\bar{\epsilon}} \times G_{\bar{E}}$ be the natural factoring. By 10.5 the fate of the cardinals in question is decided by $P_{\bar{\epsilon}}$. We note that $P_{\bar{\epsilon}} = P_{\bar{\epsilon}_2} \times R(\nu, \kappa^0(\bar{\epsilon}))$ where $p_{l..k+1} \in P_{\bar{\epsilon}_2}$. So we factor $G_{\bar{\epsilon}} = G_{\bar{\epsilon}_2} \times G_\nu$. As it stands $R(\nu, \kappa^0(\bar{\epsilon}))$ is $\nu^+$-closed. So in order to prove the claim we make some finer analysis.

We remind the reader that the $V$ we work with is a generic extension of $V^*$ for a reverse Easton forcing. Let $Q_1$ be the reverse Easton forcing up to $\kappa^0(\bar{\epsilon}_2)$ and $H_1$ be its generic. Let $Q_2$ be the forcing at stage $\kappa^0(\bar{\epsilon}_2)$ and $H_2$ be its generic over $V^*[H_1]$. Let $Q_3$ be the rest of the reverse Easton forcing up to $\kappa^0(\bar{\epsilon})$ and $H_3$ be its generic over $V^*[H_1][H_2]$. Then we have

$$\mathcal{P}^{V[G]}(\kappa^0(\bar{\epsilon})) = \mathcal{P}^{V^*[H_1][H_2][H_3][G_\nu][G_{\bar{\epsilon}_2}]}(\kappa^0(\bar{\epsilon}))$$

Comparison of the forcings used to construct $M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa][G^{\bar{E}}_{>\kappa}][I_{\bar{E}}]$ (section 3) and $V^*[H_1][H_2][H_3][G_\nu]$ shows that the cardinal structure and power function of the model $V^*[H_1][H_2][H_3][G_\nu]$ in the range $[\nu^+, \kappa^0(\bar{\epsilon})^{+3}]$ behave in the same way as the cardinal structure and power function of the model $M^*_{\bar{E}}[G_{<\kappa}][G^{\bar{E}}_\kappa][G^{\bar{E}}_{>\kappa}][I_{\bar{E}}] = M_{\bar{E}}[I_{\bar{E}}]$ in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$. From 3.4 we see that in $V^*[H_1][H_2][H_3][G_\nu]$: there are no cardinals in $[\nu^{+7}, \kappa^0(\bar{\epsilon})]$, $\nu^+, \ldots, \nu^{+6}$ are cardinals, $2^{\nu^+} = \nu^{+4}$, $2^{\nu^{++}} = \nu^{+5}$, $2^{\nu^{+3}} = \nu^{+6}$, $2^{\nu^{+4}} = \kappa^0(\bar{\epsilon})^+$, $2^{\nu^{+5}} = \kappa^0(\bar{\epsilon})^{++}$, $2^{\nu^{+6}} = \kappa^0(\bar{\epsilon})^{+3}$.

Forcing with $P_{\bar{\epsilon}_2}$ does not change the power function and does not collapse cardinals above $\kappa^0(\bar{\epsilon}_2)$ by the previous claims adapted to the current context. $\square$

**Claim 10.7.** *Assume* $\mathrm{l}(\bar{E}) > 0$. $\Vdash_{P_{\bar{E}}} \ulcorner \kappa$ *is a cardinal*$\urcorner$.

*Proof.* $\kappa$ is limit ordinal and by 10.6, there are unbounded number of cardinals below $\kappa$ which are preserved. Hence $\kappa$ is preserved. $\square$

With 9.6 at our disposal we can give a direct proof of the following theorem. It is the same one given in [6] for proving the theorem in Radin forcing context.

**Theorem 10.8.** *If* $\operatorname{cf} \mathrm{l}(\bar{E}) > \kappa$ *then* $\Vdash_{P_{\bar{E}}} \ulcorner \widehat{\kappa}$ *is regular*$\urcorner$.

*Proof.* Let $\lambda < \kappa$, $p \in P_{\bar{E}}$. Let $\chi$ be large enough. By 9.6 we have $N \prec H_\chi$, $|N| = \kappa$, $N \supset \kappa$, $N \supset N^{<\kappa}$, $p, P_{\bar{E}} \in N$, $p^* \leq^* p$ such that $p^* \Vdash_{P_{\bar{E}\upharpoonright\tau}} \ulcorner \tilde{G}$ is $\widehat{P}_{\bar{E}}$-generic over $\widehat{N}\urcorner$ for each $\sup\{\tau' + 1 \mid \tau' \in \mathrm{l}(\bar{E}) \cap N\} \leq \tau \leq \mathrm{l}(\bar{E})$.

Choose $\tau$ such that $\sup\{\tau' + 1 \mid \tau' \in \mathrm{l}(\bar{E}) \cap N\} \leq \tau \leq \mathrm{l}(\bar{E})$, $\lambda < \operatorname{cf} \tau \leq \kappa$. This is possible because $\operatorname{cf} \mathrm{l}(\bar{E}) > \kappa$, $|N| = \kappa$. Choose $q \leq_{P_{\bar{E}\upharpoonright\tau}} p^*$ such that $q \Vdash_{P_{\bar{E}\upharpoonright\tau}} \ulcorner \widehat{\lambda} < \operatorname{cf} \widehat{\kappa} < \widehat{\kappa}\urcorner$. Let $G$ be $P_{\bar{E}\upharpoonright\tau}$-generic with $q \in G$. Of course, $p^* \in G$ also. Hence, $G$ is $P_{\bar{E}}$-generic over $N$.

As $N[G] \in V[G]$ we have $\operatorname{cf}^{N[G]} \kappa \geq \operatorname{cf}^{V[G]} \kappa > \lambda$. So there is $r \in P_{\bar{E}} \cap G$, $r \leq p^*$ such that $r \Vdash_{P_{\bar{E}}} \ulcorner \operatorname{cf} \widehat{\kappa} > \widehat{\lambda}\urcorner$. □

## 11. Consistency theorem

We state the consistency theorem we worked so much for.

**Theorem 11.1.** *If there is* $\bar{E}$ *such that* $|\bar{E}| = \kappa^{+3}$, $\operatorname{cf} \mathrm{l}(\bar{E}) > \kappa$ *then it is consistent to have the power function* $2^\mu = \mu^{+3}$ *for all cardinals* $\mu$.

*Proof.* Let $p^* \in P^*_{\bar{E}}$ such that $\kappa(p^{*0})$ is inaccessible and $G$ be $P_{\bar{E}}$-generic with $p^* \in G$. (Forcing below an element of $P^*_{\bar{E}}$ eliminates a finite number of exceptions which we might otherwise have. That is if $p_1 {}^\frown p_0 \in G$ and $\kappa^0(\mathrm{mc}(p_1)) < \kappa(p^0_0)$ then the interval $[\kappa^0(\mathrm{mc}(p_1)), \kappa(p^0_0)]$ is untouched by the forcing). We set

$$M = \bigcup \{p_0^{\bar{E}_\kappa} \mid p \in G\},$$

$$C = \bigcup \{\kappa(p_0^{\bar{E}_\kappa}) \mid p \in G\}.$$

Note that $M$ is a Radin generic sequence for the extender sequence $\bar{E}_\kappa$. Hence $C \subset \kappa$ is a club. The first ordinal in this club is $\lambda = \kappa(p^{*0})$. We investigate the range $(\lambda, \kappa)$ in $V[G]$. We note that, by 10.5, for $\bar{\epsilon} \in M$ it is enough to use $P_{\bar{\epsilon}}$ in order to understand $V^{V[G]}_{\kappa^0(\bar{\epsilon})}$. So let $\mu \in C$, $\mu > \lambda$.

- $\mu \in \lim C$: Then there is $\bar{\epsilon} \in M$ such that $\mathrm{l}(\bar{\epsilon}) > 0$ and $\kappa(\bar{\epsilon}) = \mu$. By 10.7, $\mu$ remains a cardinal and by 10.3, $2^\mu = \mu^{+3}$.
- $\mu \in C \setminus \lim C$: Then there is $\bar{\epsilon} \in M$ such that $\mathrm{l}(\bar{\epsilon}) = 0$ and $\kappa(\bar{\epsilon}) = \mu$. Let $\mu_2 \in C$ be the $C$-immediate predecessor of $\mu$. By 10.6 we have: $\mu_2^+, \dots, \mu_2^{+6}$ are cardinals, there are no cardinals in $[\mu_2^{+7}, \mu]$, $2^{\bar{\mu}_2^+} = \mu_2^{+4}$, $2^{\bar{\mu}_2^{++}} = \mu_2^{+5}$, $2^{\bar{\mu}_2^{+3}} = \mu_2^{+6}$, $2^{\bar{\mu}_2^{+4}} = \mu^+$, $2^{\bar{\mu}_2^{+5}} = \mu^{++}$, $2^{\bar{\mu}_2^{+6}} = \mu^{+3}$.

In fact due to all the cardinals collapsed we have $\{\mu \text{ is a cardinal } \mid \lambda < \mu < \kappa\} = \lim C \cup \{\mu^+, \dots, \mu^{+6} \mid \mu \in C\}$. Hence if $\mu \in (\lambda, \kappa)$ is a cardinal then $2^\mu = \mu^{+3}$. By 10.8, $\kappa$ is an inaccessible cardinal. Let $H$ be $\mathrm{Col}(\aleph_0, \lambda^+)_{V[G]}$-generic over $V[G]$. In $V[G][H]$ $\kappa$ remains inaccessible and $\forall \mu < \kappa\ 2^\mu = \mu^{+3}$. So $V_\kappa^{V[G][H]}$ is a model of ZFC satisfying $\forall \mu\ 2^\mu = \mu^{+3}$. □

## 12. Concluding remarks

12.1. **Regarding The Power Function in Our Model.** Our forcing divides the cardinals into 3 categories. The first category contains the cardinals appearing in the club, $C$, generated by the normal Radin sequence. The second category contains the successors of cardinals in $C$ which are below the length of the extender we use. The third category contains the cardinals above the length of the extender.

The gap on cardinals in each of these categories can be different. We give several examples to clarify this point. In all of them we assume that $\operatorname{cf} \operatorname{l}(\bar{E}) > \kappa$ and $V_\kappa$ of the generic extension is the model of ZFC we are interested in.

Example: By just doing the extender based Radin forcing (that is, without the extra cardinal collapsing and Cohen forcings) starting from $j_{\bar{E}}: V \to M_{\bar{E}} \supset V_{\kappa+n}$ we get that there is a generic extension with a club $C \subset \kappa$ and a power function

$$2^\mu = \begin{cases} \lambda^{+n} & \lambda \in \lim C,\ \mu = \lambda^{+k},\ 0 \leq k < n \\ \mu^+ & \text{otherwise} \end{cases}.$$

Example: By adding to the previous example the collapse $\operatorname{Col}(\lambda_1^{+n+1}, \lambda_2)$ for each $\lambda_1, \lambda_2 \in C$ successive points, we get the same power function. Our gain here is that the cardinals of the new model are 'close' to $C$. Namely the cardinals are $\bigcup\{\lambda^+, \dots, \lambda^{+n+1} \mid \lambda \in C\} \cup \lim C$.

Example: The collapse we chose in the previous example is the lowest possible. We can use others if the need arises. Let $n = 3$. By doing a reverse Easton preparation on the inaccessibles of $\operatorname{C}(\lambda^+, \lambda^{+5}) \times \operatorname{C}(\lambda^{++}, \lambda^{+7}) \times \operatorname{C}(\lambda^{+3}, \lambda^{+10})$ and then invoking $P_{\bar{E}}$ with the collapse $\operatorname{Col}(\lambda_1^{+10}, \lambda_2)$ we get that the cardinals are $\bigcup\{\lambda^+, \dots, \lambda^{+10} \mid \lambda \in C\} \cup \lim C$ with power function

$$2^\mu = \begin{cases} \mu^{+3} & \mu \in \lim C \\ \mu^{+4} & \mu = \lambda^+,\ \lambda \in C \\ \mu^{+5} & \mu = \lambda^{++},\ \lambda \in C \\ \mu^{+7} & \mu = \lambda^{+3},\ \lambda \in C \\ \mu^{+6} & \mu = \lambda^{+4},\ \lambda \in C \\ \mu^{+5} & \mu = \lambda^{+5},\ \lambda \in C \\ \mu^{+4} & \mu = \lambda^{+6},\ \lambda \in C \\ \mu^{+3} & \mu = \lambda^{+7},\ \lambda \in C \\ \mu^{+2} & \mu = \lambda^{+8},\ \lambda \in C \\ \mu^{+} & \text{otherwise} \end{cases}.$$

Example: If we do the reverse Easton forcing as in the previous example and then invoke $P_{\bar{E}}$ with the forcing $\operatorname{Col}(\lambda_1^{+10}, \lambda_2) \times \operatorname{C}(\lambda_1^{+4}, \lambda_2^+) \times \operatorname{C}(\lambda_1^{+6}, \lambda_2^{+5})$ we get the same cardinals and the power function

$$2^\mu = \begin{cases} \mu^{+3} & \mu \in \lim C \\ \mu^{+4} & \mu = \lambda^+,\ \lambda \in C \\ \mu^{+5} & \mu = \lambda^{++},\ \lambda \in C \\ \mu^{+7} & \mu = \lambda^{+3},\ \lambda \in C \\ \mu^{+7} & \mu = \lambda^{+4},\ \lambda \in C \\ \mu^{+6} & \mu = \lambda^{+5},\ \lambda \in C \\ \mu^{+9} & \mu = \lambda^{+6},\ \lambda \in C \\ \mu^{+8} & \mu = \lambda^{+7},\ \lambda \in C \\ \mu^{+7} & \mu = \lambda^{+8},\ \lambda \in C \\ \mu^{+6} & \mu = \lambda^{+9},\ \lambda \in C \\ \mu^{+5} & \mu = \lambda^{+10},\ \lambda \in C \\ \mu^{+} & \text{otherwise} \end{cases}.$$

Note the following limitation. If our reverse Easton preparation would have contained $\mathrm{C}(\lambda^+,\lambda^{+4})$ then the case $\mu=\lambda^+$ in the power function would have been like this

$$2^\mu = \begin{cases} \mu^{+3} & \mu=\lambda^+,\ \lambda\in\lim C \\ \mu^{+4} & \mu=\lambda^+,\ \lambda\in C\setminus\lim C \end{cases}.$$

As can be seen we have quite a lot of freedom in setting the power of the successors in these models. However, we do have a major limitation. We get the same behavior over and over again. This is inherent to our forcing. Another point is that this freedom is on a somewhat insignificant set. It is a non-stationary set. And a very thin non-stationary. It contains no limit cardinals.

A generalization of the second example above is as follows. Assume that we have $f{:}\kappa\to\kappa$ such that $\operatorname{cf}\lambda^{+f(\lambda)}>\lambda$ on a measure 1 set and $j_{\bar E}{:}V\to M_{\bar E}\supset V_{\kappa+j(f)(\kappa)}$. If we force with $P_{\bar E}$ adding the collapse $\operatorname{Col}(\lambda_1^{+f(\lambda_1)+1},\lambda_2)$ for each $\lambda_1,\lambda_2\in C$ successive points then the cardinals in the new model are $\bigcup\{\mu \text{ is cardinal} \mid \lambda^+\le\mu\le\lambda^{+f(\lambda)+1},\ \lambda\in C\}\cup\lim C$ with power function

$$2^\mu = \begin{cases} \lambda^{+f(\lambda)} & \mu\in\lim C \\ \lambda^{+f(\lambda)} & \lambda<\mu<\lambda^{+f(\lambda)},\ \lambda\in\lim C,\ \operatorname{cf}\lambda^{+f(\lambda)}>\mu \\ \lambda^{+f(\lambda)+1} & \lambda<\mu<\lambda^{+f(\lambda)},\ \lambda\in\lim C,\ \operatorname{cf}\lambda^{+f(\lambda)}\le\mu \\ \mu^+ & \text{otherwise} \end{cases}.$$

We can, of course, do a preparation forcing and add Cohen forcings along the normal Radin sequence as before. However, if $\mu\notin C$ is a singular cardinal we have SCH on it. A different method is needed in order to generate a gap on such cardinal.

We suggest the following attack and we stress that it is a *suggestion*. Unlike the previous examples which are immediate consequences of our forcing notion, this attack require a deeper research. So, we assume that $M_{\bar E}$ thinks that there is a cardinal $\nu$ between $\kappa$ and $j_{\bar E}(\kappa)$ carrying a $\nu+3$-strong extender, $F$. Let $\bar F$ be the extender sequence of length 1 built from $F$. Hence we can define $Q_{\bar F}$ in $M_{\bar E}$, the forcing for adding $\nu{+}3$ Prikry sequences to $\nu$. Our idea is to add along $C$ reflections of $Q_{\bar F}$. Hence, if $\lambda_1,\lambda_2\in C$ are successive points then we force with $Q_{\bar\epsilon}$, the forcing notion for adding $\nu'{+}3$ Prikry sequences to $\nu'$ for $\nu'$ lying between $\lambda_1$ and $\lambda_2$. Recall that in order to have a Prikry like condition we need to have a $Q_{\bar F}$-generic filter over $M_{\bar E}$. Alas, we do not have one. However, we do have a $\langle Q_{\bar F},\le^*\rangle$-generic filter over $M_{\bar E}$. (Construct a generic filter over the normal ultrapower and then send it to $M_{\bar E}$). We think that with some modifications our proofs go through using this weaker generic filter.

12.2. **Regarding The Power Function.** Let Reg be the class of regular cardinals. We recall Easton's theorem

**Theorem.** *Assume GCH and let $F{:}Reg\to Card$ be a class function such that*

(1) $\lambda_1<\lambda_2 \implies F(\lambda_1)\le F(\lambda_2)$,
(2) $\operatorname{cf}F(\lambda)>\lambda$,

*then there is a cofinalities preserving generic extension satisfying $\forall\lambda\in Reg\ 2^\lambda=F(\lambda)$ and SCH.*

Nowadays view on the power function is that we look for ZFC theorems. Forcing is used in order to show that some theorem is not possible or to gain intuition on what is possible. Looking on Easton's theorem from this point of view, it says: The only theorems we have regarding the power function on the regular cardinals are monotonicity and König's lemma.

The question which, still, stands before us is how to include the singular cardinals in an Easton like theorem. Even the formulation of such a theorem is troublesome. For example, let $F{:}\omega+1 \to \mathrm{On}$ be defined as: $F(n)=n+1$, $F(\omega)=\omega_5$. Can we have a generic extension in which $\forall\alpha<\omega+1\ 2^{\aleph_\alpha}=\aleph_{F(\alpha)}$? The answer to this question, as posed, is positive. Neither our lack of knowledge in blowing up $2^{\aleph_\omega}$ above $\aleph_{\omega_1}$ nor Shelah's bound $2^{\aleph_\omega}<\aleph_{\omega_4}$ come into play here due to the non-absoluteness of $\aleph_{\omega_5}$. By picking a successor $\alpha<\omega_1$ and starting with $\mathrm{Col}(\alpha,\omega_5)$ we are in a position to invoke Gitik-Magidor forcing realizing the required power function.

Let us rephrase the question. For this we set $\psi(\lambda,\mu)=\ulcorner(\lambda<\aleph_\omega \implies \mu=\lambda^+)\wedge(\lambda=\aleph_{\omega+1} \implies \mu=\aleph_{\omega_5})\urcorner$. This time we ask: Does ZFC + $\ulcorner\forall\lambda,\mu\ \psi(\lambda,\mu) \implies 2^\lambda=\mu\urcorner$ consistent? And the answer is, of course, negative due to Shelah's bound.

So a possible attempt at including the singulars is: For what formulas $\psi(\lambda,\mu)$, satisfying $\forall\lambda\ \exists!\mu\ \psi(\lambda,\mu)$, the theory ZFC + $\ulcorner\forall\lambda,\mu\ \psi(\lambda,\mu) \implies 2^\lambda=\mu\urcorner$ is consistent?

Of course this theory should satisfy the 2 'trivialities':

(1) (Monotonicity) $\forall\lambda_1,\mu_1,\lambda_2,\mu_2\ \lambda_1<\lambda_2\wedge\psi(\lambda_1,\mu_1)\wedge\psi(\lambda_2,\mu_2) \implies \mu_1\le\mu_2$,
(2) (König's lemma) $\forall\lambda,\mu\ \psi(\lambda,\mu) \implies \mathrm{cf}\,\mu>\lambda$.

Let us assume the theory satisfies:

(1) (Galvin-Hajnal) If $\psi(\lambda,\mu)$ and $\lambda=\aleph_\eta$ is a singular strong limit of uncountable cofinality then $\mu<\aleph_{\xi^+}$ where $\psi(|\eta|,\xi)$,
(2) (Silver) $\omega<\mathrm{cf}\,\lambda<\lambda\wedge\{\kappa<\lambda \mid \psi(\kappa,\kappa^+)\}$ is stationary $\implies \psi(\lambda,\lambda^+)$,
(3) (Shelah) $\lambda$ is strong limit $\wedge\ \lambda=\aleph_{\xi+\zeta}\wedge\xi<\aleph_\zeta\wedge\psi(\lambda,\mu) \implies \mu<\aleph_{\xi+|\zeta|^{+4}}$.

Are these restrictions enough to ensure consistency of the theory?

Such a general theorem is beyond our knowledge at this time. Note, this $\psi$ does not preclude infinite gaps and we miss a lot of information for such gaps. Already for the first singular, $\aleph_\omega$, assuming it is strong limit, we are lacking the technology to blow up $2^{\aleph_\omega}$ above $\aleph_{\omega_1}$ while the best known bound is $2^{\aleph_\omega}<\aleph_{\omega_4}$.

So let us restrict ourselves to finite gaps. The forcing presented in this work showed the consistency of the theory ZFC + $\forall\lambda\ 2^\lambda=\lambda^{+n}$. In the previous subsection several generalizations and the principle limitations of it were shown. The main point was the appearance of a club with a fixed gap on it. And the question is: Can we do without such a club?

For example, can the cardinals be partitioned into 2 stationary classes such that on one of them we have gaps of 2 and on the other 3? We do not know the answer (also) to this question.

In fact we do not know if it is possible to realize a similar situation even below $\aleph_{\omega_1}$. That is, can we have stationary subsets of $\omega_1$, $S_1$ and $S_2$, such that $S_1\cup S_2=\omega_1$ and $\alpha\in S_1 \implies 2^{\aleph_\alpha}=\aleph_{\alpha+2}$, $\alpha\in S_2 \implies 2^{\aleph_\alpha}=\aleph_{\alpha+3}$? Note, by Silver's theorem, we must have $2^{\aleph_{\omega_1}}\le\aleph_{\omega_1+2}$.

It is interesting to note it looks as if the situation $\alpha\in S_1 \implies 2^{\aleph_\alpha}=\aleph_{\alpha+1}$, $\alpha\in S_2 \implies 2^{\aleph_\alpha}=\aleph_{\alpha+2}$ (in which case $2^{\aleph_{\omega_1}}=\aleph_{\omega_1+1}$) is simpler to attack than the previous situation.

On the other hand, for a simple enough $\psi$ the club appearance is a must due to the following. Let $\psi$ be an *absolute* formula such that $\forall\alpha\ \exists n < \omega\ \psi(\alpha, \alpha+n)$ and assume the theory we work with is ZFC + $\forall\alpha, \beta\ \psi(\alpha, \beta) \implies 2^{\aleph_\alpha} = \aleph_\beta$. (Note the change in $\psi$'s parameters: from cardinals to ordinals). We set $C_n = \{\alpha \mid \psi(\alpha, \alpha+n)\}$. Each $C_n$ is an $L$-class and On $= \bigcup\{C_n \mid n < \omega\}$. Hence there is $n < \omega$ such that $C_n$ contains one of the $L$-indiscernibles hence all of them. So $C_n$ contains a club.

12.3. **Regarding Our Forcing Notion.** We showed here only that $\kappa$ is regular if $\operatorname{cf} \operatorname{l}(\bar{E}) > \kappa$. We have some preliminary work showing that if we have a repeat point, in the sense that $P_{\bar{E}} = P_{\bar{E}\restriction\tau}$, then $\kappa$ remains measurable.

Let $G^*$ be $\langle P_{\bar{E}}, \leq^* \rangle$-generic. We think there is $H$ in $V[G^*]$ which is $j^{\xi}_{\bar{E}}(P_{\bar{E}})$-generic over $M^{\xi}_{\bar{E}}$ for a properly chosen $\xi$. So far, this is the closest we come to getting a generic by iteration.

We think it is of interest to find the connection between $M^{\xi}_{\bar{E}}[H]$ and $\bigcap_{\xi' < \xi} M^{\xi'}_{\bar{E}}$.

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