SOME APPLICATIONS OF SUPERCOMPACT EXTENDER BASED FORCINGS TO HOD.

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Abstract. Supercompact extender based forcings are used to construct models with HOD cardinal structure different from those of V. In particular, a model where all regular uncountable cardinals are measurable in HOD is constructed.

1. Introduction

In [3] the following result was proved:

Theorem. *Suppose $\kappa < \lambda$ are cardinals such that $\mathrm{cf}(\kappa) = \omega$, λ is inaccessible, and κ is a limit of λ-supercompact cardinals. Then there is a forcing poset Q that adds no bounded subsets of κ, and if G is Q-generic then:*
- $\lambda = (\kappa^+)^{V[G]}$.
- *Every cardinal $\geq \lambda$ is preserved in $V[G]$.*
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{HOD_{\{x\}}} < \lambda$.*

The supercompact extender based Prikry forcing, developed by the second author in [8], is applied to reduce largely the initial assumptions of this theorem and to give a simpler proof. Namely, we show the following:

Theorem 1. *Suppose κ is a $<\lambda$-supercompact cardinal*[1]*, and λ is an inaccessible cardinal above κ. Then there is a forcing poset Q that adds no bounded subsets of κ, and if G is Q-generic then:*
- $\lambda = (\kappa^+)^{V[G]}$.
- *Every cardinal $\geq \lambda$ is preserved in $V[G]$.*
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{HOD_{\{x\}}} < \lambda$.*

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[1] A cardinal κ is said to be $<\lambda$-supercompact if there is an elementary embedding $j : V \to M$ such that M is transitive, $\mathrm{crit}\, j = \kappa$, $j(\kappa) \geq \lambda$, and $M \supseteq {}^{<\lambda}M$.

- $\mathrm{cf}^{HOD_{\{x\}}} \kappa = \omega$

Actually, assuming the measurability (or supercompactness) of λ in V, we obtain that $(\kappa^+)^{V[G]}$ is measurable (or supercompact) in $\mathrm{HOD}_{\{x\}}$.

In [2], a model with the property $(\alpha^+)^{\mathrm{HOD}} < \alpha^+$, for every infinite cardinal α was constructed. We extend this result, using the supercompact extender based Magidor forcing of the second author [9], and show the following:

Theorem 2[2]**.** *Assume there is a Mitchell increasing sequence of extenders $\langle E_\xi \mid \xi < \lambda \rangle$ such that λ is measurable, and for each $\xi < \lambda$, $\mathrm{crit}(j_\xi) = \kappa$, $M_\xi \supseteq {}^{<\lambda}M_\xi$, and $M_\xi \supseteq V_{\lambda+2}$, where $j_\xi : V \to \mathrm{Ult}(V, E_\xi) \simeq M_\xi$ is the natural embedding. Then there is a model of ZFC where all regular uncountable cardinals are measurable in HOD.*

The work [1] obtained results similar to our last theorem using iteration of Radin forcing together with Cardinal collapsing.

This may be of some interest due to the following result of H. Woodin [10]:

Theorem (The HOD dichotomy theorem)**.** Suppose *δ is an extendible cardinal. Then exactly one of the following holds:*

(1) For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $\gamma^+ = (\gamma^+)^{HOD}$
(2) Every regular cardinal greater than δ is measurable in HOD.

However, we do not have even inaccessibles in the model of theorem 2. It is possible to modify the construction in order to have measurable cardinals (and bit more) in the model. We do not know how to get supercompacts and it is very unlikely the method used will allow model with supercompacts.

The structure of this work is as follows. In section 2 we give definitions and claims about HOD and homogeneous forcing notions which are well known. In section 3 we prove theorem 1. In section 4 we prove theorem 2.

We assume knowledge of large cardinals and forcing. In particular this work depends on the supercompact extender based Prikry-Magidor-Radin forcing.

[2]This result was presented at the Arctic Set Theory Worshop 2 in Kilpisjärvi, Finland, February 2015.

2. HOD background

Definition 2.1. Let M be a class. The class OD_M contains the sets definable using ordinals and sets from M, i.e., $A \in \mathrm{OD}_M$ iff there is a formula $\varphi(x, x_1, \ldots, x_k, y_1, \ldots, y_m)$, ordinals $\beta, \alpha_1, \ldots, \alpha_k \in \mathrm{On}$, and sets $a_1, \ldots, a_m \in M$, such that $A = \{a \in V_\beta \mid V_\beta \vDash \varphi(a, \alpha_1, \ldots, \alpha_k, a_1, \ldots, a_m)\}$.

The class HOD_M contains the sets which are hereditarily in OD_M, i.e., $A \in \mathrm{HOD}_M$ iff $\mathrm{tc}(\{A\}) \subseteq \mathrm{HOD}_M$.

We write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$, respectively.

Note, if $A \in \mathrm{OD}$ is a set of ordinals then $A \in \mathrm{HOD}$.

We will work in HOD of generic extensions, hence the relation between $V[G]$ and $\mathrm{HOD}^{V[G]}$, where $V[G]$ is a generic extension, will be our main machinery.

Our main tool will be forcing notions which are homogeneous in some sense. A forcing notion P is said to be cone homogeneous if for each pair of conditions $p_0, p_1 \in P$ there is a pair of conditions $p_0^*, p_1^* \in P$ such that $p_0^* \leq p_0$, $p_1^* \leq p_1$, and $P/p_0^* \simeq P/p_1^*$.

A forcing notion P is said to be weakly homogeneous if for each pair of conditions $p_0, p_1 \in P$ there is an automorphism $\pi : P \to P$ so that $\pi(p_0)$ and p_1 are compatible. It is evident a weakly homogeneous forcing notion is cone homogeneous.

An automorphism $\pi : P \to P$ induces an automorphism on P-terms by setting recursively $\pi(\langle \dot\tau, p\rangle) = \langle \pi(\dot\tau), \pi(p)\rangle$.

Note ground model terms are fixed by automorphisms, i.e., $\pi(\check{x}) = \check{x}$, in particular for each ordinal α, $\pi(\check{\alpha}) = \check{\alpha}$.

An essential fact about a cone homogeneous forcing notion P is that for each formula φ, either $\Vdash_P \varphi(\alpha_1, \ldots, \alpha_l)$ or $\Vdash_P \neg\varphi(\alpha_1, \ldots, \alpha_l)$. If in addition the forcing P is ordinal definable then we get $\mathrm{HOD}^{V[G]} \subseteq V$, where G is P-generic.

In [4] it was shown that an arbitrary iteration of weakly (cone) homogeneous forcing notions is weakly (cone) homogeneous under the very mild assumption that the iterand is fixed by automorphisms. For the sake of completeness, we show here a special case of this theorem, which is enough for our purpose.

Theorem 2.2 (Special case of Dobrinen-Friedman [4])**.** *Assume $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ is a backward Easton iteration such that for each $\beta < \kappa$, $\Vdash_{P_\beta}$ "$\dot{Q}_\beta$ is cone homogeneous" and for each $p_0, p_1 \in P_\beta$ and automorphism $\pi : P_\beta/p_0 \to P_\beta/p_1$, we have $\Vdash_{P_\beta/p_0}$ "$\pi^{-1}(\dot{Q}_\beta) = \dot{Q}_\beta$". Then P_κ is cone homogeneous.*

Proof. Fix two conditions $p_0, p_1 \in P_\kappa$. We will construct two conditions $p_0^* \leq p_0$ and $p_1^* \leq p_1$ such that $P_\kappa/p_0^* \simeq P_\kappa/p_1^*$, by which we will be done. The construction is done by induction on $\alpha \leq \kappa$ as follows.

Assume $\alpha = \beta + 1$, $p_0^* \restriction \beta$, $p_1^* \restriction \beta$, and $\pi_\beta : P_\beta/p_0^* \restriction \beta \simeq P_\beta/p_1^* \restriction \beta$ were constructed. We know $\Vdash_{P_\beta/p_0^* \restriction \beta}$ "$\dot{Q}_\beta = \pi_\beta^{-1}(\dot{Q}_\beta)$ is cone homogeneous". Let $\rho_\beta : \dot{Q}_\beta \to \dot{Q}_\beta$ be a function for which $\dot{\tau}[G] = \rho_\beta(\dot{\tau})[\pi''_\beta G]$ holds, whenever $G \subseteq P_\beta$ is generic and $\dot{\tau}[G] \in \dot{Q}[G]$. If both $p_0(\beta)$ and $p_1(\beta)$ are the maximal element of $\dot{Q}_\beta$ then let $p_0^*(\beta)$ and $p_1^*(\beta)$ be the maximal element of $\dot{Q}_\beta$ and let $\sigma_\beta = \mathrm{id}$ be the trivial automorphism of $\dot{Q}_\beta$. If either $p_0(\beta)$ or $p_1(\beta)$ is not the maximal element of $\dot{Q}_\beta$ then use the the cone homogeneity of $\dot{Q}_\beta$ to find P_β-names $p_0^*(\beta)$, $p_1^*(\beta)$, and $\dot{\sigma}_\beta$, such that $p_0^* \restriction \beta \Vdash_{P_\beta}$ "$p_0^*(\beta) \leq p_0(\beta)$", $p_1^* \restriction \beta \Vdash_{P_\beta}$ "$p_1^*(\beta) \leq p_1(\beta)$", and $\dot{\sigma}_\beta : \dot{Q}_\beta/p_0^*(\beta) \simeq \dot{Q}_\beta/\rho_\beta^{-1}(p_1^*(\beta))$ is an automorphism. Whatever way $\dot{\sigma}_\beta$ was constructed define the automorphism $\pi_{\beta+1}$ by letting $\pi_{\beta+1}(s) = \langle \pi_\beta(s \restriction \beta), \rho_\beta(\dot{\sigma}_\beta(s(\beta))) \rangle$, for each $s \leq p_0^* \restriction \beta + 1$.

Assume α is limit and for each $\beta < \alpha$ we have $p_0^* \restriction \beta \leq p_0 \restriction \beta$, $p_1^* \restriction \beta \leq p_1 \restriction \beta$, and $\pi_\beta : P_\beta/p_0^* \restriction \beta \simeq P_\beta/p_1^* \restriction \beta$ is an automorphism such that $\pi_\beta \restriction P_{\beta'} = \pi_{\beta'}$, whenever $\beta' \leq \beta$. For each $s \leq p_0^* \restriction \alpha$ let $\pi_\alpha(s) \in P_\alpha$ be the condition defined by setting for each $\beta < \alpha$, $\pi_\alpha(s)(\beta) = \pi_{\beta+1}(s \restriction \beta + 1)(\beta)$. □

The following claim is practically the successor case of the previous one. It is useful when we will have automorphism of forcing notions which are not necessarily cone homogeneous.

Claim 2.3. *Assume P_0 and P_1 are forcing notions with $\pi_0 : P_0 \to P_1$ being an isomorphism. Let $\dot{Q}_0$ be a P_0-name of a cone homogeneous forcing notion such that $\Vdash_{P_0}$ "$\dot{Q}_0 = \dot{Q}_1$", where $\dot{Q}_1 = \pi_0(\dot{Q}_0)$.*

Then for each pair $1 * \dot{q}_0 \in P_0 * \dot{Q}_0$ and $1 * \dot{q}_1 \in P_1 * \dot{Q}_1$ there are stronger conditions $1 * \dot{q}_0^* \leq 1 * \dot{q}_0$ and $1 * \dot{q}_1^* \leq 1 * \dot{q}_1$ such that $P_0 * \dot{Q}_0/1 * \dot{q}_0^* \simeq P_1 * \dot{Q}_1/1 * \dot{q}_1^*$.

Proof. Note there is a function ρ taking P_0-names to P_1-names such that $\dot{q}_0[G_0] = \rho(\dot{q}_0)[G_1]$, where $G_0 \subseteq P_0$ is generic and $G_1 = \pi''_0 G_0$.

Set $\dot{q}'_1 = \rho^{-1}(\dot{q}_1)$. By the cone homogeneity of $\dot{Q}_0$ in V^{P_0} there are stronger conditions $\dot{q}_0^* \leq \dot{q}_0$ and $\dot{q}_1'^* \leq \dot{q}'_1$, for which there is (a name of) an automorphism $\pi_1 : \dot{Q}_0/\dot{q}_0^* \to \dot{Q}_0/\dot{q}_1'^*$. Set $\dot{q}_1^* = \rho(\dot{q})_1'^*$. Since for generics G_0, G_1 as above we have $\dot{Q}_0/\dot{q}_1'^*[G_0] = \dot{Q}_1/\dot{q}_1^*[G_1]$ we get $\pi(p * \dot{q}) = \pi_0(p) * (\rho \circ \pi_1(\dot{q}))$ is the required automorphism. □

While the forcing notions we will use are cone homogeneous we will deliberately break down some of their homogeneity. The relation between $\mathrm{HOD}^{V[G]}$ and V will be as follows.

Claim 2.4. *Assume P is an ordinal definable cone homogeneous forcing notion. Let $\pi : P \to Q$ be a projection. Assume that for each condition $p \in P$, ordinals $\alpha_1, \dots, \alpha_l \in On$, and formula φ, if $p \Vdash_P \varphi(\alpha_1, \dots, \alpha_l)$ then $\pi(p) \Vdash_Q \varphi(\alpha_1, \dots, \alpha_l)$. Then $HOD^{V[G]} \subseteq V[\pi(G)]$, where $\pi(G)$ is the upward closure of $\pi''G$.*

Proof. Assume $\Vdash_P$ "$\dot{A} \subseteq \mathrm{On}$ and $\dot{A} \in \mathrm{HOD}$". Let $G \subseteq P$ be generic. Then in $V[G]$ there are ordinals $\alpha_1, \dots, \alpha_l, \beta$ such that for each $\alpha \in \mathrm{On}$,

$$\alpha \in \dot{A}[G] \iff V_\beta \vDash \varphi(\alpha, \alpha_1, \dots, \alpha_l).$$

Let $X_0^\alpha \cup X_1^\alpha \subseteq P$ be a maximal antichain such that for each $p \in X_0^\alpha$,

$$p \Vdash \text{“}V_\beta \vDash \neg\varphi(\alpha, \alpha_1, \dots, \alpha_l)\text{”},$$

and for each $p \in X_1^\alpha$,

$$p \Vdash \text{“}V_\beta \vDash \varphi(\alpha, \alpha_1, \dots, \alpha_l)\text{”}.$$

Let $\dot{A}'$ be a Q-name defined by setting for each $p \in X_0^\alpha \cup X_1^\alpha$.

$$\pi(p) \Vdash_Q \text{“}\alpha \in \dot{A}'\text{”} \iff p \Vdash_P \text{“}\alpha \in \dot{A}\text{”}.$$

Since $\pi''(X_0^\alpha \cup X_1^\alpha)$ is predense in Q we get $\dot{A}'[\pi(G)] = \dot{A}[G]$, by which we are done. ☐

Let $C(\tau, \mu)$ be the Cohen forcing for adding μ subsets to τ, i.e., $C(\tau, \mu) = \{f : a \to 2 \mid a \subseteq \mu,\ |a| < \tau\}$. The following is well known.

Claim 2.5. $\mathrm{C}(\tau, \mu)$ *is cone homogeneous.*

Proof. Assume $f, g \in \mathbb{C}(\tau, \mu)$ are conditions. Choose stronger conditions, $f^* \leq f$ and $g^* \leq g$, such that $\operatorname{dom} f^* = \operatorname{dom} g^* = \operatorname{dom} f \cup \operatorname{dom} g$. Define $\pi : \mathbb{C}(\tau, \mu)/f^* \to \mathbb{C}(\tau, \mu)/g^*$ by setting $\pi(f') = g^* \cup (f' \setminus f^*)$ for each $f' \leq f^*$. It is obvious π is an automorphism. ☐

The following is immediate from the previous claim and theorem 2.2.

Claim 2.6. *The Easton product of Cohen forcing notions is cone homogeneous.*

3. The cofinality ω case

Let us switch to the cone-homogeneity of the Extender Based Prikry forcing. Extender based Prikry forcing was originally developed in [5]. We use a revision of the forcing where the extender can witness supercompactness. This was first developed in [8]. At the suggestion[3] of the referee we add intuitive explanation of the extender based Prikry

[3]The suggestion was for a *short* intuitive explanation, really.

forcing. We will do so by introducing definitions going gradually from the Prikry forcing to the extender based Prikry forcing.

We begin with a definition of Prikry forcing which is more cumbersome than the standard definition. When generalizing to the extender base Prikry forcing this cumbersome definition becomes simpler than the standard definition of the extender based forcing.

Assume $j : V \to M$ is an elementary embedding such that $\mathrm{crit}(j) = \kappa$, $M \supseteq {}^{\kappa}M$, and κ is its sole generator. Define the measure U by letting for each $A \subseteq \kappa$,

$$A \in U \iff \kappa \in j(A).$$

Recall that a condition in Prikry forcing is of the form $\langle t, A\rangle$, where $t \in {}^{<\omega}\kappa$ is a finite increasing sequence and $A \in U$. We can always assume that for each ordinal $\nu \in A$, $\nu > \max t$. The condition $\langle t, B\rangle$ is said to be a direct extension of the condition $\langle t, A\rangle$ ($\langle t, B\rangle \leq^* \langle t, A\rangle$) in this forcing if $B \subseteq A$. Let $\nu \in A$ be an ordinal. Then $\langle t, A\rangle_{\langle\nu\rangle} = \langle t^\frown\langle\nu\rangle, A\setminus(\nu+1)\rangle$. We say the condition $\langle t, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle t, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle t, A\rangle$ to be $(\langle t, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle s, B\rangle$ is stronger than the condition $\langle t, A\rangle$ if there are $\langle\nu_0,\ldots,\nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle s, B\rangle \leq^* \langle t, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}$. This is clearly a valid definition of Prikry forcing. If G is the generic object then letting $t_G = \bigcup\{t \mid \langle t, A\rangle \in G\}$ we get that t_G is an ω-sequence cofinal in κ.

Let us define again Prikry forcing, increasing the level of cumbersomeness. Starting from the same assumption as above proceed as follows. Define the measure U by letting for each $A \subseteq {}^{\{\kappa\}}\kappa$,

$$A \in U \iff \{\langle j(\kappa), \kappa\rangle\} \in j(A).$$

Note a typical function $\nu \in A$ is of the form $\nu : \{\kappa\} \to \kappa$. Define now a condition in Prikry forcing to be of the form $\langle f, A\rangle$, where $f : \{\kappa\} \to {}^{<\omega}\kappa$ is a function such that $f(\nu)$ is a finite increasing sequence, and $A \in U$. Note we can assume for each $\nu \in A$, $\nu(\kappa) > \max f(\kappa)$. The condition $\langle f, B\rangle$ is said to be a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing is if $B \subseteq A$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. Define the function $f_{\langle\nu\rangle}$ by letting $f_{\langle\nu\rangle}(\kappa) = f(\kappa)^\frown\langle\nu(\kappa)\rangle$. Define the set of functions in A which are above ν as $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\kappa)\}$. A 1-point extension $\langle f, A\rangle_{\langle\nu\rangle}$ of $\langle f, A\rangle$ is defined to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ if there is $\langle\nu_0,\ldots,\nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}$. This is clearly a valid, if somewhat bizarre, definition of Prikry forcing. If we let G be the generic object and define the function $f_G : \{\kappa\} \to {}^{\omega}\kappa$ by setting $f_G(\kappa) = \bigcup\{f(\kappa) \mid \langle f, A\rangle \in G\}$, then $f_G(\kappa)$ is an ω-sequence cofinal in κ.

Staying in the previous context, let us choose an ordinal $\alpha < j(\kappa)$. There is a function F in the ground model such that $j(F)(\kappa) = \alpha$. Then $F'' f_G(\kappa)$ is a sequence Prikry generic for the measure W generated by α. We can, however, generalize somewhat the forcing so as to be able to read directly the generic sequence corresponding to α from the generic object. For this define the measure U by letting for each $A \subseteq {}^{\{\kappa,\alpha\}}\kappa$,

$$A \in U \iff \{\langle j(\kappa), \kappa\rangle, \langle j(\alpha), \alpha\rangle\} \in j(A).$$

Note a typical function $\nu \in A$ is of the form $\nu : \{\kappa, \alpha\} \to \kappa$ such that $\nu(\kappa) < \nu(\alpha)$. Define now a condition in Prikry forcing to be of the form $\langle f, A\rangle$, where $f : \{\kappa, \alpha\} \to {}^{<\omega}\kappa$ is a function such that both $f(\kappa)$ and $f(\alpha)$ are finite increasing sequences, and $A \in U$. Note we can assume for each $\nu \in A$, $\max f(\kappa) < \nu(\kappa)$ and $\max f(\alpha) < \nu(\alpha)$. The condition $\langle f, B\rangle$ is a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing if $B \subseteq A$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. Define the function $f_{\langle\nu\rangle}$ by setting

$$f_{\langle\nu\rangle}(\kappa) = f(\kappa) ^\frown \langle\nu(\kappa)\rangle$$

and

$$f_{\langle\nu\rangle}(\alpha) = f(\alpha) ^\frown \langle\nu(\alpha)\rangle.$$

Set $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\alpha)\}$. Then define the condition $\langle f, A\rangle_{\langle\nu\rangle}$ to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. We say the condition $\langle f, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle f, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ ($\langle g, B\rangle \leq \langle f, A\rangle$) if there is $\langle\nu_0, \ldots, \nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}$. The above is clearly a valid, if strange, definition of Prikry forcing. Letting G be the generic object we define the function $f_G : \{\kappa, \alpha\} \to {}^{\omega}\kappa$ by setting

$$f_G(\kappa) = \bigcup\{f(\kappa) \mid \langle f, A\rangle \in G\}$$

and

$$f_G(\alpha) = \bigcup\{f(\alpha) \mid \langle f, A\rangle \in G\}.$$

Then both $f_G(\kappa)$ and $f_G(\alpha)$ are ω-sequences cofinal in κ.

Of course, seeing the above one can immediately generalize to any less than κ ordinals below $j(\kappa)$, which still leaves us in the realm of

Prikry forcing. Thus fix $d \in {}^{<\kappa}j(\kappa)$. It is technically useful to assume $\kappa \in d$. Define the measure U by letting for each $A \subseteq {}^{d}\kappa$,

$$A \in U \iff \{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\} \in j(A).$$

A typical function $\nu \in A$ is of the form $\nu : d \to \kappa$ and for each $\alpha, \beta \in d$ such that $\alpha < \beta$ we have $\nu(\alpha) < \nu(\beta)$. While it is not terribly important now, it should be observed that we also have $|\nu| < \nu(\kappa)$. Define now a condition in Prikry forcing to be of the form $\langle f, A\rangle$, where $f : d \to {}^{<\omega}\kappa$ is a function such that for each $\alpha \in d$ we have $f(\alpha)$ is increasing, and $A \in U$. Note if $\nu \in A$ we can assume for that each $\alpha \in d$ we have $\max f(\alpha) < \nu(\alpha)$. The condition $\langle f, B\rangle$ is said to be a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing if $B \subseteq A$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$. The function $f_{\langle\nu\rangle}$ is the function defined by setting for each $\alpha \in d$,

$$f_{\langle\nu\rangle}(\alpha) = f(\alpha) {}^\frown \langle\nu(\alpha)\rangle.$$

Set $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\alpha) \text{ for each } \alpha \in d\}$. Then define the condition $\langle f, A\rangle_{\langle\nu\rangle}$ to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. We say the condition $\langle f, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle f, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ ($\langle g, B\rangle \leq \langle f, A\rangle$) if there is $\langle\nu_0, \dots, \nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle}$. The above is still a valid definition of Prikry forcing. Letting G be the generic object and defining the function $f_G : d \to {}^{\omega}\kappa$ by setting for each $\alpha \in d$, $f_G(\alpha) = \bigcup\{f(\alpha) \mid \langle f, A\rangle \in G\}$, we get that for each $\alpha \in d$, $f_G(\alpha)$ is an ω-sequence cofinal in κ. Note the function f_G defined here is found in the generic extension of the standard Prikry forcing.

In fact, one can use in the previous definition also sets d of size κ. The usefulness of the ultrafilter U is more apparent in this case, when one views a typical function in a measure one set. Thus fix $d \in {}^{\kappa}j(\kappa)$. It is technically useful to assume $\kappa \in d$. Define the measure U by letting for each $A \subseteq \bigcup\{{}^{d'}\kappa \mid d' \subseteq d,\ |d'| < \kappa\}$,

$$A \in U \iff \{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\} \in j(A).$$

A typical function $\nu \in A$ is of the form $\nu : \operatorname{dom}\nu \to \kappa$ (note $\mathbf{dom}\,\boldsymbol{\nu}$ and **not** $\boldsymbol{d}$!), where $\operatorname{dom}\nu \subseteq d$, and for each $\alpha, \beta \in \operatorname{dom}\nu$ such that $\alpha < \beta$ we have $\nu(\alpha) < \nu(\beta)$. Moreover, $|\operatorname{dom}\nu| < \nu(\kappa)$. Define now a condition in Prikry forcing to be of the form $\langle f, A\rangle$, where $f : d \to {}^{<\omega}\kappa$ is a function such that for each $\alpha \in d$ we have $f(\alpha)$ is a finite increasing sequence, and $A \in U$. We can always assume that for each $\nu \in A$ we have $\max f(\alpha) < \nu(\alpha)$ for each $\alpha \in \operatorname{dom}\nu$ The condition $\langle f, B\rangle$ is said to be a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing if $B \subseteq A$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. Define the function $f_{\langle\nu\rangle}$ by setting for each $\alpha \in d$,

$$f_{\langle\nu\rangle}(\alpha) = \begin{cases} f(\alpha) \frown \langle \nu(\alpha)\rangle & \alpha \in \operatorname{dom}\nu, \\ f(\alpha) & \alpha \notin \operatorname{dom}\nu. \end{cases}$$

Set $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\alpha) \text{ for each } \alpha \in \operatorname{dom}\nu\}$. Then define the condition $\langle f, A\rangle_{\langle\nu\rangle}$ to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. We say the condition $\langle f, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle f, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ if there is $\langle\nu_0,\ldots,\nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}$. The above is still a valid definition of Prikry forcing. Letting G be the generic object and defining the function $f_G : d \to {}^{\omega}\kappa$ by setting for each $\alpha \in d$, $f_G(\alpha) = \bigcup\{f(\alpha) \mid \langle f, A\rangle \in G,\ \alpha \in \operatorname{dom} f\}$, we get that for each $\alpha \in d$, $f_G(\alpha)$ is an ω-sequence cofinal in κ. Defining the function f_G as in the previous paragraph we get κ cofinal ω-sequences, but still all of them are generated by $f_G(\kappa)$.

Since the ultrapower M is closed only under κ-sequences, one cannot enlarge d to be of size greater than κ while keeping the nice properties of Prikry type forcing notions. However, we can use conditions with different domains, thus adding sequence corresponding to each of the ordinals below $j(\kappa)$. The domain change, however, causes the forcing to be non-isomorphic to Prikry forcing. Thus if $d \in {}^{\kappa}j(\kappa)$ then define the measure U_d by letting for each $A \subseteq \bigcup\{{}^{d'}\kappa \mid d' \subseteq d,\ |d'| < \kappa\}$,

$$A \in U_d \iff \{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\} \in j(A).$$

Define now a condition in the forcing to be of the form $\langle f, A\rangle$, where $f : \operatorname{dom} f \to {}^{<\omega}\kappa$ is a function such that for each $\alpha \in \operatorname{dom} f$, $f(\alpha)$ is a finite increasing sequence, $\operatorname{dom} f \in {}^{\kappa}j(\kappa)$, and $A \in U_{\operatorname{dom} f}$. If $d \subseteq e$ and $A \in U_e$ then set $A\restriction d = \{\nu\restriction d \mid \nu \in A\}$. The condition $\langle g, B\rangle$ is said to be a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing if $g \supseteq f$ and $B\restriction \operatorname{dom} f \subseteq A$. **Note this definition of the direct order is a major change from all previous definitions. In fact the direct order is a Cohen forcing for adding $\boldsymbol{j(\kappa)}$ subsets to $\boldsymbol{\kappa^+}$.** Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. The function $f_{\langle\nu\rangle}$ is defined by setting for each $\alpha \in \operatorname{dom} f$,

$$f_{\langle\nu\rangle}(\alpha) = \begin{cases} f(\alpha) \frown \langle \nu(\alpha)\rangle & \alpha \in \operatorname{dom}\nu, \\ f(\alpha) & \alpha \notin \operatorname{dom}\nu. \end{cases}$$

Given a set $A \in U_d$ set $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\alpha) \text{ for each } \alpha \in \operatorname{dom}\nu\}$. Then define the condition $\langle f, A\rangle_{\langle\nu\rangle}$ to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. We say the condition $\langle f, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle f, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ ($\langle g, B\rangle \leq \langle f, A\rangle$) if there is $\langle\nu_0,\dots,\nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle}$. **This forcing notion is no longer Prikry forcing.** Using the same definition of f_G as before we get $j(\kappa)$ cofinal ω-sequence into κ. However, while each of the sequences appears in a generic extension by Prikry forcing, the function f_G itself does not belong to a Prikry generic extension.

The point of the previous forcing is that nothing restricts us from using it with elementary embeddings with many generators, thus we get the extender based Prikry forcing. Thus assume $j : V \to M$ is an elementary embedding such that $\operatorname{crit}(j) = \kappa$, $M \supseteq {}^{\kappa}M$, and $\kappa < \lambda < j(\kappa)$ is a cardinal in V. Using the previous definition with conditions $\langle f, A\rangle$ such that $\operatorname{dom} f \in {}^{\kappa}\lambda$, we get that λ-many new ω-sequences cofinal in κ appear in the generic extension, thus $2^\kappa = \kappa^\omega \geq \lambda$. Working out the proof we get that no cardinals are collapsed, $2^\kappa = \lambda$, and $\operatorname{cf}\kappa = \omega$.

The final generalization achieved so far, along the lines above, is to begin with elementary embedding with even more closure properties, i.e., $j : V \to M$ such that $\operatorname{crit}(j) = \kappa$ and $M \supseteq {}^{<\lambda}M$, where $\lambda > \kappa$ and then work out the definition above to use d of arbitrary size below λ. This yields a generic extension in which $\operatorname{cf}\kappa = \omega$ and 2^κ blows up to whatever cardinal the model M catches beginning with λ up to $j(\kappa)$. However, the cardinals above κ and below λ are collapsed in this extension., which is to be expected when using a $< \lambda$-supercompact cardinal.

Let E be an extender as in [8]. Let $\mathbb{P}_E$ be the extender based Prikry forcing derived from E. We show $\mathbb{P}_E$ is cone homogeneous.

Claim 3.1. *For each pair of conditions $p_0, p_1 \in \mathbb{P}_E$ there are direct extensions $p_0^* \leq^* p_0$ and $p_1^* \leq^* p_1$ such that $\mathbb{P}_E/p_0^* \simeq \mathbb{P}_E/p_1^*$.*

Proof. Set $d = \operatorname{dom} f^{p_0} \cup \operatorname{dom} f^{p_1}$. Set $f_0^* = f^{p_0} \cup \{\langle\alpha, \langle\rangle\rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_0}\}$ and $f_1^* = f^{p_1} \cup \{\langle\alpha, \langle\rangle\rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_1}\}$. Choose a set $A \subseteq \pi^{-1}_{d,\operatorname{dom} f^{p_0}}(A^{p_0}) \cap \pi^{-1}_{d,\operatorname{dom} f^{p_1}}(A^{p_1})$ so that both $p_0^* = \langle f_0^*, A\rangle$ and $p_1^* = \langle f_1^*, A\rangle$ are conditions. Define $\pi : \mathbb{P}_E/p_0^* \to \mathbb{P}_E/p_1^*$ by setting for each $p \leq p_0^*$, $\pi(p) = \langle f^{p_1^*}_{\langle\nu_0,\dots,\nu_{n-1}\rangle} \cup (f^p \restriction (\operatorname{dom} f^p \setminus d)), A^p\rangle$, where $\langle\nu_0,\dots,\nu_{n-1}\rangle \in {}^{<\omega}A^{p_0^*}$ and $p \leq^* p^*_{0\langle\nu_0,\dots,\nu_{n-1}\rangle}$. We claim π is an isomorphism. Note the condition p and $\pi(p)$ are mostly identical. The condition $\pi(p)$ can differ from p when $\alpha \in d$ and $f^{\pi(p)}(\alpha)$ differs from

$f^p(\alpha)$. Thus the one point which is not trivial is that π is order preserving.

Assume $p \le q \le p^*_0$. We show $\pi(p) \le \pi(q)$. We need to show there is $\vec{\nu} \in {}^{<\omega}A^{\pi(q)}$ such that $\pi(p) \le^* \pi(q)_{\langle\vec{\nu}\rangle}$. Choose $\vec{\nu} \in A^q = A^{\pi(q)}$ such that $p \le^* q_{\langle\vec{\nu}\rangle}$. For the measure one sets we get at once

$$A^{\pi(p)} \restriction \operatorname{dom} f^q = A^p \restriction \operatorname{dom} f^q \subseteq A^q_{\langle\vec{\nu}\rangle} = A^{\pi(q)}_{\langle\vec{\nu}\rangle}.$$

For the functions we have the following. If $\alpha \in \operatorname{dom} f^{\pi(p)} \setminus d$ then

$$f^{\pi(p)}(\alpha) = f^p(\alpha) = f^q(\alpha) \frown \langle\vec{\nu}(\alpha)\rangle = f^{\pi(q)}(\alpha) \frown \langle\vec{\nu}(\alpha)\rangle = f^{\pi(q)_{\langle\vec{\nu}\rangle}}(\alpha).$$

If $\alpha \in d$ then there is $\vec{\mu} \in {}^{<\omega}A^{p^*_0} = {}^{<\omega}A^{p^*_1}$ such that $q \le^* p^*_{0\langle\vec{\mu}\rangle}$, thus $p \le^* p^*_{0\langle\vec{\mu}\frown\vec{\nu}\rangle}$, hence

$$f^{\pi(p)}(\alpha) = f^{p^*_{1\langle\vec{\mu}\frown\vec{\nu}\rangle}}(\alpha) = f^{p^*_1}_{\langle\vec{\mu}\frown\vec{\nu}\rangle}(\alpha) = f^{\pi(q)}_{\langle\vec{\nu}\rangle}(\alpha) = f^{\pi(q)}(\alpha) \frown \langle\vec{\nu}(\alpha)\rangle = f^{\pi(q)_{\langle\vec{\nu}\rangle}}(\alpha).$$

□

For a generic filter $G \subseteq \mathbb{P}_E$ define the function f_G by setting $f_G(\alpha) = \bigcup\{f^p(\alpha) \mid p \in G, \alpha \in \operatorname{dom} f^p\}$.

Let us define the Easton products we are going to work with. Let $A \subseteq \mathrm{On}$ be a set of ordinals. Let $\mathbb{C}_{\chi,A}$ be the Easton product of the Cohen forcing notions yielding, in the generic extension, for each $\xi < \sup A$,

$$2^{\chi^{+\xi+1}} = \begin{cases} \chi^{+\xi+3} & \xi \in A, \\ \chi^{+\xi+2} & \xi \notin A. \end{cases}$$

When forcing with $\mathbb{C}_{\chi,A}$ we will choose χ to be large enough so as not to interfere with our intended usage. Due to the (cone) homogeneity of $\mathbb{P}_E$, the sequences forced by $\mathbb{P}_E$ are not in $\mathrm{HOD}^{V^{\mathbb{P}_E}}$. We would like to break the homogeneity of $\mathbb{P}_E$ so as to have the Prikry sequence enter $\mathrm{HOD}^{V^{\mathbb{P}_E}}$. We will achieve this by coding the Prikry sequence into the power set function. We will want the Cohen forcing used to be stabilized by reasonable automorphisms of $\mathbb{P}_E$. Let $\mathbb{P}_{E(\kappa)}$ be Prikry forcing using the measure $E(\kappa)$. Define the function $s : \mathbb{P}_E \to \mathbb{P}_{E(\kappa)}$ by setting $s(p) = \langle f^p\restriction\{\kappa\}, A^p\restriction\{\kappa\}\rangle$, where $A^p\restriction\{\kappa\} = \{\nu\restriction\{\kappa\} \mid \nu \in A^p\}$. Note $\mathbb{P}_{E(\kappa)} \subseteq \mathbb{P}_E$. Assume a pair of conditions $p, q \in \mathbb{P}_{E(\kappa)}$ are compatible in $\mathbb{P}_E$. I.e., there is a condition $r \le_{\mathbb{P}_E} p, q$. Then $s(r) \le_{\mathbb{P}_E} p, q$, hence $s(r) \le_{\mathbb{P}_{E(\kappa)}} p, q$. Hence a maximal antichain in $\mathbb{P}_{E(\kappa)}$ is also a maximal antichain in $\mathbb{P}_E$, thus the function s is a projection. Thus if $G \subseteq \mathbb{P}_E$ is generic then $s''G$ is $\mathbb{P}_{E(\kappa)}$-generic.

Until the end of the section set $\mathbb{P} = \mathbb{P}_E * \dot{\mathbb{C}}_{\chi, \dot{f}_G(\kappa)}$.

Claim 3.2. *Assume $\langle p_0, \dot{q}_0 \rangle, \langle p_1, \dot{q}_1 \rangle \in \mathbb{P}$ are conditions such that $s(p_0)$ and $s(p_1)$ are compatible. Then there are extensions, $\langle p_0^*, \dot{q}_0^* \rangle \leq \langle p_0, \dot{q}_0 \rangle$ and $\langle p_1^*, \dot{q}_1^* \rangle \leq \langle p_1, \dot{q}_1 \rangle$, such that $\mathbb{P}/\langle p_0^*, \dot{q}_0^* \rangle \simeq \mathbb{P}/\langle p_1^*, \dot{q}_1^* \rangle$.*

Proof. Since $s(p_0)$ and $s(p_1)$ are compatible, we can choose conditions $p'_0 \leq p_0$ and $p'_1 \leq p_1$ such that $f^{p'_0} \restriction \{\kappa\} = f^{p'_1} \restriction \{\kappa\}$. By claim 3.1 there are direct extensions $p_0^* \leq^* p'_0$ and $p_1^* \leq^* p'_1$ such that $\pi_0 : \mathbb{P}_E/p_0^* \simeq \mathbb{P}_E/p_1^*$ is an automorphism. Since $\mathbb{C}_{\chi, f_G(\kappa)} = \pi(\mathbb{C}_{\chi, f_G(\kappa)})$, where $G \subseteq \mathbb{P}_E$ is generic, we are done by claim 2.3. □

The following is immediate from the previous claim.

Corollary 3.3. *Assume $\alpha, \alpha_1, \ldots, \alpha_n \in On$ and $\langle p, q \rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$. Then $\langle s(p), 1 \rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$.*

Proof. In order to show $\langle s(p), 1 \rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \ldots, \alpha_n)$ we will show a dense subset of conditions below $\langle s(p), 1 \rangle$ forces $\varphi(\alpha, \alpha_1, \ldots, \alpha_n)$. Let $\langle p_0, \dot{q}_0 \rangle \leq \langle s(p), 1 \rangle$ be an arbitrary condition. By claim 3.2 there is $\langle p'_0, \dot{q}'_0 \rangle \leq \langle p_0, \dot{q}_0 \rangle$ and $\langle p'_1, \dot{q}'_1 \rangle \leq \langle p, \dot{q} \rangle$ such that $\mathbb{P}/p'_0 * \dot{q}'_0 \simeq \mathbb{P}/p'_1 * \dot{q}'_1$. Thus $\langle p'_0, \dot{q}'_0 \rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_n)$. □

The previous corollary together with claim 2.4 yields the following.

Corollary 3.4. *Assume $G * H$ is $\mathbb{P}$-generic. Then $\mathrm{cf}^{V[G*H]} \kappa = \omega$ and $f_G(\kappa) \in HOD^{V[G*H]} \subseteq V[s''G]$.*

We will get a special case of theorem 1 by invoking the last corollary in a model of the form $L[A]$.

Corollary 3.5. *Assume $V = L[A]$, where $A \subseteq On$ is a set of ordinals, and E is an extender witnessing κ is a $<\lambda$-supercompact cardinal. There is a forcing notion R preserving the extender E such that in $V[I][G * H]$, where $I * G * H$ is $R * \mathbb{P}$-generic, $\kappa^+ = \lambda$, $\mathrm{cf}\, \kappa = \omega$, and $HOD^{V[I][G][H]} = V[I][s''G]$.*

Proof. We will begin by defining the forcing notion R so that for an R-generic filter I we will have $\mathrm{HOD}^{V[I]} = V[I]$.

Define by induction the forcing notions $\langle R_n \mid n \leq \omega \rangle$ and sets $\langle A_n \mid n < \omega \rangle$, as follows. Set $R_0 = 1$ and $A_0 = A$. For each $n < \omega$ define R_{n+1} as follows. In $V[G_n]$, where $G_n \subseteq R_n$ is generic over V, let $\mathbb{C}_n$ be the forcing notion $\mathbb{C}_{\chi_n, A_n}$ for a large enough χ_n. Let A_{n+1} be $\mathbb{C}_n$-generic over $V[G_n]$, i.e., A_{n+1} is a code for A_n. Set $R_{n+1} = R_n * \dot{\mathbb{C}}_n$, where $\dot{\mathbb{C}}_n$ is an R_n-name for $\mathbb{C}_n$. Let R be the inverse limit of $\langle R_n \mid n < \omega \rangle$. Let $I \subseteq R$ be generic.

Invoking *corollary* 3.4 inside $V[I]$ and calculating $\mathrm{HOD}^{V[I][G][H]}$ we get $f_G(\kappa) \in \mathrm{HOD}^{V[I][G][H]} \subseteq V[I][s''G]$. For each $n < \omega$, $A_n \in \mathrm{HOD}^{V[I][G][H]}$, thus $\mathrm{HOD}^{V[I][G][H]} \supseteq L[A][I][s''G] = V[I][s''G]$. $\square$

Hence we get:

Corollary 3.6. *Assume λ is measurable and κ is $<\lambda$-supercompact. Then there is a generic extension in which $\mathrm{cf}^{HOD}\kappa = \omega$, and κ^+ (of the generic extension) is measurable in HOD.*

In order to analyze $\mathrm{HOD}_{\{a\}}$, where $a \subseteq \kappa$, let us derive another line of corollaries stemming from claim 3.2. The problem we face when dealing with $\mathrm{HOD}_{\{a\}}$ is an automorphism π of $\mathbb{P}$ might move $\dot{a}$, the name of a. Thus we will need to fine tune the projection s.

First we recall the notion of a good pair from [9]. We say the pair $\langle N, f\rangle$ is a good pair if $N \prec H_\chi$ is a κ-internally approachable elementary substructure, $|N| < \lambda$, and there is a sequence $\langle\langle N_\xi, f_\xi\rangle \mid \xi < \kappa\rangle$ such that $\langle N_\xi \mid \xi < \kappa\rangle$ witnesses the κ-internal approachability of N, $f = \bigcup\{f_\xi \mid \xi < \kappa\}$, $\langle f_\xi \mid \xi < \kappa\rangle$ is a $\leq^*$-decreasing continuous sequence in $\mathbb{P}^*_f$, and for each $\xi < \kappa$, $f_\xi \in \bigcap\{D \in N_\xi \mid D \text{ is a dense open subset of } \mathbb{P}^*_f\}$, $f_\xi \subseteq N_{\xi+1}$, and $f_\xi \in N_{\xi+1}$.

Set $\mathbb{P}^N_E = \{\langle f, A\rangle \in \mathbb{P}_E \mid \mathrm{dom}\, f \subseteq N\}$. Define the function $s_N : \mathbb{P}_E \to \mathbb{P}^N_E$ by setting for each $p \in \mathbb{P}_E$, $s_N(p) = \langle f^p \restriction N, A^p \restriction N\rangle$. Note $\mathbb{P}^N_E \subseteq \mathbb{P}_E$. Fix two conditions $p, q \in \mathbb{P}^N_E$. Assume they are compatible in $\mathbb{P}_E$, i.e., there is a condition $r \in \mathbb{P}_E$ such that $r \leq_{\mathbb{P}_E} p, q$. Thus there are $\vec{\nu} \in {}^{<\omega}A^p$ and $\vec{\mu} \in {}^{<\omega}A^q$ such that $r \leq^*_{\mathbb{P}_E} p_{\langle\vec{\nu}\rangle}, q_{\langle\vec{\mu}\rangle}$. Immediately we get $s_N(r) \leq^*_{\mathbb{P}^N_E} p_{\langle\vec{\nu}\rangle}, q_{\langle\vec{\mu}\rangle}$. Hence a maximal antichain in $\mathbb{P}^N_E$ is a maximal antichain in $\mathbb{P}_E$, hence the function s_N is a projection.

Corollary 3.7. *Assume $N \prec H_\chi$ is an elementary substructure such that p^* is an $\langle N, \mathbb{P}_E\rangle$-generic condition and $\langle N, f^{p^*}\rangle$ is a good pair. Let $\dot{a} \in N$ be a $\mathbb{P}_E$-name such that $\Vdash_{\mathbb{P}_E}$ "$\dot{a} \subseteq \kappa$". If $\alpha, \alpha_1, \dots, \alpha_n \in On$, $p \leq p^*$, and $\langle p, \dot{q}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$, then $\langle s_N(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$.*

Proof. In order to show $\langle s_N(p), 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$ we will show a dense subset of conditions below $\langle s_N(p), 1\rangle$ forces $\varphi(\alpha, \alpha_1, \dots, \alpha_n, \dot{a})$.

Let $\langle p_0, \dot{q}_0\rangle \leq \langle s_N(p), 1\rangle$ be arbitrary condition. We can choose $p_1 \leq p$ such that $s_N(p_0) = s_N(p_1)$. By claim 3.1 there is $p^*_0 \leq^* p_0$ and $p^*_1 \leq^* p_1$ such that $\mathbb{P}_E/p^*_0 \simeq \mathbb{P}_E/p^*_1$.

Recall that if $r \leq p^*$, $\alpha < \kappa$, and $r \Vdash_{\mathbb{P}_E}$ "$\alpha \in \dot{a}$", then $p^*_{\langle\nu_0,\dots,\nu_{l-1}\rangle} \Vdash$ "$\alpha \in \dot{a}$", where $\langle\nu_0, \dots, \nu_{l-1}\rangle \in {}^{<\omega}A^{p^*}$ is such that $r \leq^* p^*_{\langle\nu_0,\dots,\nu_{l-1}\rangle}$.

Thus for each $\langle\nu_0,\dots,\nu_{l-1}\rangle \in A^{p^*_0} = A^{p^*_1}$, $\alpha<\kappa$, and $r\in\mathbb{P}_E/p^*_0$,

$$\begin{aligned}
&r \leq^* p^*_{0\langle\nu_0,\dots,\nu_{l-1}\rangle} \text{ and } r \Vdash_{\mathbb{P}_E} \text{“}\alpha\in\dot a\text{”} \iff\\
&p_{\langle\nu_0,\dots,\nu_{l-1}\rangle\restriction\operatorname{dom} f^p} \Vdash_{\mathbb{P}_E} \text{“}\alpha\in\dot a\text{”} \iff\\
&\pi(r) \leq^* p^*_{1\langle\nu_0,\dots,\nu_{l-1}\rangle} \text{ and } \pi(r) \Vdash_{\mathbb{P}_E} \text{“}\alpha\in\pi(\dot a)\text{”}.
\end{aligned}$$

Thus $p^*_0 \Vdash$ “$\dot a = \pi^{-1}(\dot a)$”. Use claim 3.2 to find stronger conditions $\langle p'_0,\dot q'_0\rangle \leq \langle p^*_0,\dot q_0\rangle$ and $\langle p'_1,\dot q'_1\rangle \leq \langle p^*_0,\dot q\rangle$ such that $\tilde\pi : \mathbb{P}/p'_0 * \dot q'_0 \simeq \mathbb{P}/p'_1 * \dot q'_1$ is an automorphism. Since $\langle p'_1,\dot q'_1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1,\dots,\alpha_n,\dot a)$ we get $\langle p'_0,\dot q'_0\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1,\dots,\alpha_n,\pi^{-1}(\dot a))$. We are done since $p'_0 \Vdash$ “$\dot a = \pi^{-1}(\dot a)$”. □

Corollary 3.8. *Assume $G*H$ is $\mathbb{P}$-generic, $a\in V[G*H]$, and $a\subseteq\kappa$. Then $\operatorname{cf}^{V[G*H]}\kappa=\omega$ and $f_G(\kappa)\in HOD^{V[G*H]}_{\{a\}} \subseteq V[s''_X G]$ for a set $X\subseteq\operatorname{dom} E$ such that $|X|<\lambda$.*

We will get theorem 1 by beginning with a model where HOD $\supseteq V_{\lambda+2}$. For this let us define the following coding. Let $\mathfrak{A}=\langle A_\alpha \mid \alpha<\lambda^{+3}\rangle$ be an enumeration of all subsets of λ^{++}. Let $\mathbb{C}_{\chi,\mathfrak{A}}$ be the Easton product of the Cohen forcing notions yielding, in the generic extension, for each $\alpha<\lambda^{+3}$ and $\xi<\lambda^{++}$,

$$2^{\chi^{+\lambda^{++}\cdot\alpha+\xi+1}} = \begin{cases}\chi^{\lambda^{++}\cdot\alpha+\xi+3} & \xi\in A_\alpha,\\ \chi^{\lambda^{++}\cdot\alpha+\xi+2} & \xi\notin A_\alpha.\end{cases}$$

Corollary 3.9. *Let E is an extender witnessing κ is a $<\lambda$-supercompact cardinal. In $V[I][G*H]$, where $I*G*H$ is $\mathbb{C}_{\chi,\mathfrak{A}}*\mathbb{P}$-generic, $\kappa^+=\lambda$, and for each set $a\subseteq\kappa$, $\operatorname{cf}^{HOD^{V[I][G*H]}_{\{a\}}}\kappa=\omega$ and λ is measurable in $HOD^{V[I][G][H]}_{\{a\}}$.*

Proof. Let $U\in V$ be a measure on λ. Then $U\in V_{\lambda+2}$, hence $U\in \mathrm{HOD}^{V[I]}$, where I is $\mathbb{C}_{\chi,\mathfrak{A}}$-generic.

Working in $V[I]$ let $G*H$ be $\mathbb{P}$-generic. By corollary 3.8 there is $X\subseteq\operatorname{dom}E$ such that $|X|<\lambda$, $X\in V[I]$, and $f_G(\kappa)\in\mathrm{HOD}^{V[I][G*H]}_{\{a\}}\subseteq V[I][s''_X G]$. The filter $s''_X G$ is $s''_X\mathbb{P}_E$-generic. Since $|X|<\lambda$ we have $|s''_X\mathbb{P}_E|<\lambda$, hence any measure (in V) over λ trivially lifts to a measure in $V[s_X(G)]$ over λ. In particular U lifts to $\bar{\bar U}$, which is definable by $\bar{\bar U}=\{B\in V[I][s''_X G]\cap\mathcal{P}(\lambda)\mid \exists A\in U\, B\supseteq A\}$. Since $U\in\mathrm{HOD}^{V[I][G*H]}_{\{a\}}$ we can define in $\mathrm{HOD}^{V[I][G*H]}_{\{a\}}$, $\bar U=\{B\in\mathrm{HOD}^{V[I][G*H]}_{\{a\}}\cap\mathcal{P}(\lambda)\mid\exists A\in U\, B\supseteq A\}$. Since $\mathrm{HOD}^{V[I][G*H]}_{\{a\}}\subseteq V[I][s''_X G]$ we necessarily have $\bar U\subseteq\bar{\bar U}$. Thus $\bar U$ is a measure on λ in $\mathrm{HOD}^{V[I][G*H]}_{\{a\}}$. □

4. The global result

In this section we prove theorem 2. The extender based Radin forcing was originally developed in [7]. We use a generalization of the forcing where the extenders can witness supercompactness. This was developed in [9]. Since the introduction in the previous section was very detailed, we will give here only our version of Radin forcing.

Let us begin with with defining the Magidor forcing [6] using two measures. Assume $U'_0 \lhd U'_1$ are two normal measures over κ. For each $i < 2$ let $j_i : V \to M_i \simeq \mathrm{Ult}(V, U'_i)$ be the natural elementary embeddings. Define the measure U_0 by letting for each $A \subseteq {}^{\{\kappa\}}\kappa$,

$$A \in U_0 \iff \{\langle j_0(\kappa), \langle\kappa\rangle\rangle\} \in j_0(A).$$

Note a typical function $\nu \in A$ is of the form $\nu : \{\kappa\} \to \kappa$. Define the measure U_1 by letting for each $A \subseteq {}^{\{\kappa\}}V_\kappa$,

$$A \in U_1 \iff \{\langle j_1(\kappa), \langle\kappa, U_0\rangle\rangle\} \in j_1(A).$$

A typical function $\nu \in A$ is of the form $\nu : \{\kappa\} \to V_\kappa$, where $\nu(\kappa) = \langle\xi, \mu\rangle$ and μ is a measure over ξ. Define by recursion the conditions and ordering of the forcing notion as follows. A basic condition in Magidor forcing is of the form $\langle f, A\rangle$, where $A \in U_0 \cap U_1$ and $f : \{\kappa\} \to V_\kappa$ is a function such that $f(\kappa) = \langle\langle\xi_0, \mu_0\rangle, \ldots, \langle\xi_{k-1}, \mu_{k-1}\rangle, \langle\xi_k\rangle, \ldots, \langle\xi_{n-1}\rangle\rangle$, where $\xi_0 < \cdots < \xi_{n-1} < \kappa$ and for each $i < k$, μ_i is a measure over ξ_i. A sequence of the form $\langle\langle\xi_0, \mu_0\rangle, \ldots, \langle\xi_{k-1}, \mu_{k-1}\rangle, \langle\xi_k\rangle, \ldots, \langle\xi_{n-1}\rangle\rangle$ is said to be o-decreasing since we consider $\mathrm{o}(\langle\xi_i, \mu_i\rangle) = 1$ and $\mathrm{o}(\langle\xi_i\rangle) = 0$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. We define the functions $f_{\langle\nu\rangle\downarrow}$ and $f_{\langle\nu\rangle\uparrow}$ as follows. Assume $\mathrm{o}(\nu) = 0$. In this case we work essentialy as in the Prikry forcing case. We let $f_{\langle\nu\rangle\downarrow} = \emptyset$. Define the function $f_{\langle\nu\rangle\uparrow}$ by letting $f_{\langle\nu\rangle\uparrow}(\kappa) = f(\kappa) \frown \langle\nu(\kappa)\rangle$. Note that since $\mathrm{o}(\nu) = 0$ the sequence $f(\kappa) \frown \langle\nu(\kappa)\rangle$ is o-decreasing if $f(\kappa)$ is o-decreasing. Assume $\nu \in A$ is a function such that $\mathrm{o}(\nu) = 1$. In this case we define two functions $f_{\langle\nu\rangle\uparrow}$ and $f_{\langle\nu\rangle\downarrow}$. If we would have let $f_{\langle\nu\rangle\uparrow}(\kappa) = f(\nu) \frown \langle\nu(\kappa)\rangle$ then we might have ended with a non o-decreasing sequence. Thus we cut the possible problematic tail of $f(\kappa)$ as follows. Set $l = \max\{l' \mid \mathrm{o}(f_{l'}(\kappa)) = 1\} + 1$. If the set over which the max above is operating is empty then set $l = 0$. Then let $f_{\langle\nu\rangle\uparrow}(\kappa) = f(\kappa)\restriction l \frown \langle\nu(\kappa)\rangle$. Whatever is the value of $\mathrm{o}(\nu)$ let $A_{\langle\nu\rangle\uparrow} = \{\tau \in A \mid \mathring{\tau}(\kappa) > \mathring{\nu}(\kappa)\}$. The tail removed from $f(\kappa)$ is 'pushed down' by letting $f_{\langle\nu\rangle\downarrow} : \{\mathring{\nu}(\kappa)\} \to \mathring{\nu}$ be a function such that $f_{\langle\nu\rangle\downarrow}(\mathring{\nu}) = f(\kappa) \setminus l$, where $\mathring{\nu}$ is ξ if $\nu = \langle\xi, \mu\rangle$. Together with the 'pushed down function' we set the pushed down part of A to be

$A_{\langle\nu\rangle\downarrow} = \{\tau \downarrow \nu \mid \tau \in A,\ \mathrm{o}(\tau) = 0, \mathring{\tau} < \mathring{\nu}\}$, where $\tau \downarrow \nu(\xi) = \tau(\xi)$. Finally set $\langle f, A\rangle_{\langle\nu\rangle\uparrow} = \langle f_{\langle\nu\rangle\uparrow}, A_{\langle\nu\rangle\uparrow}\rangle$ and $\langle f, A\rangle_{\langle\nu\rangle\downarrow} = \langle f_{\langle\nu\rangle\downarrow}, A_{\langle\nu\rangle\downarrow}\rangle$. Note $\langle f, A\rangle_{\langle\nu\rangle\downarrow}$ is a condition in a Prikry forcing. A 1-point extension of $\langle f, A\rangle$ is $\langle f, A\rangle_{\langle\nu\rangle} = \langle f, A\rangle_{\langle\nu\rangle\downarrow} {}^\frown \langle f, A\rangle_{\langle\nu\rangle\uparrow}$.

Assume G is generic with $\langle f^*, A^*\rangle \in G$, where $f^*(\kappa) = \langle\rangle$. Letting $f_G(\kappa) = \bigcup\{f(\kappa) \mid s {}^\frown \langle f, A\rangle \in G\}$ and $\mathring{f}_G(\kappa) = \langle\mathring{\nu} \mid \nu \in f_G(\kappa)\rangle$ we get that $\mathring{f}_G(\kappa)$ is an ω^2 sequence cofinal in κ. Moreover if $s {}^\frown \langle g, B\rangle {}^\frown t {}^\frown \langle f, A\rangle \in G$ then setting $g_G(\kappa) = \bigcup\{\langle g', B'\rangle \leq \langle g, B\rangle \mid s' {}^\frown \langle g', B'\rangle {}^\frown t' {}^\frown \langle f, A\rangle \in G\}$ and $\mathring{g}_G(\tau) = \langle\mathring{\nu} \mid \nu \in g_G(\kappa)\rangle$. Then $\mathring{g}(\tau)$ is an ω-sequence cofinal in $\operatorname{dom} g$.

Let us switch to the extender based Magidor forcing using two extenders. Assume $E'_0 \triangleleft E'_1$ are two extenders over κ. For each $i < 2$ let $j_i : V \to M_i \simeq \mathrm{Ult}(V, E'_i)$ be the natural elementary embeddings.

For each $d \in {}^\kappa j(\kappa)$ such that $\kappa \in d$ define the measure $E_0(d)$ as follows. For each $A \subseteq \bigcup\{{}^{d'}\kappa \mid d' \subseteq d,\ |d'| < \kappa\}$,

$$A \in E_0(d) \iff \{\langle j_0(\alpha), \langle\alpha\rangle\rangle \mid \alpha \in d\} \in j_0(A).$$

A typical function $\nu \in A$ is of the form $\nu : \operatorname{dom}\nu \to \kappa$ where $\operatorname{dom}\nu \subseteq d$, and for each $\alpha, \beta \in \operatorname{dom}\nu$ such that $\alpha < \beta$ we have $\mathring{\nu}(\alpha) < \mathring{\nu}(\beta)$. Moreover, $|\operatorname{dom}\nu| < \nu(\kappa)$. For each $d \in {}^\kappa j_1(\kappa)$ such that $\kappa \in d$ define the ultrafilter $E_1(d)$ as follows. For each $A \subseteq \bigcup\{{}^{d'}V_\kappa \mid d' \subseteq d,\ |d'| < \kappa\}$,

$$A \in E_1(d) \iff \{\langle j_1(\alpha), \langle\alpha, E_0\rangle\rangle \mid \alpha \in d\} \in j_1(A).$$

A typical function $\nu \in A$ is of the form $\nu : \operatorname{dom}\nu \to V_\kappa$ where $\operatorname{dom}\nu \subseteq d$, and for each $\alpha, \beta \in \operatorname{dom}\nu$ such that $\alpha < \beta$ we have $\mathring{\nu}(\alpha) < \mathring{\nu}(\beta)$. Moreover, $|\operatorname{dom}\nu| < \nu(\kappa)$. Define by recursion the conditions and order on the forcing notion as follows. A basic condition in the extender based Magidor forcing is of the form $\langle f, A\rangle$, where $f : d \to {}^{<\omega}V_\kappa$ is a function such that for each $\alpha \in d$ we have $f(\alpha)$ is a finite o-decreasing sequence, and $A \in E_0(d) \cap E_1(d)$. We can always assume that for each $\nu \in A$ we have $\max f(\alpha) < \mathring{\nu}(\alpha)$ for each $\alpha \in \operatorname{dom}\nu$. Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. We define the functions $f_{\langle\nu\rangle\downarrow}$ and $f_{\langle\nu\rangle\uparrow}$, employing the same idea used on $f(\kappa)$ to each of the $f(\alpha)$'s, as follows. Assume $\mathrm{o}(\nu) = 0$. Let $f_{\langle\nu\rangle\downarrow} = \emptyset$. Define the function $f_{\langle\nu\rangle\uparrow}$ by setting for each $\alpha \in d$,

$$f_{\langle\nu\rangle\uparrow}(\alpha) = \begin{cases} f(\alpha) {}^\frown \langle\nu(\alpha)\rangle & \alpha \in \operatorname{dom}\nu, \\ f(\alpha) & \alpha \notin \operatorname{dom}\nu. \end{cases}$$

Assume $\mathrm{o}(\nu)=1$. Define the function $f_{\langle\nu\rangle\uparrow}$ by setting for each $\alpha\in d$,

$$f_{\langle\nu\rangle\uparrow}(\alpha)=\begin{cases} f(\alpha)\restriction l_\alpha \frown \langle\nu(\alpha)\rangle & \alpha\in\operatorname{dom}\nu,\\ f(\alpha) & \alpha\notin\operatorname{dom}\nu,\end{cases}$$

where

$$l_\alpha=\begin{cases}\max\{l'\mid \mathrm{o}(f_{l'}(\alpha))=1\}+1 & \exists l\ \ \mathrm{o}(f_l(\alpha))=1,\\ 0 & \text{Otherwise.}\end{cases}$$

Let $f_{\langle\nu\rangle\downarrow}:\operatorname{ran}\mathring{\nu}\to V_{\mathring{\nu}(\kappa)}$ be the function defined by setting for each $\alpha\in\operatorname{dom}\nu$, $f_{\langle\mathring{\nu}(\alpha)\rangle}=f(\alpha)\setminus l_\alpha$. Set $A_{\langle\nu\rangle\uparrow}=\{\tau\in A\mid \mathring{\tau}(\kappa)>\mathring{\nu}(\alpha)$ for each $\alpha\in\operatorname{dom}\nu\}$. If $\mathrm{o}(\nu)=0$ then set $A_{\langle\nu\rangle\downarrow}=\langle\rangle$. If $\mathrm{o}(\nu)=1$ then set $A_{\langle\nu\rangle\downarrow}=\{\tau\downarrow\nu\mid \tau\in A,\ \mathrm{o}(\tau)=0,\ \operatorname{dom}\tau\subseteq\operatorname{dom}\nu,\ \mathring{\tau}(\alpha)<\mathring{\nu}(\kappa)$ for each $\alpha\in\operatorname{dom}\tau\}$, where $\tau\downarrow\nu:\operatorname{ran}\mathring{\tau}\to\mathring{\nu}(\kappa)$ defined by setting for each $\alpha\in\operatorname{dom}\tau$, $\tau\downarrow\nu(\mathring{\nu}(\alpha))=\tau(\alpha)$. Then define $\langle f,A\rangle_{\langle\nu\rangle\uparrow}$ and $\langle f,A\rangle_{\langle\nu\rangle\uparrow}$ to be $\langle f_{\langle\nu\rangle\uparrow},A_{\langle\nu\rangle\uparrow}\rangle$ and $\langle f_{\langle\nu\rangle\downarrow},A_{\langle\nu\rangle\downarrow}\rangle$, respectively. We say the condition $s'\frown\langle f',A'\rangle$ is stronger than the condition $s\frown\langle f,A\rangle$ ($s'\frown\langle f',A'\rangle\leq s\frown\langle f,A\rangle$), if $s'\leq s$ and $\langle f',A'\rangle\leq^*\langle f,A\rangle_{\langle\vec{\nu}\rangle}$.

Assume G is generic with $\langle f^*,A^*\rangle\in G$, where $f(\kappa)=\langle\rangle$. Letting $f_G(\alpha)=\bigcup\{f(\alpha)\mid s\frown\langle f,A\rangle\in G\}$ and $\mathring{f}(\alpha)=\langle\mathring{\nu}\mid\nu\in f_G(\alpha)\rangle$ we get that $\mathring{f}(\alpha)$ is an ω^2 sequence cofinal in κ. Moreover if $s\frown\langle g,B\rangle\frown t\frown\langle f,A\rangle\in G$ then setting $g_G(\tau)=\bigcup\{\langle g',B'\rangle\leq\langle g,B\rangle\mid s'\frown\langle g',B'\rangle\frown t'\frown\langle f,A\rangle\in G\}$ and $\mathring{g}_G(\tau)=\langle\mathring{\nu}\mid\nu\in g_G(\tau)\rangle$. Then $\mathring{g}(\tau)$ is an ω-sequence cofinal in τ. Note there are $|j_1(\kappa)|$ new ω^2-sequences cofinal into κ. For each of the reflections down we get the reflected amount of ω-sequences. E.g., if $|j_0(\kappa)|=\kappa^{+3}$ then there are τ_n^{+3} new ω-sequences cofinal in τ_n.

Letting G be the generic object and defining the function $f_G:d\to{}^{\omega}\kappa$ by setting for each $\alpha\in d$, $f_G(\alpha)=\bigcup\{f(\alpha)\mid\langle f,A\rangle\in G,\ \alpha\in\operatorname{dom}f\}$, we get that for each $\alpha\in d$, $f_G(\alpha)$ is an ω-sequence cofinal in κ. Defining the function f_G as in the previous paragraph we get κ cofinal ω-sequences, but still all of them are generated by $f_G(\kappa)$.

Since the ultrapower M is closed only under κ-sequences, one cannot enlarge d to be of size greater than κ while keeping the nice properties of Prikry type forcing notions. However, we can use conditions with different domains, thus adding sequence corresponding to each of the ordinals below $j(\kappa)$. The domain change, however, causes the forcing to be non-isomorphic to Prikry forcing. Thus if $d\in{}^{\kappa}j(\kappa)$ then define the measure U_d by letting for each $A\subseteq\bigcup\{{}^{d'}\kappa\mid d'\subseteq d\}$,

$$A\in U_d \iff \{\langle j(\alpha),\alpha\rangle\mid\alpha\in d\}\in j(A).$$

Define now a condition in the forcing to be of the form $\langle f, A\rangle$, where $f : \operatorname{dom} f \to {}^{<\omega}\kappa$ is a function such that for each $\alpha \in \operatorname{dom} f$, $f(\alpha)$ is a finite increasing sequence, $\operatorname{dom} f \in {}^{\kappa}j(\kappa)$, and $A \in U_{\operatorname{dom} f}$. If $d \subseteq e$ and $A \in U_e$ then set $A\restriction d = \{\nu\restriction d \mid \nu \in A\}$. The condition $\langle g, B\rangle$ is said to be a direct extension of the condition $\langle f, A\rangle$ ($\langle f, B\rangle \leq^* \langle f, A\rangle$) in this forcing if $g \supseteq f$ and $B\restriction \operatorname{dom} f \subseteq A$. **Note this definition of the direct order is a major change from all previous definitions. In fact the direct order is a Cohen forcing for adding $\boldsymbol{j(\kappa)}$ subsets to $\boldsymbol{\kappa^+}$.** Assume $\langle f, A\rangle$ is a condition and $\nu \in A$ is a function. The function $f_{\langle\nu\rangle}$ is defined by setting for each $\alpha \in \operatorname{dom} f$,

$$f_{\langle\nu\rangle}(\alpha) = \begin{cases} f(\alpha) ^\frown \langle\nu(\alpha)\rangle & \alpha \in \operatorname{dom}\nu, \\ f(\alpha) & \alpha \notin \operatorname{dom}\nu. \end{cases}$$

Given a set $A \in U_d$ set $A_{\langle\nu\rangle} = \{\mu \in A \mid \mu(\kappa) > \nu(\alpha) \text{ for each } \alpha \in \operatorname{dom}\nu\}$. Then define the condition $\langle f, A\rangle_{\langle\nu\rangle}$ to be $\langle f_{\langle\nu\rangle}, A_{\langle\nu\rangle}\rangle$. We say the condition $\langle f, A\rangle_{\langle\nu\rangle}$ is a 1-point extension of $\langle f, A\rangle$. By recursion define the $n+1$-point extension of the condition $\langle f, A\rangle$ to be $(\langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$. We say the condition $\langle g, B\rangle$ is stronger than the condition $\langle f, A\rangle$ ($\langle g, B\rangle \leq \langle f, A\rangle$) if there is $\langle\nu_0, \dots, \nu_{n-1}\rangle \in {}^{<\omega}A$ such that $\langle g, B\rangle \leq^* \langle f, A\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle}$. **This forcing notion is no longer Prikry forcing.** Using the same definition of f_G as before we get $j(\kappa)$ cofinal ω-sequence into κ. However, while each of the sequences appears in a generic extension by Prikry forcing, the function f_G itself does not below to a Prikry generic extension.

The point of the previous forcing is that nothing restricts us from using it with elementary embeddings with many generators, thus we get the extender based Prikry forcing. Thus assume $j : V \to M$ is an elementary embedding such that $\operatorname{crit}(j) = \kappa$, $M \supseteq {}^{\kappa}M$, and $\kappa < \lambda < j(\kappa)$ is a cardinal in V. Using the previous definition with conditions $\langle f, A\rangle$ such that $\operatorname{dom} f \in {}^{\kappa}\lambda$, we get that λ new ω-sequences cofinal in κ appear in the generic extension, thus $2^\kappa = \kappa^\omega \geq \lambda$. Working out the proof we get that no cardinals are collapsed, $2^\kappa = \lambda$, and $\operatorname{cf}\kappa = \omega$.

The final generalization achieved so far is to begin with elementary embedding with even more closure properties, i.e., $j : V \to M$ such that $\operatorname{crit}(j) = \kappa$ and $M \supseteq {}^{<\lambda}M$, where $\lambda > \kappa$ and then work out the definition above to use d of arbitrary size below λ. This yields a generic extension in which $\operatorname{cf}\kappa = \omega$ and 2^κ blows up to whatever cardinal the model M catches beginning with λ up to $j(\kappa)$. However, the cardinals above κ and below λ are collapsed in this extension.

Thus throughout this section assume $\vec{E} = \langle E_\xi \mid \xi < \lambda \rangle$ is a Mitchell increasing sequence of extenders such that λ is measurable, and for each $\xi < \lambda$, $\operatorname{crit}(j_\xi) = \kappa$, $M_\xi \supseteq {}^{<\lambda}M_\xi$, and $M_\xi \supseteq V_{\lambda+2}$, where $j_\xi : V \to \operatorname{Ult}(V, E_\xi) \simeq M_\xi$ is the natural embedding. (We demand $M_\xi \supseteq V_{\lambda+2}$ since we want λ to be measurable in all ultrapowers, not only in V).

Let $\mathbb{P}_{\vec{E}}$ be the supercompact extender based Radin forcing using $\vec{E}$. (see [9]).

Let us recall the cardinal structure in $V^{\mathbb{P}_{\vec{E}}}$. κ remains an inaccessible cardinal, hence $(V_\kappa)^{V^{\mathbb{P}_{\vec{E}}}}$ is a model of ZFC. while λ remains a cardinal, the cardinals between κ and λ are collapsed. Both κ and λ are reflected down using the different extenders. Let τ_κ be reflection of κ which a limit cardinal in $V^{\mathbb{P}_{\vec{E}}}$. Let τ_λ be the matching reflection of λ. Then τ_λ is preserved while the V-cardinals between τ_κ and τ_λ are collapsed.

Let us deal with the homogeneity of the Extender Based Radin forcing.

Lemma 4.1. *For a pair of conditions $p_0, p_1 \in \mathbb{P}^*_{\vec{E}}$ there are direct extensions $p^*_0 \leq^* p_0$ and $p^*_1 \leq^* p_1$ such that $\mathbb{P}_{\vec{E}}/p^*_0 \simeq \mathbb{P}_{\vec{E}}/p^*_1$.*

Proof. Set $d = \operatorname{dom} f^{p_0} \cup \operatorname{dom} f^{p_1}$. Set $f^*_0 = f^{p_0} \cup \{\langle \alpha, \langle\rangle\rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_0}\}$ and $f^*_1 = f^{p_1} \cup \{\langle \alpha, \langle\rangle\rangle \mid \alpha \in d \setminus \operatorname{dom} f^{p_1}\}$. Choose a set T so that $p^*_0 = \langle f^*_0, T\rangle$ and $p^*_1 = \langle f^*_1, T\rangle$ are conditions, $T\restriction \operatorname{dom} f^{p^0} \subseteq T^{p^0}$ and $T\restriction \operatorname{dom} f^{p^1} \subseteq T^{p^1}$. Define the isomorphism $\pi : \mathbb{P}_{\vec{E}}/p^*_0 \to \mathbb{P}_{\vec{E}}/p^*_1$ as follows. Thus assume $p^0 \leq p^*_0$. By the definition of the order there is $\langle \nu_0, \dots, \nu_{n-1}\rangle \in {}^{<\omega}T^{p^*_0}$ such that $p^0 \leq^* p^*_{0\langle \nu_0,\dots,\nu_{n-1}\rangle}$. I.e., $p^0 = p^0_0 {}^\frown \cdots {}^\frown p^0_n$, where $p^0_i \leq^* p^*_{0\langle \nu_0,\dots,\nu_{i-1}\rangle\uparrow\langle\nu_i\rangle\downarrow}$ for each $i < n$, and $p^0_n \leq^* p^*_{0\langle \nu_0,\dots,\nu_n\rangle\uparrow}$. Consider the condition $p^*_{1\langle \nu_0,\dots,\nu_{n-1}\rangle} = p^*_{1,0} {}^\frown \cdots {}^\frown p^*_{1,n}$. Note $\operatorname{dom} f^{p^0_i} \supseteq \operatorname{dom} f^{p^*}_{1,i}$. Let $p^1_i = \langle f^1_i, T^{p^0_i}\rangle$, where $f^1_i = f^{p^*_{1,i}} \cup f^{p^0_i}\restriction(\operatorname{dom} f^{p^0_i} \setminus \operatorname{dom} f^{p^*_{1,i}})$. Finally set $\pi(p^0) = p^1_0 {}^\frown \cdots {}^\frown p^1_n$. Let us show the function π is order preserving. Fix $q \leq p \leq p^*_0$. We will show $\pi(q) \leq \pi(p)$.

Since $p \leq p^*_0$ there is $\langle \nu_0, \dots, \nu_{n-1}\rangle \in {}^{<\omega}T^{p^*_0}$ such that $p = p_0 {}^\frown \cdots {}^\frown p_n$, where $p_i \leq^* p^*_{0\langle \nu_0,\dots,\nu_{i-1}\rangle\uparrow\langle\nu_i\rangle\downarrow}$ for each $i < n$, and $p_n \leq^* p^*_{0\langle \nu_0,\dots,\nu_{n-1}\rangle\uparrow}$. Since $q \leq p^*_0$ there is $\langle \mu_0, \dots, \mu_{m-1}\rangle \in {}^{<\omega}T^{p^*_0}$ such that $q = q_0 {}^\frown \cdots {}^\frown q_m$, where $q_i \leq^* p^*_{0\langle \mu_0,\dots,\mu_{i-1}\rangle\uparrow\langle\mu_i\rangle\downarrow}$ for each $i < m$, and $q_m \leq^* p^*_{0\langle \mu_0,\dots,\nu_{m-1}\rangle\uparrow}$.

Thus $\pi(p) = p' = p'_0 \frown \cdots \frown p'_n$ and $\pi(q) = q' = q'_0 \frown \cdots \frown q'_m$, where

$$f^{p'_i} = f^{p^*_{0\langle\nu_0,\dots,\nu_{i-1}\rangle\uparrow\langle\nu_i\rangle\downarrow}} \cup f^{p_i} \restriction (\operatorname{dom} f^{p_i} \setminus \operatorname{dom} f^{p^*_{0,i}}) \text{ for each } i < n,$$
$$f^{p'_n} = f^{p^*_{0\langle\nu_0,\dots,\nu_{n-1}\rangle\uparrow}} \cup f^{p_n} \restriction (\operatorname{dom} f^{p_n} \setminus \operatorname{dom} f^{p^*_0}),$$
$$T^{p'_i} = T^{p_i} \text{ for each } i \leq n,$$
$$f^{q'_i} = f^{p^*_{0\langle\mu_0,\dots,\mu_{i-1}\rangle\uparrow\langle\mu_i\rangle\downarrow}} \cup f^{q_i} \restriction (\operatorname{dom} f^{p_i} \setminus \operatorname{dom} f^{p^*_{0,i}}) \text{ for each } i < m,$$
$$f^{q'_m} = f^{p^*_{0\langle\nu_0,\dots,\nu_{m-1}\rangle\uparrow}} \cup f^{q_m} \restriction (\operatorname{dom} f^{q_n} \setminus \operatorname{dom} f^{p^*_0}),$$

and

$$T^{q'_i} = T^{q_i} \text{ for each } i \leq m.$$

Since $q \leq p$ there $k < \omega$ such that $q_0 \frown \cdots \frown q_k \leq p_0$. Thus there is $\langle\tau_0, \dots, \tau_{k-1}\rangle \in {}^{<\omega}T^{p_0}$ such that $q_0 \frown \cdots \frown q_k \leq^* p_{0\langle\tau_0,\dots,\tau_{k-1}\rangle}$. I.e., $q_i \leq^* p_{0\langle\tau_0,\dots,\tau_{i-1}\rangle\uparrow\langle\tau_i\rangle\downarrow}$ for each $i < k$, and $q_k \leq^* p_{0\langle\tau_0,\dots,\tau_{k-1}\rangle\uparrow}$. Noting that $p^*_{0\langle\mu_0,\dots,\mu_{i-1}\rangle\uparrow\langle\mu_i\rangle\downarrow} = p^*_{0\langle\nu_0\rangle\downarrow\langle\tau_0,\dots,\tau_{i-1}\rangle\uparrow\langle\tau_i\rangle\downarrow}$ for each $i < k$, and $p^*_{0\langle\mu_0,\dots,\mu_{k-1}\rangle\uparrow} = p^*_{0\langle\nu_0\rangle\downarrow\langle\tau_0,\dots,\tau_{k-1}\rangle\uparrow}$, we conclude that $q'_0 \frown \cdots \frown q'_k \leq^* p'_{0\langle\tau_0,\dots,\tau_{k-1}\rangle}$, hence $q'_0 \frown \cdots \frown q'_k \leq p'_0$.

Proceeding as above for each p_i, (e.g., there is $k^1 \leq \omega$ such that $q_{k+1} \frown \cdots \frown q_{k^1} \leq p_1$) we get that $q' \leq p'$. □

Recall that for a condition $p = p_0 \frown \cdots \frown p_n$ we have $\mathbb{P}_{\vec{E}}/p \simeq \mathbb{P}_{\vec{e}_0}/p_0 \cdots \frown \mathbb{P}_{\vec{e}_n}/p_n$, where $p_i \in \mathbb{P}^*_{\vec{e}_i}$ and $\vec{e}_n = \vec{E}$. Thus the following is an immediate corollary of the above lemma by recursion.

Corollary 4.2. *Assume $p^0, p^1 \in \mathbb{P}_{\vec{E}}$ are conditions such that $p^0, p^1 \in \prod_{0 \leq i \leq n} P^*_{\vec{e}_i}$. Then there are direct extensions $p^{0*} \leq^* p^0$ and $p^{1*} \leq^* p^1$ such that $\mathbb{P}_{\vec{E}}/p^{0*} \simeq \mathbb{P}_{\vec{E}}/p^{1*}$.*

For a condition $p \in \mathbb{P}^*_{\vec{E}}$ define its projection $s(p)$ to the normal measure by setting $s(p) = \langle f^p \restriction \{\kappa\}, T^p \restriction \{\kappa\}\rangle$. Define by recursion the projection of arbitrary condition $p = p_0 \frown \cdots \frown p_n \in \mathbb{P}_{\vec{E}}$ by setting $s(p) = s(p_0 \frown \cdots \frown p_{n-1}) \frown s(p_n)$. It is obvious $s''\mathbb{P}_{\vec{E}}$ is the Radin forcing using the measures $\langle E_\xi(\kappa) \mid \xi < \mathrm{o}(\vec{E})\rangle$. Moreover, if G is $\mathbb{P}_{\vec{E}}$-generic then $s''G$ is $s''\mathbb{P}_{\vec{E}}$-generic.

Let G be $\mathbb{P}_{\vec{E}}$-generic. Work in $V[G]$. Let $\langle\kappa_\alpha \mid \alpha < \kappa\rangle$ be the increasing enumeration of $f_G(\kappa)$. Define the sequence $\langle\mu_\alpha, U_\alpha \mid \alpha < \kappa\rangle$ by setting for each $\alpha < \kappa$,

$$\mu_\alpha = \begin{cases} \kappa_\alpha^+ & \alpha \text{ is limit,} \\ \kappa_\alpha & \alpha \text{ is successor.} \end{cases}$$

Note: If α is limit, then $\mu_\alpha = \kappa_\alpha^+$ is measurable in V since it is a reflection of λ being measurable in one of the V-ultrapowers. On the other hand, if α is successor then $\mu_\alpha = \kappa_\alpha$ is measurable in V since E_0 concentrates on measurables. Thus for each $\alpha < \kappa$ we can choose $U_\alpha \in V$ which is a measure in V over μ_α. Define the backward Easton iteration $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ by setting for each $\alpha < \kappa$, $\dot{Q}_\alpha = \mathrm{Col}(\mu_\alpha, {<}\kappa_{\alpha+1})$. By theorem 2.2 the iteration P_κ is cone homogeneous. Let $H \subseteq P_\kappa$ be generic.

Working in $V[G * H]$ we want to pull into the HOD of a generic extension the measures U_α's. Define the backward Easton iteration $\langle R_\alpha, \dot{S}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$ by setting for each $\beta < \kappa$, $\dot{S}_\beta = \mathrm{C}_{\chi_\beta, \mathfrak{A}_\beta}$, where, $\mathfrak{A}_\beta = \{A \in V \mid A \subseteq (\mu_\beta^{++})_V\}$ and $\sup_{\gamma<\beta} \chi_\gamma < \chi_\beta < \kappa$. By theorem 2.2 R_κ is cone homogeneous.

One final definition is in order before the following claim. If $p \in \mathbb{P}^*_{\vec{E}}$ then set $\kappa(p) = \operatorname{ran} f^p(\kappa)$. If $p = p_0 {}^\frown \cdots {}^\frown p_n \in \mathbb{P}_{\vec{E}}$ then set by recursion $\kappa(p) = \kappa(p_0 {}^\frown \cdots {}^\frown p_{n-1}) {}^\frown \kappa(p_n)$. Note $\kappa(p)$ is the subset of $f_G(\kappa)$ decided by the condition p.

Claim 4.3. *Let $\mathbb{P} = \mathbb{P}_{\vec{E}} * \dot{P}_\kappa * \dot{R}_\kappa$. Assume $\langle p_0, \dot{q}_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1, \dot{r}_1\rangle \in \mathbb{P}$ are conditions such that $s(p_0)$ and $s(p_1)$ are compatible. Then there are stronger conditions, $\langle p_0^*, \dot{q}_0^*, \dot{r}_0^*\rangle \leq \langle p_0, \dot{q}_0, \dot{r}_0\rangle$ and $\langle p_1^*, \dot{q}_1^*, \dot{q}_1^*\rangle \leq \langle p_1, \dot{q}_1, \dot{r}_1\rangle$, such that $\mathbb{P}/\langle p_0^*, \dot{q}_0^*, \dot{r}_0^*\rangle \simeq \mathbb{P}/\langle p_1^*, \dot{q}_1^*, \dot{r}_1^*\rangle$.*

Proof. Since $s(p_0)$ and $s(p_1)$ are compatible there are stronger conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ and Mitchell increasing sequences $\{\vec{e}_i \mid i \leq k\}$ such that $p_0', p_1' \in \prod_{i \leq k} \mathbb{P}_{\vec{e}_i}$ and $\kappa(p_0') = \kappa(p_1')$. By the previous corollary there are direct extensions $p_0^* \leq^* p_0'$ and $p_1^* \leq^* p_1'$ such that $\pi : \mathbb{P}_{\vec{E}}/p_0^* \simeq \mathbb{P}_{\vec{E}}/p_1^*$. Most importantly we have $\pi(\dot{P}_\kappa * \dot{Q}_\kappa) = \dot{P}_\kappa * \dot{Q}_\kappa$ is cone homogeneous. Thus by claim 2.3 we are done. □

Corollary 4.4. *If $\langle p, \dot{q}, \dot{r}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_l)$, then $\langle s(p), 1, 1\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_1, \ldots, \alpha_l)$.*

Proof. We will prove a dense subset of conditions below $\langle s(p), 1, 1\rangle$ force $\varphi(\alpha_0, \ldots, \alpha_l)$. Assume $\langle p^0, \dot{q}^0, \dot{r}^0\rangle \leq \langle s(p), 1, 1\rangle$. Trivially $s(p^0)$ and $s(p)$ are compatible, hence by the previous corollary there are stronger conditions $\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \leq \langle p^0, \dot{q}^0, \dot{r}^0\rangle$ and $\langle p^{1*}, \dot{q}^{1*}, \dot{r}^{1*}\rangle \leq \langle p, \dot{q}, \dot{r}\rangle$ such that $\mathbb{P}/\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \simeq \mathbb{P}/\langle p^{1*}, \dot{q}^{1*}, \dot{r}^{1*}\rangle$. Necessarily $\langle p^{0*}, \dot{q}^{0*}, \dot{r}^{0*}\rangle \Vdash_{\mathbb{P}} \varphi(\alpha_0, \ldots, \alpha_l)$. □

Letting I be R_κ-generic over $V[G][H]$ we get the following from the previous corollary together with claim 2.4.

Corollary 4.5. *$HOD^{V[G][H][I]} \subseteq V[s''G]$.*

Claim 4.6. *In $V_\kappa^{V[G][H][I]}$ all regulars above κ_0 are measurable in $HOD^{V_\kappa^{V[G][H][I]}}$.*

Proof. Since the regulars in the range $[\kappa_0, \kappa)$ are $\{\mu_\alpha \mid \alpha < \kappa\}$, we will be done by showing for each $\alpha < \kappa$ the measure U_α (in V) lifts to a measure in $\mathrm{HOD}^{V_\kappa^{V[G][H][I]}}$. In V, μ_α is measurable. The set $s''\mathbb{P}_{\vec{E}}$ is the plain Radin forcing, hence any measure in V over μ_α lifts trivially to a measure on μ_α in $V[s''G]$. In particular the measure U_α in V lifts to the measure $\bar{\bar{U}}_\alpha$ in $V[s''G]$, which is definable by $\bar{\bar{U}}_\alpha = \{B \in V[s''G] \mid \exists A \in U_\alpha\ A \subseteq B \subseteq \mu_\alpha\}$.

Since $\mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \supseteq V_{(\mu_\alpha^{++})^V}$ we get $U_\alpha \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \subseteq \mathrm{HOD}^{V[G][H][I]} \subseteq V[s''G]$. Let $\bar{U}_\alpha = \{B \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}} \mid \exists A \in U_\alpha\ A \subseteq B \subseteq \mu_\alpha\}$. Then $\bar{U}_\alpha \in \mathrm{HOD}^{V_\kappa^{V[G][H][I]}}$ and $\bar{U}_\alpha \subseteq \bar{\bar{U}}_\alpha$. Necessarily $\bar{U}_\alpha$ is a measure on μ_α. ☐

We get theorem 2 by forcing in $V[G][H][I]$ with $\mathrm{Col}(\omega, {<}\kappa_0)$.

References

[1] Omer Ben-Neria and Spencer Unger. Homogenerous changes in cofinalities with applications to HOD. Preprint.

[2] James Cummings, Sy David Friedman, and Mohammad Golshani. Collapsing the cardinals of hod. *Journal of Mathematical Logic*, 15(02):1550007, 2015.

[3] James Cummings, Sy-David Friedman, Menachem Magidor, Dima Sinapova, and Assaf Rinot. Ordinal definable subsets of singular cardinals. Preprint.

[4] Natash Dobrinen and Sy-David Friedman. Homogeneous iteration and measure one covering relative to HOD. *Archive for Mathematical logic*, 47(7–8):711–718, November 2008. doi:10.1007/s00153-008-0103-5.

[5] Moti Gitik and Menachem Magidor. The Singular Cardinal Hypothesis revisited. In Haim Judah, Winfried Just, and W. Hugh Woodin, editors, *Set theory of the continuum*, volume 26 of *Mathematical Sciences Research Institute publications*, pages 243–279. Springer, 1992.

[6] Menachem Magidor. Changing the cofinality of Cardinals. *Fundamenta Mathematicae*, 99(1):61–71, 1978.

[7] Carmi Merimovich. Extender-based Radin forcing. *Transactions of the American Mathematical Society*, 355:1729–1772, 2003. doi:10.1090/S0002-9947-03-03202-1.

[8] Carmi Merimovich. Supercompact Extender Based Prikry forcing. *Archiv for Mathematical Logic*, 50(5-6):592—601, June 2011. doi:10.1007/s00153-011-0234-y.

[9] Carmi Merimovich. Supercompact extender based Magidor-Radin forcing. *Annals of Pure and Applied Logic*, 168(8):1571–1587, 2017. doi:10.1016/j.apal.2017.02.006.

[10] W. Hugh Woodin. Suitable extender models I. *Journal of Mathematical Logic*, 10(01n02):101–339, 2010. doi:10.1142/S021906131000095X.

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