EXTENDER BASED MAGIDOR-RADIN FORCING

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Abstract. We define the extender based Magidor-Radin forcing notion from a Mitchell increasing sequence of extenders. We prove the basic properties of this forcing.

1. Introduction

The extender based Radin forcing was defined in [5]. It generalizes Radin forcing [9] in the same way as the extender based Prikry forcing notion [2] generalizes Prikry forcing [8]. In [5] the existence of a large enough extender is assumed and a sequence of extenders is generated from it recursively, very much like the way a measure sequence is generated recursively from an elementary embedding in [9] and [1]. In the current work instead of assuming the existence of 'a large extender above them all', we assume the existence of a Mitchell increasing sequence of extenders, and work directly from it. This generalizes [4] where a coherent sequence of extenders was used to defined the extender based Magidor forcing. This, in turn, generalized [7], where a presentation of Radin forcing using a coherent sequence of measuers is given. In some sense the current work is the top-down version of the forcing notion presented in [3].

The general theme of the forcing notion we present is as follows. Given a Mithcell increasing extender sequence, 2^κ is controlled by the size of the extenders, the cofinality of κ is controlled by the length of the sequence, and a club is added to κ so that the power and cofinality of cardinals in the club is controlled by reflections of the extender sequence. In the generic extension κ can become singular, can remain regular, or measurable.

Let us show a typical example. Assume that $\langle E_\xi \mid \xi < \operatorname{o}(\bar{E})\rangle$ is a Mitchell increasing sequence of extenders of length $\operatorname{o}(\bar{E})$, and let j_ξ be the corresponding natural embeddings. Furthermore assume that for each $\xi < \operatorname{o}(\bar{E})$, $|j_\xi(\kappa)| \geq \kappa^{+3}$. Then there is a generic extension preserving all cardinals, in which $2^\kappa = \kappa^{+3}$, and there is a club of κ on which $2^\mu = \mu^{+3}$. Now, if $\operatorname{cf}(\operatorname{o}(\bar{E})) > \kappa^{+3}$ then κ is measurable in the extension. If $\operatorname{cf}(\operatorname{o}(\bar{E})) > \kappa$ then κ is regular in the extension. Finally if $\operatorname{cf}(\operatorname{o}(\bar{E})) = \kappa$ then $\operatorname{cf}\kappa = \omega$, and if $\operatorname{cf}(\operatorname{o}(\bar{E})) < \kappa$ then $\operatorname{cf}\kappa = \operatorname{cf}(\operatorname{o}(\bar{E}))$.

The general theorem we prove is as follows.

Theorem. *Assume $\langle E_\xi \mid \xi < \operatorname{o}(\bar{E})\rangle$ is a Mitchell increasing sequence of short extenders on κ, and $\kappa^+ \leq \epsilon \leq \sup\{j_{E_\xi}(\kappa) \mid \xi < \operatorname{o}(\bar{E})\}$. Then there is a generic extension in which:*

(1) $2^\kappa = |\epsilon|$.

Date: December 20, 2007.
1991 *Mathematics Subject Classification.* Primary 03E35, 03E55.
The writing of this paper was partially supported by the Israel Science Foundation.

(2) *The cofinality of* κ *depends on the cofinality of* $\mathrm{o}(\bar{E})$ *as follows:*

$$\mathrm{cf}(\kappa) = \begin{cases} \mathrm{cf}(\mathrm{o}(\bar{E})) & \mathrm{cf}(\mathrm{o}(\bar{E})) < \kappa, \\ \omega & \mathrm{cf}(\mathrm{o}(\bar{E})) = \kappa, \\ \kappa & \mathrm{cf}(\mathrm{o}(\bar{E})) > \kappa. \end{cases}$$

(3) *If* $\mathrm{cf}(\mathrm{o}(\bar{E})) > |\epsilon|$, *then* κ *is measurable.*

(4) *There is a club* $C \subseteq \kappa$ *such that for each* $\mu \in C$, $\mathrm{cf}\,\mu$ *and* 2^μ *are computed recursively from reflections downward of the sequence* $\langle E_\xi \mid \xi < \mathrm{o}(\bar{E})\rangle$.

The structure of this work is as follows. In section 2 we give the definition of the extender based Prikry forcing notion. We quote the necessary results from [2]. In section 3 we give the definition of the extender based Magidor forcing using two extenders. This section demonstrates in a simpler setting the ideas appearing in later sections. In section 4 the extender based Magidor-Radin forcing is being presented. All the proofs are complete, i.e., no proof demonstrations on, say, 1-point extensions are given.

Our notation is standard. We use the convention that $p \leq q$ for condition p, q in some forcing notion means that p is stronger (i.e., has more information) than q. We assume large cardinals knowledge, namely extenders, measures, and elementary embeddings, in addition to forcing knowledge. Given a set A and a cardinal λ, define $\mathcal{P}_\lambda(A) = \{a \subseteq A \mid |a| < \lambda\}$. A partial order $\langle T, <\rangle$ is a tree if for each $t \in T$ the structure $\langle \{s < t \mid s \in T\}, <\rangle$ is a well order. The ξ-th level of the tree T, $\mathrm{Lev}_\xi(T)$, is the set $\{t \in T \mid \mathrm{ot}\{s < t \mid s \in T\} = \xi\}$. The height of T is the minimal ordinal ξ such that $\mathrm{Lev}_\xi(T) = \emptyset$.

2. Forcing with $\bar{E} = \langle E \rangle$

The forcing defined in this section is the extender based Prikry forcing notion [2]. The form presented here is in essence the one from [6]. The general forcing notion appearing in section 4 is defined by recursion, and the non-recursive step is the extender based Prikry forcing notion.

Assume E is a short extender on κ. Let $j_E : V \to M \simeq \mathrm{Ult}(V, E)$ be the corresponding natural embedding, and let $\kappa^+ \leq \epsilon \leq j_E(\kappa)$.

Definition 2.1. The set of coordinates appearing in a condition is

$$\mathfrak{D} = \{\langle \alpha, E\rangle \mid \kappa \leq \alpha < \epsilon\}.$$

For each $\kappa \leq \alpha < j_E(\kappa)$ we write $\bar{\alpha}$ for $\langle \alpha, E\rangle$. Define the order $<$ on $\mathfrak{D}$ naturally by: $\bar{\alpha} < \bar{\beta} \iff \alpha < \beta$.

Definition 2.2. Assume $d \in \mathcal{P}_{\kappa^+}\,\mathfrak{D}$ and $\bar{\kappa} \in d$. Then $\nu \in \mathrm{OB}(d) \iff$

(1) $\nu : \mathrm{dom}\,\nu \to \kappa$;
(2) $\bar{\kappa} \in \mathrm{dom}\,\nu \subseteq d$;
(3) $|\nu| \leq \nu(\bar{\kappa})$;
(4) $\forall \bar{\alpha}, \bar{\beta} \in \mathrm{dom}\,\nu\ \big(\bar{\alpha} < \bar{\beta} \implies \nu(\bar{\alpha}) < \nu(\bar{\beta})\big)$.

On $\mathrm{OB}(d)$ the partial order $<$ is defined by:

$$\nu < \mu \iff \big(\forall \bar{\alpha} \in \mathrm{dom}\,\nu\ \nu(\bar{\alpha}) < \nu(\bar{\alpha})\big).$$

Definition 2.3. (1) Assume $T \subseteq \mathrm{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$. Then for each $n < \xi$,

$$\mathrm{Lev}_n(T) = \{\langle \nu_0, \ldots, \nu_n\rangle \in \mathrm{OB}(d)^{n+1} \mid \langle \nu_0, \ldots, \nu_n\rangle \in T\},$$

and

$$\mathrm{Suc}_T(\nu_0,\dots,\nu_{n-1}) = \{\mu \in \mathrm{OB}(d) \mid \langle \nu_0,\dots,\nu_{n-1},\mu\rangle \in T\}.$$

For notational convenience let $\mathrm{Suc}_T(\langle\rangle) = \mathrm{Lev}_0(T)$. Assume $\langle\nu\rangle \in T$. Define

$$T_{\langle\nu\rangle} = \{\langle \nu_0,\dots,\nu_{k-1}\rangle \mid k<\xi,\ \langle \nu,\nu_0,\dots,\nu_{k-1}\rangle \in T\},$$

and by recursion when $\langle \nu_0,\dots,\nu_n\rangle \in T$ define

$$T_{\langle \nu_0,\dots,\nu_n\rangle} = (T_{\langle \nu_0,\dots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}.$$

(2) A measure $E(d)$, where $d \in \mathcal{P}_{\kappa^+}\mathfrak{D}$, is defined on $\mathrm{OB}(d)$ as follows:

$$\forall X \subseteq \mathrm{OB}(d)\ \big(X \in E(d) \iff \mathrm{mc}(d) \in j(X)\big),$$

where $\mathrm{mc}(d)$ is defined by

$$\mathrm{mc}(d) = \{\langle j(\bar\alpha),\alpha\rangle \mid \bar\alpha \in d\}.$$

The measure $E^{(n+1)}(d)$ $(n<\omega)$ on $\mathrm{OB}(d)^{n+1}$ is defined by recursion as follows.

$$X \in E^{(n+1)}(d) \iff$$
$$\{\langle \nu_0,\dots,\nu_{n-1}\rangle \in \mathrm{Lev}_{n-1}(X) \mid \mathrm{Suc}_X(\nu_0,\dots,\nu_{n-1}) \in E(d)\} \in E^{(n)}(d),$$

where we set $E^{(0)} = \{\langle\rangle\}$ and consider it a measure on $\mathrm{OB}(d)^0 = \{\langle\rangle\}$. Note that essentially $E^{(1)}(d) = E(d)$. The measure $E^{(\omega)}(d)$ on $\mathrm{OB}(d)^{<\omega}$ is defined by recursion as follows:

$$X \in E^{(\omega)}(d) \iff \forall n<\omega\ \mathrm{Lev}_n(X) \in E^{(n+1)}(d).$$

(3) A set $T \subseteq \mathrm{OB}(d)^{<\xi}$ $(1<\xi\le\omega)$ ordered by end-extension and closed under initial segments is a tree. A tree $T \subseteq \mathrm{OB}(d)^{<\omega}$ is called a d-tree if for each $\langle \nu_0,\dots,\nu_{n-1}\rangle \in T$ $(n<\omega)$ we have $\nu_k < \nu_{k+1}$ $(k<n-1)$, and

$$\forall \langle \nu_0,\dots,\nu_{n-1}\rangle \in T\ \ \mathrm{Suc}_T(\nu_0,\dots,\nu_{n-1}) \in E(d).$$

Note that if T is a d-tree then $T \in E^{(\omega)}(d)$. Moreover, if a tree T belongs to $E^{(\omega)}(d)$ then there is a subtree $S \subseteq T$ which is a d-tree.

(4) Assume $c,d \in \mathcal{P}_{\kappa^+}\mathfrak{D}$, $c \subseteq d$, and T is a tree with elements from $\mathrm{OB}(d)$. Then the projection of T to a a tree with elements from $\mathrm{OB}(c)$ is

$$T \restriction c = \{\langle \nu_0\restriction c,\dots,\nu_n \restriction c\rangle \mid n<\mathrm{ht}(T), \langle \nu_0,\dots,\nu_n\rangle \in T\}.$$

Definition 2.4. The following list of points leads to the definition of $\langle \mathbb{P}_{\langle E\rangle,\epsilon}, \le, \le^*\rangle$.

- A condition f is in the forcing notion $\mathbb{P}^*_{\langle E\rangle,\epsilon}$ if $f : d \to {}^{<\omega}\kappa$ is a function such that:
  (1) $\bar\kappa \in d \in \mathcal{P}_{\kappa^+}\mathfrak{D}$;
  (2) For each $\bar\alpha \in d$, $f(\bar\alpha) = \langle f_0(\bar\alpha),\dots,f_{k-1}(\bar\alpha)\rangle$ is an increasing sequence in κ.
- Assume $f,g \in \mathbb{P}^*_{\langle E\rangle,\epsilon}$. Then f is an extension of g ($f \le^*_{\mathbb{P}^*_{\langle E\rangle,\epsilon}} g$) if $f \supseteq g$.
- For $f \in \mathbb{P}^*_{\langle E\rangle,\epsilon}$ we write $\mathrm{mc}(f)$ and $E(f)$ for $\mathrm{mc}(\mathrm{dom} f)$ and $E(\mathrm{dom} f)$, respectively. We will also say that a tree is an f-tree if it is a $\mathrm{dom} f$-tree.
- Assume $f \in \mathbb{P}^*_{\langle E\rangle,\epsilon}$ and $\nu \in \mathrm{OB}(f)$. The condition $g = f_{\langle\nu\rangle} \in \mathbb{P}^*_{\langle E\rangle,\epsilon}$ is defined by:

(1) $\operatorname{dom} g = \operatorname{dom} f$;
(2) For each $\bar\alpha \in \operatorname{dom} g$,

$$g(\bar\alpha) = \begin{cases} f(\bar\alpha) ^\frown \langle \nu(\bar\alpha) \rangle & \bar\alpha \in \operatorname{dom} \nu,\ \nu(\bar\alpha) > f_{|f(\bar\alpha)|-1}(\bar\alpha), \\ f(\bar\alpha) & \text{Otherwise.} \end{cases}$$

Observe that if $X \in E(f)$ then there is a subset $Y \subseteq X$ such that $Y \in E(f)$, and for each $\nu \in Y$ and $\bar\alpha \in \operatorname{dom} \nu$, $\nu(\bar\alpha) > f_{|f(\bar\alpha)|-1}(\bar\alpha)$.

- A condition p in the forcing notion $\mathbb{P}_{\langle E \rangle, \epsilon}$ is of the form $\langle f, A \rangle$ where
  (1) $f \in \mathbb{P}^*_{\langle E \rangle, \epsilon}$;
  (2) A is an f-tree such that for each $\langle \nu \rangle \in A$ and $\bar\alpha \in \operatorname{dom} \nu$,
  $$f_{|f(\bar\alpha)|-1}(\bar\alpha) < \nu(\bar\alpha).$$
  We write f^p, A^p, and $\operatorname{mc}(p)$, for f, A, and $\operatorname{mc}(f)$.
- Let $p, q \in \mathbb{P}_{\langle E \rangle, \epsilon}$. We say that p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_{\langle E \rangle, \epsilon}} q$) if
  (1) $f^p \leq^*_{\mathbb{P}^*_{\langle E \rangle, \epsilon}} f^q$;
  (2) $A^p \restriction \operatorname{dom} f^q \subseteq A^q$.
- Let $q \in \mathbb{P}_{\langle E \rangle, \epsilon}$ and $\langle \nu \rangle \in A^q$. Define the one point extension of q by $\langle \nu \rangle$ to be $p = q_{\langle \nu \rangle} \in \mathbb{P}_{\langle E \rangle, \epsilon}$ where:
  (1) $f^p = f^q_{\langle \nu \rangle}$;
  (2) $A^p = A^q_{\langle \nu \rangle}$.
  Define $q_{\langle \nu_0, \dots, \nu_k \rangle}$ recursively as $(q_{\langle \nu_0, \dots, \nu_{k-1} \rangle})_{\langle \nu_k \rangle}$, where $\nu_0 < \dots < \nu_k$.
  Whenever the notation $\langle \nu_0, \dots, \nu_{n-1} \rangle$ is used, where $\nu_k \in \operatorname{OB}(d)$ ($k < n$), it is implicitly assumed that $\nu_{k-1} < \nu_k$ ($k < n$).
- Let $p, q \in \mathbb{P}_{\langle E \rangle, \epsilon}$. Then p is an extension of q ($p \leq_{\mathbb{P}_{\langle E \rangle, \epsilon}} p$) if there is $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^q$ such that $p \leq^*_{\mathbb{P}_{\langle E \rangle, \epsilon}} q_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$.

Before quoting results about this forcing we give some pictorial representation of the forcing notion.

The weakest condition in $\mathbb{P}_{\langle E \rangle, \epsilon}$ is $\langle \langle \langle \kappa, E \rangle, \langle \rangle \rangle, T \rangle$, where T is the 'full' tree, i.e., $\operatorname{Lev}_0(T) = \{\kappa\} \times \kappa$, and the higher levels are similar. Let us call it q. We present q graphically in figure 1.

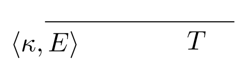

Figure 1. The weakest condition $q \in \mathbb{P}_{\langle E \rangle, \epsilon}$.

Let $\kappa < \alpha < j_E(\kappa)$. The weakest condition in $\mathbb{P}_{\langle E \rangle, \epsilon}$ mentioning κ and α is

$$\langle \{ \langle \langle \kappa, E \rangle, \langle \rangle \rangle, \langle \langle \alpha, E \rangle, \langle \rangle \rangle \}, A \rangle.$$

Let us call it p. Then $p \leq^* q$. Note that the form of an arbitrary $\langle \nu \rangle \in A$ is

$$\{ \langle \langle \kappa, E \rangle, \tau \rangle, \langle \langle \alpha, E \rangle, \mu \rangle \}.$$

Note that $\{ \nu(\langle \kappa, E \rangle) \mid \langle \nu \rangle \in A \} \in E(\kappa)$ and $\{ \nu(\langle \alpha, E \rangle) \mid \langle \nu \rangle \in A \} \in E(\alpha)$. Figure 2 shows p graphically.

Let $\langle \nu_0 \rangle \in A$. The weakest 1-point extension of p using $\langle \nu_0 \rangle$, i.e. $p_{\langle \nu_0 \rangle}$, is

$$\langle \{ \langle \langle \kappa, E \rangle, \langle \nu_0(\langle \kappa, E \rangle) \rangle \rangle, \langle \langle \alpha, E \rangle, \langle \nu_0(\langle \alpha, E \rangle) \rangle \rangle \}, A_{\langle \nu_0 \rangle} \rangle.$$

$$\begin{array}{ccc} \hline \langle\kappa,E\rangle & \langle\alpha,E\rangle & A \end{array}$$

FIGURE 2. The weakest condition mentioning α and κ, $p \leq^* q$.

$$\begin{array}{ccc} \tau_0 & \mu_0 & \\ \hline \langle\kappa,E\rangle & \langle\alpha,E\rangle & A_{\langle\nu_0\rangle} \end{array}$$

FIGURE 3. 1-point extension of p, $p_{\langle\nu_0\rangle}$.

Letting $\nu_0(\langle\kappa, E\rangle) = \tau_0$ and $\nu_0(\langle\alpha, E\rangle) = \mu_0$, the condition is shown graphically in figure 3.

Let $\langle\nu_1\rangle \in A_{\langle\nu_0\rangle}$. The weakest 2-point extension of p using $\langle\nu_0, \nu_1\rangle$ (which is the same as the weakest 1-point extension of $p_{\langle\nu_0\rangle}$ using $\langle\nu_1\rangle$) is

$$\langle\{\langle\langle\kappa, E\rangle, \langle\nu_0(\langle\kappa, E\rangle), \nu_1(\langle\kappa, E\rangle)\rangle\rangle, \langle\langle\alpha, E\rangle, \langle\nu_0(\langle\alpha, E\rangle), \nu_1(\langle\alpha, E\rangle)\rangle\rangle\}, A_{\langle\nu_0,\nu_1\rangle}\rangle.$$

Assuming $\nu_1(\langle\kappa, E\rangle) = \tau_1$ and $\nu_1(\langle\alpha, E\rangle) = \mu_1$, $p_{\langle\nu_0,\nu_1\rangle}$ is shown graphically in figure 4.

$$\begin{array}{ccc} \tau_1 & \mu_1 & \\ \tau_0 & \mu_0 & \\ \hline \langle\kappa,E\rangle & \langle\alpha,E\rangle & A_{\langle\nu_0,\nu_1\rangle} \end{array}$$

FIGURE 4. 2-point extension of p, $p_{\langle\nu_0,\nu_1\rangle} = (p_{\langle\nu_0\rangle})_{\langle\nu_1\rangle}$.

Let $\kappa < \gamma < \epsilon$ such that $\gamma \neq \alpha$. We want $r \in \mathbb{P}_{\langle E\rangle,\epsilon}$ to be a Prikry extension of $p_{\langle\nu_0,\nu_1\rangle}$ mentioning γ. We note that there is a complete freedom in choosing $f^r(\langle\gamma, E\rangle)$. We *can* have $f^r(\langle\kappa, E\rangle) = \langle\rangle$, but this time we prefer something else. Thus let $\zeta_0 < \kappa$. We let the extension be

$$\langle\{\langle\langle\kappa, E\rangle, \langle\nu_0(\langle\kappa, E\rangle), \nu_1(\langle\kappa, E\rangle)\rangle\rangle, \langle\langle\alpha, E\rangle, \langle\nu_0(\langle\alpha, E\rangle), \nu_1(\langle\alpha, E\rangle)\rangle\rangle, \\ \langle\langle\gamma, E\rangle, \langle\zeta_0\rangle\rangle\}, B\rangle.$$

This B must satisfy the following.

(1) $\{\nu \restriction \{\langle\kappa, E\rangle, \langle\alpha, E\rangle\} \mid \nu \in B\} \in E(\{\kappa, \alpha\})$.
(2) $\{\nu \restriction \{\langle\kappa, E\rangle, \langle\alpha, E\rangle\} \mid \nu \in B\} \subseteq A$.
(3) $\{\nu(\langle\gamma, E\rangle) \mid \nu \in A\} \in E(\gamma)$.

Graphically r appears in Figure 5.

$$\begin{array}{cccc} \tau_1 & \mu_1 & & \\ \tau_0 & \mu_0 & \zeta_0 & \\ \hline \langle\kappa,E\rangle & \langle\alpha,E\rangle & \langle\gamma,E\rangle & B \end{array}$$

FIGURE 5. $r \leq^* p_{\langle\nu_0,\nu_1\rangle}$.

Take $\langle \nu_2 \rangle \in B$ such that $\langle \gamma, E\rangle \notin \operatorname{dom} \nu_2$.

$$\begin{aligned} r_{\langle \nu_2 \rangle} = \langle \{ & \langle \langle \kappa, E\rangle, \langle \nu_0(\langle \kappa, E\rangle), \nu_1(\langle \kappa, E\rangle), \nu_2(\langle \kappa, E\rangle)\rangle\rangle, \\ & \langle \langle \alpha, E\rangle, \langle \nu_0(\langle \alpha, E\rangle), \nu_1(\langle \alpha, E\rangle), \langle \alpha, E\rangle, \nu_2(\langle \alpha, E\rangle)\rangle\rangle, \\ & \langle \langle \gamma, E\rangle, \langle \zeta_0\rangle\rangle\}, B_{\langle \nu_2\rangle}\rangle, \end{aligned}$$

which appears graphically in figure 6, where $\tau_2 = \nu_2(\bar{\kappa})$ and $\mu_2 = \nu_2(\bar{\alpha})$.

τ_2	μ_2		
τ_1	μ_1		
τ_0	μ_0	ζ_0	
$\langle \kappa, E\rangle$	$\langle \alpha, E\rangle$	$\langle \gamma, E\rangle$	$B_{\langle \nu_2\rangle}$

Figure 6. $r_{\langle \nu_2\rangle}$.

Take $\langle \nu_3 \rangle \in B$ such that $\operatorname{dom} \nu_3 = \operatorname{dom} f^r$. Then

$$\begin{aligned} r_{\langle \nu_2, \nu_3 \rangle} = \langle \{ & \langle \langle \kappa, E\rangle, \langle \nu_0(\langle \kappa, E\rangle), \nu_1(\langle \kappa, E\rangle), \nu_2(\langle \kappa, E\rangle), \nu_3(\langle \kappa, E\rangle)\rangle\rangle, \\ & \langle \langle \alpha, E\rangle, \langle \nu_0(\langle \alpha, E\rangle), \nu_1(\langle \alpha, E\rangle), \nu_2(\langle \alpha, E\rangle), \nu_3(\langle \alpha, E\rangle)\rangle\rangle, \\ & \langle \langle \gamma, E\rangle, \langle \zeta_0, \nu_3(\langle \gamma, E\rangle)\rangle\rangle\}, B_{\langle \nu_2, \nu_3\rangle}\rangle, \end{aligned}$$

which appears graphically in figure 7, where $\tau_3 = \nu_3(\langle \kappa, E\rangle)$, $\mu_3 = \nu_3(\langle \alpha, E\rangle)$, and $o_3 = \nu_3(\langle \gamma, E\rangle)$.

τ_3	μ_3		
τ_2	μ_2		
τ_1	μ_1	o_3	
τ_0	μ_0	ζ_0	
$\langle \kappa, E\rangle$	$\langle \alpha, E\rangle$	$\langle \gamma, E\rangle$	$B_{\langle \nu_2, \nu_3\rangle}$

Figure 7. $r_{\langle \nu_2, \nu_3\rangle}$.

Lemma 2.5. *(Gitik-Magidor)*

1. $\langle \mathbb{P}_{\langle E\rangle, \epsilon}, \leq^* \rangle$ *is* κ*-closed.*
2. $\langle \mathbb{P}_{\langle E\rangle, \epsilon}, \leq, \leq^* \rangle$ *is of Prikry type.*
3. $\mathbb{P}_{\langle E\rangle, \epsilon}$ *has the* κ^{++}*-c.c.*
4. $\Vdash_{\mathbb{P}_{\langle E\rangle, \epsilon}} \ulcorner (\kappa^+)_V$ *remains cardinal*$\urcorner$.

Theorem 2.6. *(Gitik-Magidor) Let* G *be* $\mathbb{P}_{\langle E\rangle, \epsilon}$*-generic. Then:*

1. V *and* $V[G]$ *have the same cardinals and the same bounded subsets of* κ*.*
2. $\operatorname{cf}^{V[G]}(\kappa) = \omega$ *and* $(\kappa^\omega = |\epsilon|)^{V[G]}$*.*
3. $\forall \mu \in [\kappa, \kappa^\omega)\ \mu^\omega = \kappa^\omega$*.*
4. *Outside of* $[\kappa, \kappa^\omega)$ *the GCH is holds.*

The following definition is needed in order to bootstrap the recursive definition of the Magidor-Radin forcing in later sections.

Definition 2.7. Assume $\langle e_i \mid i < n\rangle$ $(n < \omega)$ is a sequence of extenders such that $e_i \in V_{\operatorname{crit}(e_{i+1})}$. The product forcing notion $\mathbb{P} = \prod_{i<n} \mathbb{P}_{e_i}$ is defined by applying the definitions of the Prikry with extenders forcing notion coordinatewise. That is for each $\langle p_i \mid i < n\rangle, \langle q_i \mid i < n\rangle \in \mathbb{P}$,

$$\langle p_i \mid i < n\rangle \leq_{\mathbb{P}} \langle q_i \mid i < n\rangle \iff \forall i < n\ p_i \leq q_i,$$

and

$$\langle p_i \mid i < n\rangle \leq^*_{\mathbb{P}} \langle q_i \mid i < n\rangle \iff \forall i < n\ p_i \leq^* q_i.$$

For $p = \langle p_i \mid i < n\rangle \in \mathbb{P}$ we use the notation $p_{\leftarrow} = p_0{}^\frown \cdots {}^\frown p_{n-2}$ and $p_{\rightarrow} = p_{n-1}$. Assume $\langle \nu\rangle \in A^{p_{\rightarrow}}$. Define the condition $p_{\langle\nu\rangle}$ recursively as follows.

$$p_{\langle\nu\rangle} = p_{\leftarrow}{}^\frown p_{\rightarrow\langle\nu\rangle}.$$

Note that with $p_{\leftarrow}$ and $p_{\rightarrow}$ defined we have for each $p, q \in \mathbb{P}$,

$$p \leq p \iff \big(p_{\leftarrow} \leq q_{\leftarrow} \text{ and } p_{\rightarrow} \leq q_{\rightarrow}\big),$$

and

$$p \leq^* p \iff \big(p_{\leftarrow} \leq^* q_{\leftarrow} \text{ and } p_{\rightarrow} \leq^* q_{\rightarrow}\big).$$

It is a standard fact that $\langle \mathbb{P}, \leq_{\mathbb{P}}, \leq^*_{\mathbb{P}}\rangle$ is a Prikry type forcing notion. Since the extenders e_i are disjoint, factoring of $\mathbb{P}$ is easily achieved, thus a generic extension by $\mathbb{P}$ can be analyzed by inspecting generic extensions by each factor $\mathbb{P}_{e_i}$.

3. Forcing with $\bar{E} = \langle E_0, E_1\rangle$ $(E_0 \lhd E_1)$

The aim of this section is to introduce the techniques used in section 4, where the general forcing notion is defined. Thus we show the step immediately following the extender based Prikry forcing notion.

Assume E_0 and E_1 are (short) extenders on κ such that $E_0 \lhd E_1$ (i.e., $E_0 \in \operatorname{Ult}(V, E_1)$). Let $j_{E_k} : V \to M_k \simeq \operatorname{Ult}(V, E_k)$ $(k < 2)$ be the corresponding natural elementary embeddings. Note that $j_{E_0}(\kappa) < j_{E_1}(\kappa)$. Let $\kappa^+ \leq \epsilon \leq j_{E_1}(\kappa)$.

Definition 3.1. An extender sequence $\bar{\nu}$ has one of the following three forms.

(1) $\langle \tau\rangle$ where $\tau \in \mathrm{On}$.
(2) $\langle \tau, e_0\rangle$ where e_0 is an extender such that $\operatorname{crit} e_0 \leq \tau < j_{e_0}(\operatorname{crit}(e_0))$.
(3) $\langle \tau, e_0, e_1\rangle$ where e_0 and e_1 are extenders such that $e_0 \lhd e_1$, $\operatorname{crit}(e_0) = \operatorname{crit}(e_1)$, and $\operatorname{crit}(e_0) \leq \tau < j_{e_0}(\operatorname{crit}(e_0))$.

We write $\bar{\nu}_0$ for τ. Each extender sequence $\bar{\nu}$ has an order $\mathrm{o}(\bar{\nu})$ defined by:

$$\mathrm{o}(\bar{\nu}) = \begin{cases} 0 & \text{if } \bar{\nu} = \langle \tau\rangle, \\ 1 & \text{if } \bar{\nu} = \langle \tau, e_0\rangle, \\ 2 & \text{if } \bar{\nu} = \langle \tau, e_0, e_1\rangle. \end{cases}$$

On the extender sequences a partial order $<$ is defined by: $\bar{\nu} < \bar{\mu} \iff \bar{\nu}_0 < \bar{\mu}_0$.

Definition 3.2. The set of coordinates $\mathfrak{D}$ used in a condition is defined to be $\mathfrak{D}_0 \cup \mathfrak{D}_1$, where

$$\mathfrak{D}_0 = \{\langle \alpha, E_0, E_1\rangle \mid \kappa \leq \alpha < \min(j_{E_0}(\kappa), \epsilon)\},$$

and

$$\mathfrak{D}_1 = \{\langle \alpha, E_1 \rangle \mid j_{E_0}(\kappa) \leq \alpha < \epsilon\}.$$

Note that $\mathfrak{D}_1$ might be empty (if $\epsilon \leq j_{E_0}(\kappa)$). For each $\kappa \leq \alpha < j_{E_0}(\kappa)$ we write $\bar{\alpha}$ for $\langle \alpha, E_0, E_1 \rangle$, and for each $j_{E_0}(\kappa) \leq \alpha < j_{E_1}(\kappa)$ we write $\bar{\alpha}$ for $\langle \alpha, E_1 \rangle$.

On $\mathfrak{D}$ the order $<$ is defined by: $\bar{\alpha} < \bar{\beta} \iff \alpha < \beta$.

Definition 3.3. The set of ranges $\mathfrak{R}$ appearing in conditions is set to be $\mathfrak{R}_0 \cup \mathfrak{R}_1$, where

$$\mathfrak{R}_0 = \{\langle \tau \rangle \mid \tau < \kappa\},$$

and

$$\mathfrak{R}_1 = \{\langle \tau, e \rangle \mid e \text{ is an extender}, \ \operatorname{crit}(e) \leq \tau < j_e(\operatorname{crit}(e)) < \kappa\}.$$

On $\mathfrak{R}$ the partial order $<$ is defined by $\bar{\nu} < \bar{\mu} \iff \bar{\nu}_0 < \bar{\mu}_0$.

Definition 3.4. Assume $d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$. Then $\nu \in \operatorname{OB}(d) \iff$

1. $\nu : \operatorname{dom} \nu \to \mathfrak{R}$;
2. $\bar{\kappa} \in \operatorname{dom} \nu \subset d$;
3. $|\nu| \leq \nu(\bar{\kappa})$;
4. $\forall \bar{\alpha} \in \operatorname{dom} \nu \ \big(\mathrm{o}(\nu(\bar{\alpha})) < \mathrm{o}(\bar{\alpha})\big)$;
5. If $\mathrm{o}(\nu(\bar{\kappa})) = 0$ then for each $\bar{\alpha} \in \operatorname{dom} \nu$, $\alpha < j_{E_0}(\kappa)$ and $\mathrm{o}(\nu(\bar{\alpha})) = 0$. If $\mathrm{o}(\nu(\bar{\kappa})) = 1$ then $\nu(\bar{\kappa}) = \langle \tau, e_0 \rangle$, where $\tau = \operatorname{crit}(e_0)$, and for each $\bar{\alpha} \in \operatorname{dom} \nu$, if $\alpha \in (\kappa, j_{E_0}(\kappa))$ then $\nu(\bar{\alpha}) = \langle \rho, e_0 \rangle$ for some $\rho \in (\tau, j_{e_0}(\tau))$, and if $\alpha \in [j_{E_0}(\kappa), j_{E_1}(\kappa))$ then $\nu = \langle \rho \rangle$ for some $\rho \in [j_{e_0}(\tau), j_{e_1}(\tau))$.
6. $\forall \bar{\alpha}, \bar{\beta} \in \operatorname{dom} \nu \ \big(\bar{\alpha} < \bar{\beta} \implies \nu(\bar{\alpha}) < \nu(\bar{\beta})\big)$.

On $\operatorname{OB}(d)$ the partial order $<$ is defined by:

$$\nu < \mu \iff \big(\forall \bar{\alpha} \in \operatorname{dom} \nu \ \nu(\bar{\alpha}) < \mu(\bar{\alpha})\big).$$

Definition 3.5.
1. Assume $T \subseteq \operatorname{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$. Define $\operatorname{Lev}_n(T)$, $\operatorname{Suc}_T(\nu_0, \ldots, \nu_{n-1})$, and $T_{\langle \nu_0, \ldots, \nu_{n-1} \rangle}$ as in definition 2.3 taking into consideration the current definition of $\operatorname{OB}(d)$.
2. The measures $E_0(d)$ and $E_1(d)$ are defined on $\operatorname{OB}(d)$ as follows:

$$\forall X \subseteq \operatorname{OB}(d) \ \big(X \in E_0(d) \iff \operatorname{mc}_0(d) \in j_{E_0}(X)\big),$$

and

$$\forall X \subseteq \operatorname{OB}(d) \ \big(X \in E_1(d) \iff \operatorname{mc}_1(d) \in j_{E_1}(X)\big),$$

where

$$\operatorname{mc}_0(d) = \{\langle j_{E_0}(\bar{\alpha}), R_0(\bar{\alpha}) \rangle \mid \bar{\alpha} \in d, \ \alpha < j_0(\kappa)\},$$
$$\operatorname{mc}_1(d) = \{\langle j_{E_1}(\bar{\alpha}), R_1(\bar{\alpha}) \rangle \mid \bar{\alpha} \in d\},$$

and the functions R_0 and R_1 are defined by:

$$\forall \kappa \leq \alpha < j_{E_0}(\kappa) \ R_0(\bar{\alpha}) = \langle \alpha \rangle,$$

and

$$\forall \kappa \leq \alpha < j_{E_1}(\kappa)\ R_1(\bar{\alpha}) = \begin{cases} \langle \alpha, E_0 \rangle & \kappa \leq \alpha < j_{E_0}(\kappa), \\ \langle \alpha \rangle & j_{E_0}(\kappa) \leq \alpha < j_{E_1}(\kappa). \end{cases}$$

Name the intersection of the measures $E(d)$. That is

$$E(d) = E_0(d) \cap E_1(d).$$

(3) Since there are two basic measures $E_0(d)$ and $E_1(d)$ on $\mathrm{OB}(d)$, there are several product measures possible. The basic property of a tree $T \subseteq \mathrm{OB}(d)^{n+1}$ we need is that the splittings are in a big set. That is for each $k \leq n$,

$$\forall \langle \nu_0, \dots, \nu_{k-1} \rangle \in T\ \exists i < 2\ \mathrm{Suc}_T(\nu_0, \dots, \nu_{k-1}) \in E_i(d).$$

Thus to characterize the product measure a function $\iota : T \to 2$ is needed such that for each $k \leq n$,

$$\forall \langle \nu_0, \dots, \nu_{k-1} \rangle \in T\ \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{k-1}) \in E_{\iota(\nu_0, \dots, \nu_{k-1})}(d).$$

Observe that $() \in \mathrm{dom}\, \iota$. Note that by removing a measure zero set from T the function ι is the constant function on each level of T. Thus a product measure on $\mathrm{OB}(d)^{n+1}$ is characterized by a sequence from ${}^{n+1}2$. Thus define by recursion the product measure $E_{(\bar{\iota})}(d)$ on $\mathrm{OB}(d)^{n+1}$, where $\bar{\iota} = \langle \iota_0, \dots, \iota_n \rangle$ is the characteristic of the measure as follows:

$$X \in E_{(\bar{\iota})}(d) \iff \{ \langle \nu_0, \dots, \nu_{n-1} \rangle \in \mathrm{Lev}_{n-1}(X) \mid \mathrm{Suc}_X(\nu_0, \dots, \nu_{n-1}) \in E_{(\iota_n)}(d) \} \in E_{(\iota_0, \dots, \iota_{n-1})}(d),$$

where we set $E_{()} = \{\langle\rangle\}$ and consider it a measure on $\mathrm{OB}(d)^0 = \{\langle\rangle\}$. Note that essentially $E_{(\iota)}(d) = E_\iota(d)$, where $\iota \in 2$. For $\bar{\iota} \in {}^{\omega}2$, the measure $E_{(\bar{\iota})}(d)$ on $\mathrm{OB}(d)^{<\omega}$ is defined by recursion as follows:

$$X \in E_{(\bar{\iota})}(d) \iff \forall n < \omega\ \ \mathrm{Lev}_n(X) \in E_{(\bar{\iota} \restriction (n+1))}(d).$$

The intersection of all the measures on $\mathrm{OB}(d)^n$ is named $E^{(n)}(d)$, i.e.,

$$E^{(n)}(d) = \bigcap \{ E_{(\bar{\iota})}(d) \mid \bar{\iota} \in {}^{n}2 \},$$

and the intersection of all the measures on $\mathrm{OB}(d)^{<\omega}$ is named $E^{(\omega)}(d)$, i.e.,

$$E^{(\omega)}(d) = \bigcap \{ E_{(\bar{\iota})}(d) \mid \bar{\iota} \in {}^{<\omega}2 \}.$$

Note that $E^{(1)}(d)$ and $E(d)$ are essentialy the same filter.

(4) A set $T \subseteq \mathrm{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$ ordered by end-extension is called a tree if it is closed under initial segments. A tree $T \subseteq \mathrm{OB}(d)^{<\omega}$ is called a d-tree if for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$ $(n < \omega)$ we have $\nu_{k-1} < \nu_k$ $(k < n)$, and

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in T\ \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{n-1}) \in E(d).$$

Note that if T is a d-tree then $T \in E^{(\omega)}(d)$. Hence for each $n < \omega$, $\mathrm{Lev}_n(T) \in E^{(n+1)}(d)$. Note also that if T is a tree such that $T \in E^{(\omega)}(d)$, then we can find a subtree $S \subseteq T$ such that S is a d-tree.

(5) Assume $c, d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$, $c \subseteq d$, and T is a tree with elements from $\mathrm{OB}(d)$. Then the projection of T to a a tree with elements from $\mathrm{OB}(c)$ is

$$T \restriction c = \{\langle \nu_0 \restriction c, \dots, \nu_n \restriction c\rangle \mid n < \mathrm{ht}(T), \langle \nu_0, \dots, \nu_n\rangle \in T\}.$$

Definition 3.6. The following list of points leads to the definition of $\langle \mathbb{P}_{\langle E_0, E_1\rangle, \epsilon}, \leq, \leq^*\rangle$.

- A condition f is in the forcing notion $\mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$, if $f : d \to {}^{<\omega}\mathfrak{R}$ is a function such that:
  (1) $\bar{\kappa} \in d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$;
  (2) For each $\bar{\alpha} \in d$, $f(\bar{\alpha}) = \langle f_0(\bar{\alpha}), \dots, f_{k-1}(\bar{\alpha})\rangle$ is an increasing sequence in $\mathfrak{R}$.
  (3) $\forall \bar{\alpha} \in d\ \forall i < |f(\bar{\alpha})|\ \big(\mathrm{o}(f_i(\bar{\alpha})) < \mathrm{o}(\bar{\alpha})\big)$.
  (4) For each $\bar{\alpha} \in d$, $\langle \mathrm{o}(f_i(\bar{\alpha})) \mid i < |f(\bar{\alpha})|\rangle$ is non-increasing.
- Assume $f, g \in \mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$,. Then f is an extension of g ($f \leq^*_{\mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}}, g$) if $f \supseteq g$.
- By $\mathrm{OB}(f)$, $\mathrm{mc}(f)$, $E(f)$, and f-tree, where $f \in \mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$,, we refer toi $\mathrm{OB}(\operatorname{dom} f)$, $\mathrm{mc}(\operatorname{dom} f)$, $E(\operatorname{dom} f)$, and $\operatorname{dom} f$-tree.
- Assume $f \in \mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$, and $\nu \in \mathrm{OB}(f)$. We define $f_{\langle \nu\rangle}$ according to the form of $\nu(\bar{\kappa})$. If $\mathrm{o}(\nu(\bar{\kappa})) = 0$ then $g = f_{\langle\nu\rangle} \in \mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$, is defined by:
  (1) $\operatorname{dom} g = \operatorname{dom} f$;
  (2) For each $\bar{\alpha} \in \operatorname{dom} g$,

$$g(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) {}^\frown \langle \nu(\bar{\alpha})\rangle & \bar{\alpha} \in \operatorname{dom} \nu,\ \nu(\bar{\alpha}) > f_{|f(\bar{\alpha})|-1}(\bar{\alpha}), \\ f(\bar{\alpha}) & \text{Otherwise.} \end{cases}$$

  If $\mathrm{o}(\nu(\bar{\kappa})) = 1$ then $g_{\leftarrow} {}^\frown g_{\rightarrow} = f_{\langle\nu\rangle}$ is defined by:
  (1) $\operatorname{dom} g_{\rightarrow} = \operatorname{dom} f$;
  (2) For each $\bar{\alpha} \in \operatorname{dom} g_{\rightarrow}$,

$$g(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) \restriction k {}^\frown \langle \nu(\bar{\alpha})\rangle & \bar{\alpha} \in \operatorname{dom} \nu,\ \nu(\bar{\alpha}) > f_{|f(\bar{\alpha})|-1}(\bar{\alpha}), \\ f(\bar{\alpha}) & \text{Otherwise,} \end{cases}$$

  where

$$(*) \qquad k = \min\{l \leq |f^q(\bar{\alpha})| \mid \forall l \leq i < |f^q(\bar{\alpha})|\ \ \mathrm{o}(f^q_i(\bar{\alpha})) < \mathrm{o}(\nu(\bar{\alpha}))\}.$$

  Note that k is defined so that $\langle \mathrm{o}(f^q_i(\bar{\alpha})) \mid i < k\rangle {}^\frown \langle \mathrm{o}(\nu(\bar{\alpha}))\rangle$ will be non-increasing;
  (3) $\operatorname{dom} g_{\leftarrow} = \{\nu(\bar{\alpha}) \mid \bar{\alpha} \in \operatorname{dom} \nu,\ \mathrm{o}(\nu(\bar{\alpha})) = 1\}$;
  (4) For each $\bar{\alpha} \in \operatorname{dom} \nu$ such that $\mathrm{o}(\nu(\bar{\alpha})) = 1$, $g_{\leftarrow}(\nu(\bar{\alpha})) = f(\bar{\alpha}) \restriction (|f^q(\bar{\alpha})| \setminus k)$, where k is defined by (*).

  Observe that one can use the definition of the case $\mathrm{o}(\nu(\bar{\kappa})) = 1$ also for the case $\mathrm{o}(\nu(\bar{\kappa})) = 0$. One gets in this case $g_{\leftarrow} = \emptyset$.
- A condition p in the forcing notion $\mathbb{P}_{\langle E_0, E_1\rangle}, \epsilon \to$ is of the form $\langle f, A\rangle$ where
  (1) $f \in \mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}$,;
  (2) A is an f-tree.

  We write f^p, A^p, $\mathrm{mc}_0(p)$, and $\mathrm{mc}_1(p)$, for f, A, $\mathrm{mc}_0(f)$, and $\mathrm{mc}_1(f)$.
- Let $p, q \in \mathbb{P}_{\langle E_0, E_1\rangle}, \epsilon \to$. Then p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_{\langle E_0, E_1\rangle}, \epsilon \to} q$) if
  (1) $f^p \leq^*_{\mathbb{P}^*_{\langle E_0, E_1\rangle, \epsilon}}, f^q$;

(2) $A^p \restriction \operatorname{dom} f^q \subseteq A^q$.

- A condition p in the forcing notion $\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$ is of the form $p_{\leftarrow} \frown p_{\rightarrow}$ where
  (1) $p_{\rightarrow} \in \mathbb{P}_{\langle E_0,E_1\rangle}, \epsilon \rightarrow$;
  (2) $p_{\leftarrow}$ is a condition in a product of extender base Prikry forcing notions (as defined in 2.7) which is in V_κ. I.e., $p_{\leftarrow} \in \prod_{i<n} \mathbb{P}_{e_i}$, where $\langle e_i \mid i < n\rangle$ $(n < \omega)$ are extenders such that $e_i \in V_{\operatorname{crit}(e_{i+1})}$ and $e_{n-1} \in V_\kappa$.

  Define recursively f^p to be $f^{p_\leftarrow} \frown f^{p_\rightarrow}$. We write also $f^p_\leftarrow$ and $f^p_\rightarrow$ for $f^{p_\leftarrow}$ and $f^{p_\rightarrow}$.
- Let $p, q \in \mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$. We say that p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}} q$) if:
  (1) $p_\rightarrow \leq^*_{P_{\langle E_0,E_1\rangle}\rightarrow} q_\rightarrow$.
  (2) $p_\leftarrow \leq^* q_\leftarrow$. (This partial order is defined in 2.7.)
- Assume $q \in \mathbb{P}_{\langle E_0,E_1\rangle}, \epsilon \rightarrow$ and $\langle \nu \rangle \in A^{q_\rightarrow}$. The one point extension of q by $\langle \nu \rangle$ is a condition $p = q_{\langle \nu \rangle} \in \mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$ of the form $p_\rightarrow$ (if $\operatorname{o}(\nu(\bar\kappa)) = 0$) or $p_\leftarrow \frown p_\rightarrow$ ($\operatorname{o}(\nu(\bar\kappa)) = 1$) where $p_\rightarrow \in \mathbb{P}_{\langle E_0,E_1\rangle}, \epsilon \rightarrow$ and $p_\leftarrow \in \mathbb{P}_{\langle e_0\rangle}$, if applicable, are defined as follows. If $\operatorname{o}(\nu(\bar\kappa)) = 0$ then:
  (1) $f^p = f^q_{\langle \nu \rangle}$;
  (2) $A^p = A^q_{\langle \nu \rangle}$.

  If $\operatorname{o}(\nu(\bar\kappa)) = 1$ then:
  (1) $f^p = f^q_{\langle \nu \rangle}$;
  (2) $A^{p_\rightarrow} = A^q_{\langle \nu \rangle}$;
  (3) $A^{p_\leftarrow} = \{\langle T(\mu_0), \dots, T(\mu_n)\rangle \mid n < \omega,\ \langle \mu_0, \dots, \mu_n\rangle \in A^q,\ \mu_n < \nu\}$, where the function T, used to 'translate coordinates', is defined as follows:
  $$\operatorname{dom}\big(T(\mu)\big) = \{\nu(\bar\alpha) \mid \bar\alpha \in \operatorname{dom}\mu,\ \operatorname{o}(\nu(\bar\alpha)) = 1\},$$
  and
  $$\big(T(\mu)\big)\big(\nu(\bar\alpha)\big) = \mu(\bar\alpha).$$
- Assume $p \in \mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$ and $\langle \nu \rangle \in A^{p_\rightarrow}$. The one point extension of p by $\langle \nu \rangle$ is defined to be
  $$p_{\langle \nu \rangle} = p_\leftarrow \frown p_{\rightarrow\langle \nu \rangle}.$$
  Define $p_{\langle \nu_0,\dots,\nu_k\rangle}$ recursively as $(p_{\langle \nu_0,\dots,\nu_{k-1}\rangle})_{\langle \nu_k\rangle}$, where $\nu_0 < \cdots < \nu_k$.

  Whenever the notation $\langle \nu_0, \dots, \nu_{n-1}\rangle$ is used, where $\nu_k \in \operatorname{OB}(d)$ $(k < n)$, it is implicitly assumed that $\nu_k < \nu_{k+1}$ $(k < n-1)$.
- Let $p, q \in \mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$. Then p is an extension of q ($p \leq_{\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}} p$) if there are $\langle \nu_0, \dots, \nu_{n-1}\rangle \in A^q$ such that $p \leq^*_{\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}} q_{\langle \nu_0,\dots,\nu_{n-1}\rangle}$.

We give a pictorial representation of $\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$ before continuing to the full fledged forcing in the next section.

The weakest condition in $\mathbb{P}_{\langle E_0,E_1\rangle,\epsilon}$ is $\langle\langle\bar\kappa, \langle\rangle\rangle, A\rangle$, where A is a tree with first level

$$\{\bar\kappa\} \times (\kappa \cup \{\langle \nu, e_0\rangle \mid \nu < \kappa,\ e_0 \text{ is an extnder with} \operatorname{crit}(e_0) = \nu\}),$$

and similary sets for the higher levels. Let us call it q. We present q graphically in figure 8.

$$\langle \kappa, E_0, E_1 \rangle \qquad A$$

Figure 8. The weakest condition $q \in \mathbb{P}_{\langle E_0, E_1 \rangle, \epsilon}$.

Let $\kappa < \alpha < j_{E_0}(\kappa)$ and $j_{E_0(\kappa)} \leq \beta < \epsilon$. The weakest condition in $\mathbb{P}_{\langle E_0, E_1 \rangle, \epsilon}$ mentioning κ, α, and β is

$$\langle \{ \langle \bar{\kappa}, \langle \rangle \rangle, \langle \bar{\alpha}, \langle \rangle \rangle, \langle \bar{\beta}, \langle \rangle \rangle \}, B \rangle.$$

Let us call it p. Then $p \leq^* q$. Note that the form of $\langle \nu \rangle \in B$ is either

$$\nu = \{ \langle \bar{\kappa}, \tau \rangle, \langle \bar{\alpha}, \mu \rangle \},$$

or

$$\nu = \{ \langle \bar{\kappa}, \langle \tau, e_0 \rangle \rangle, \langle \bar{\alpha}, \langle \mu, e_0 \rangle \rangle, \langle \bar{\beta}, \pi \rangle \},$$

Note that

$$\{ \nu(\bar{\kappa}) \mid \nu \in B \} \in E(\{\kappa\}),$$
$$\{ \nu(\bar{\alpha}) \mid \nu \in B \} \in E(\{\alpha\}),$$

and

$$\{ \nu(\bar{\beta}) \mid \nu \in B \} \in E_1(\beta).$$

Figure 9 shows p graphically.

$$\langle \kappa, E_0, E_1 \rangle \quad \langle \alpha, E_0, E_1 \rangle \qquad \langle \beta, E_1 \rangle \quad B$$

Figure 9. The weakest condition mentioning κ, α and β, $p \leq^* q$.

Let $\langle \nu_0 \rangle \in B$ be of the form

$$\nu_0 = \{ \langle \bar{\kappa}, \langle \tau_0 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_0 \rangle \rangle \}.$$

The weakest 1-point extension of p using $\langle \nu_0 \rangle$, i.e., $p_{\langle \nu_0 \rangle}$, is

$$\langle \{ \langle \bar{\kappa}, \langle \tau_0 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_0 \rangle \rangle, \langle \bar{\beta}, \langle \rangle \rangle \}, B_{\langle \nu_0 \rangle} \rangle.$$

The condition is shown graphically in figure 10.

$$\begin{array}{cccc} \tau_0 & \mu_0 & & \\ \hline \langle \kappa, E_0, E_1 \rangle & \langle \alpha, E_0, E_1 \rangle & \langle \beta, E_1 \rangle & B_{\langle \nu_0 \rangle} \end{array}$$

Figure 10. 1-point extension of p, $p_{\langle \nu_0 \rangle}$.

Let $\langle \nu_1 \rangle \in B_{\langle \nu_0 \rangle}$ be such that and

$$\nu_1 = \{ \langle \bar{\kappa}, \langle \tau_1, e_0 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_1, e_0 \rangle \rangle, \langle \bar{\beta}, \langle \zeta_0 \rangle \rangle \}.$$

The weakest 2-point extension of p using $\langle \nu_0, \nu_1 \rangle$ (which is the same as the weakest 1-point extension of $p_{\langle \nu_0 \rangle}$ using $\langle \nu_1 \rangle$) is

$$\langle \{ \langle \langle \tau_1, e_0 \rangle, \langle \tau_0 \rangle \rangle, \langle \langle \mu_1, e_0 \rangle, \langle \mu_0 \rangle \rangle \}, B \restriction \operatorname{crit}(e_0) \rangle ^\frown \\ \langle \{ \langle \bar{\kappa}, \langle \langle \tau_1, e_0 \rangle \rangle \rangle, \langle \bar{\alpha}, \langle \langle \mu_1, e_0 \rangle \rangle \rangle, \langle \bar{\beta}, \langle \langle \zeta_0 \rangle \rangle \rangle \}, B_{\langle \nu_0, \nu_1 \rangle} \rangle.$$

The condition $p_{\langle \nu_0, \nu_1 \rangle}$ is shown graphically in figure 11.

$$\begin{array}{ccc} \tau_0 & \mu_0 & \\ \hline \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & S \restriction \operatorname{crit}(e_0) \end{array} \quad \begin{array}{cccc} \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & \zeta_0 & \\ \hline \langle \kappa, E_0, E_0 \rangle & \langle \alpha, E_0, E_1 \rangle & \langle \beta, E_1 \rangle & B_{\langle \nu_0, \nu_1 \rangle} \end{array}$$

FIGURE 11. 2-point extension of p, $p_{\langle \nu_0, \nu_1 \rangle} = (p_{\langle \nu_0 \rangle})_{\langle \nu_1 \rangle}$.

We take the following from B:

$$\begin{aligned} \nu_2 &= \{ \langle \bar{\kappa}, \langle \tau_2 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_2 \rangle \rangle \}, \\ \nu_3 &= \{ \langle \bar{\kappa}, \langle \tau_3 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_3 \rangle \rangle \}, \\ \nu_4 &= \{ \langle \bar{\kappa}, \langle \tau_4, e_1 \rangle \rangle, \langle \bar{\alpha}, \langle \mu_4, e_1 \rangle \rangle, \langle \bar{\beta}, \langle \zeta_1 \rangle \rangle \}. \end{aligned}$$

We show $p_{\langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle}$ in figure 12, and $p_{\langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle}$ in 13.

$$\begin{array}{ccc} & & \\ & & \\ \tau_0 & \mu_0 & \\ \hline \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & B \restriction \operatorname{crit}(e_0) \end{array} \quad \begin{array}{cccc} \langle \tau_3 \rangle & \langle \mu_3 \rangle & & \\ \langle \tau_2 \rangle & \langle \mu_2 \rangle & & \\ \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & \zeta_0 & \\ \hline \langle \kappa, E_0, E_0 \rangle & \langle \alpha, E_0, E_1 \rangle & \langle \beta, E_1 \rangle & B_{\langle \nu_0, \dots, \nu_3 \rangle} \end{array}$$

FIGURE 12. 4-point extension of p, i.e., $p_{\langle \nu_0, \nu_1, \nu_2, \nu_3 \rangle}$

$$\begin{array}{ccc} & & \\ \tau_0 & \mu_0 & \\ \hline \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & B \restriction \operatorname{crit}(e_0) \end{array} \quad \begin{array}{ccc} \tau_3 & \mu_3 & \\ \tau_2 & \mu_2 & \\ \hline \langle \tau_4, e_1 \rangle & \langle \mu_4, e_1 \rangle & B \restriction \operatorname{crit}(e_1) \end{array}$$

$$\begin{array}{cccc} \langle \tau_4, e_1 \rangle & \langle \mu_4, e_0 \rangle & \zeta_1 & \\ \langle \tau_1, e_0 \rangle & \langle \mu_1, e_0 \rangle & \zeta_0 & \\ \hline \langle \kappa, E_0, E_0 \rangle & \langle \alpha, E_0, E_1 \rangle & \langle \beta, E_1 \rangle & B_{\langle \nu_0, \dots, \nu_4 \rangle} \end{array}$$

FIGURE 13. 5-point extension of p, i.e., $p_{\langle \nu_0, \nu_1, \nu_2, \nu_3, \nu_4 \rangle}$.

Note that each of the generated blocks operates independently of the others. Thus points and domain enlargement can be done on all blocks, not only on the highest one.

4. FORCING WITH $\bar{E} = \langle E_\xi \mid \xi < \operatorname{o}(\bar{E}) \rangle$

Assume $\bar{E} = \langle E_\xi \mid \xi < \operatorname{o}(\bar{E}) \rangle$ is a Mitchell increasing sequence of (short) extenders on κ, i.e., $\forall \xi < \operatorname{o}(\bar{E})\ \langle E_\zeta \mid \zeta < \xi \rangle \in \operatorname{Ult}(V, E_\xi)$. Let $j_{E_\xi} : V \to M_\xi \simeq \operatorname{Ult}(V, E_\xi)$ $(\xi < \operatorname{o}(\bar{E}))$ be the correspondng natural elementary embeddings. Note that $j_{E_{\xi_1}}(\kappa) < j_{E_{\xi_2}}(\kappa)$ $(\xi_1 < \xi_2 < \operatorname{o}(\bar{E}))$. Let $\kappa^+ \leq \epsilon \leq \sup_{\xi < \operatorname{o}(\bar{E})} j_{E_\xi}(\kappa)$.

Definition 4.1. An extender sequence $\bar{\nu}$ has the form $\langle \tau, e_0, \dots, e_\xi, \dots \rangle$ $(\xi < \mu)$ where $\langle e_\xi \mid \xi < \mu \rangle$ is a Mitchell increasing sequence of (short) extenders with identical critical points, and $\operatorname{crit}(e_0) \leq \nu < j_{e_0}(\operatorname{crit}(e_0))$. The order of the extender sequence $\bar{\nu}$ is μ ($\operatorname{o}(\bar{\nu}) = \mu$). We write $\bar{\nu}_0$ for τ.

Note that formally the Mitchell order function $\operatorname{o}(\dots)$ is defined on different type of objects. The first object is of the form $\langle E_\xi \mid \xi < \lambda \rangle$, and the second is of the form $\langle \nu, e_0, \dots, e_\xi, \dots \mid \xi < \mu \rangle$. In either case only the extenders are considered, thus there is no confusion.

Definition 4.2. The set $\mathfrak{D}$ is a base set used in domain of functions. For each $\kappa \leq \alpha < \sup\{ j_{E_\xi}(\kappa) \mid \xi < \operatorname{o}(\bar{E})\}$ define

$$\bar{\alpha} = \langle \alpha \rangle ^\frown \langle E_\zeta \mid \zeta < \operatorname{o}(\bar{E}),\ \alpha < j_{E_\zeta}(\kappa) \rangle.$$

Then define

$$\mathfrak{D} = \{ \bar{\alpha} \mid \kappa \leq \alpha < \epsilon \}.$$

On $\mathfrak{D}$ the order $<$ is defined by $\bar{\alpha} < \bar{\beta} \iff \alpha < \beta$. The set $\mathfrak{R}$ is used as the base set for range of functions.

$$\mathfrak{R} = \{ \bar{\nu} \in V_\kappa \mid \bar{\nu} \text{ is an extender sequence} \}.$$

On $\mathfrak{R}$ the order $<$ is defined by $\bar{\nu} < \bar{\mu} \iff \bar{\nu}_0 < \bar{\mu}_0$.

Definition 4.3. Assume $d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$. Then $\nu \in \operatorname{OB}(d) \iff$

(1) $\nu : \operatorname{dom} \nu \to \mathfrak{R}$;
(2) $\bar{\kappa} \in \operatorname{dom} \nu \subseteq d$;
(3) $|\nu| \leq \nu(\bar{\kappa})_0$;
(4) $\forall \bar{\alpha} \in \operatorname{dom} \nu \ \big(\operatorname{o}(\nu(\bar{\alpha})) < \operatorname{o}(\bar{\alpha}) \big)$;
(5) For each $\bar{\alpha} \in \operatorname{dom} \nu$ such that $\bar{\alpha} \neq \bar{\kappa}$ the following is satisfied. Assume

$$\nu(\bar{\kappa}) = \langle \tau, e_0, \dots, e_\xi, \dots \mid \xi < \zeta_\kappa \rangle, \quad (\text{where } \operatorname{crit}(e_0) = \tau)$$

and

$$\nu(\bar{\alpha}) = \langle \tau', e'_0, \dots, e'_\xi, \dots \mid \xi < \zeta_\alpha \rangle.$$

Then $\langle e_{\zeta+\xi} \mid \xi < \zeta_\alpha \rangle = \langle e'_\xi \mid \xi < \zeta_\alpha \rangle$, where $\zeta < \zeta_\kappa$ is minimal such that $\tau' \in [\sup_{\zeta' < \zeta} j_{e_{\zeta'}}(\tau), j(e_\zeta)(\tau))$.
(6) $\forall \bar{\alpha}, \bar{\beta} \in \operatorname{dom} \nu \ \big(\bar{\alpha} < \bar{\beta} \implies \nu(\bar{\alpha}) < \nu(\bar{\beta}) \big)$.

On $\operatorname{OB}(d)$ the partial order $<$ is defined by:

$$\nu < \mu \iff \big(\forall \bar{\alpha} \in \operatorname{dom} \nu \ \nu(\bar{\alpha}) < \mu(\bar{\alpha}) \big).$$

Definition 4.4. (1) Assume $T \subseteq \operatorname{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$. Then for each $n < \xi$,

$$\operatorname{Lev}_n(T) = \{ \langle \nu_0, \dots, \nu_n \rangle \in \operatorname{OB}(d)^{n+1} \mid \langle \nu_0, \dots, \nu_n \rangle \in T \},$$

and

$$\operatorname{Suc}_T(\nu_0, \dots, \nu_{n-1}) = \{ \mu \in \operatorname{OB}(d) \mid \langle \nu_0, \dots, \nu_{n-1}, \mu \rangle \in T \}.$$

For notational convenience let $\operatorname{Suc}_T(\langle\rangle) = \operatorname{Lev}_0(T)$. Assume $\langle \nu \rangle \in T$. Define

$$T_{\langle \nu \rangle} = \{ \langle \nu_0, \dots, \nu_{k-1} \rangle \mid k < \omega,\ \langle \nu, \nu_0, \dots, \nu_{k-1} \rangle \in T \},$$

and by recursion when $\langle \nu_0, \dots, \nu_n \rangle \in T$ define

$$T_{\langle \nu_0, \dots, \nu_n \rangle} = (T_{\langle \nu_0, \dots, \nu_{n-1} \rangle})_{\langle \nu_n \rangle}.$$

(2) The measures $E_\xi(d)$ ($d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$, $\xi < \mathrm{o}(\bar{E})$) on $\mathrm{OB}(d)$ are defined as follows:

$$\forall X \subseteq \mathrm{OB}(d) \ \big(X \in E_\xi(d) \iff \mathrm{mc}_\xi(d) \in j_{E_\xi}(X)\big),$$

where

$$\mathrm{mc}_\xi(d) = \{\langle j_{E_\xi}(\bar{\alpha}), R_\xi(\bar{\alpha})\rangle \mid \bar{\alpha} \in d,\ \bar{\alpha} < j_{E_\xi}(\kappa)\},$$

where R_ξ is defined for each $\kappa \leq \alpha <\geq$ by

$$R_\xi(\bar{\alpha}) = \langle \alpha \rangle ^\frown \langle E_{\xi'} \mid \xi' < \xi,\ \alpha < j_{E_{\xi'}}(\kappa)\rangle.$$

The intersection of the measures is named $E(d)$. That is

$$E(d) = \bigcap \{E_\xi(d) \mid \xi < \mathrm{o}(\bar{E})\}.$$

(3) A set $T \subseteq \mathrm{OB}(d)^{<\xi}$ ($1 < \xi \leq \omega$) ordered by end-extension and closed under initial segments is a tree. A tree $T \subseteq \mathrm{OB}(d)^{<\omega}$ is called a d-tree if for each $\langle \nu_0, \dots, \nu_{n-1}\rangle \in T$ we have $\nu_k < \nu_{k+1}$ ($k < n-1$), and for each $n < \omega$,

$$\forall \langle \nu_0, \dots, \nu_{n-1}\rangle \in T \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{n-1}) \in E(d).$$

(4) Assume $c, d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$, $c \subseteq d$, and T is a tree with elements from $\mathrm{OB}(d)$. Then the projection of T to a a tree with elements from $\mathrm{OB}(c)$ is

$$T \restriction c = \{\langle \nu_0 \restriction c, \dots, \nu_n \restriction c\rangle \mid n < \mathrm{ht}(T), \langle \nu_0, \dots, \nu_n\rangle \in T\}.$$

Definition 4.5. The following list of points leads to the definition of $\langle \mathbb{P}_{\bar{E},\epsilon}, \leq, \leq^* \rangle$.

- A condition f is in the forcing notion $\mathbb{P}^*_{\bar{E},\epsilon}$ if $f : d \to {}^{<\omega}\mathfrak{R}$ is a function such that:
  (1) $\bar{\kappa} \in d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$;
  (2) For each $\bar{\alpha} \in d$, $f(\bar{\alpha}) = \langle f_0(\bar{\alpha}), \dots, f_{k-1}(\bar{\alpha})\rangle$ is an increasing sequence in $\mathfrak{R}$;
  (3) For each $\bar{\alpha} \in d$ and $i < |f(\bar{\alpha})|$, $\big(\mathrm{o}(f_i(\bar{\alpha})) < \mathrm{o}(\bar{\alpha})\big)$.
  (4) For each $\bar{\alpha} \in d$, the sequence $\langle \mathrm{o}(f_i(\bar{\alpha})) \mid i < |f(\bar{\alpha})|\rangle$ is non-increasing.
- Let $f, g \in \mathbb{P}^*_{\bar{E},\epsilon}$. We say that f is an extension of g ($f \leq^*_{\mathbb{P}^*_{\bar{E},\epsilon}} g$) if $f \supseteq g$.
- As usual we write $\mathrm{OB}(f)$, $E_\xi(f)$, $E(f)$, $\mathrm{mc}_\xi(f)$, and f-tree, for $\mathrm{OB}(\mathrm{dom}\, f)$, $E_\xi(\mathrm{dom}\, f)$, $E(\mathrm{dom}\, f)$, $\mathrm{mc}_\xi(\mathrm{dom}\, f)$, and $\mathrm{dom}\, f$-tree, respectively, where $f \in \mathbb{P}^*_{\bar{E},\epsilon}$.
- Assume $f \in \mathbb{P}^*_{\bar{E},\epsilon}$ and $\nu \in \mathrm{OB}(f)$. Define $g = f_{\langle \nu \rangle}$ to be of the form $g = g_{\leftarrow} {}^\frown g_{\rightarrow}$ (The case $g_{\leftarrow} = \emptyset$ is allowed) where:
  (1) $\mathrm{dom}\, g_{\rightarrow} = \mathrm{dom}\, f$;
  (2) For each $\bar{\alpha} \in \mathrm{dom}\, g_{\rightarrow}$,

$$g_{\rightarrow}(\bar{\alpha}) = \begin{cases} f(\bar{\alpha}) \restriction k ^\frown \langle \nu(\bar{\alpha})\rangle, & \bar{\alpha} \in \mathrm{dom}\, \nu,\ \nu(\bar{\alpha}) > f_{|f(\bar{\alpha})|-1}(\bar{\alpha}), \\ f(\bar{\alpha}) & \text{Otherwise}, \end{cases}$$

  where

(*) $$k = \min\{l \leq |f(\bar{\alpha})| \mid \forall l \leq i < |f(\bar{\alpha})| \ \mathrm{o}(f_i(\bar{\alpha})) < \mathrm{o}(\nu(\bar{\alpha}))\}.$$

The above value of k is defined so as to ensure that $\langle \mathrm{o}(f_i(\bar\alpha)) \mid i < k\rangle \frown \mathrm{o}(\nu(\bar\alpha))$ is non-increasing. The part removed from $f(\bar\alpha)$, i.e., $f(\bar\alpha) \restriction (|f^q(\bar\alpha)| \setminus k)$, will appear in $g_\leftarrow$;

(3) $\operatorname{dom} g_\leftarrow = \{\nu(\bar\alpha) \mid \bar\alpha \in \operatorname{dom}\nu,\ \mathrm{o}(\nu(\bar\alpha)) > 0\}$;

(4) For each $\bar\alpha \in \operatorname{dom}\nu$ such that $\mathrm{o}(\nu(\bar\alpha)) > 0$ we have

$$g_\leftarrow(\nu(\bar\alpha)) = f(\bar\alpha) \restriction \big(|f(\bar\alpha)| \setminus k\big),$$

where k is defined by (*).

- A condition p in the forcing notion $\mathbb{P}_{\bar E,\epsilon\to}$ is of the form $\langle f, A\rangle$ where:
  (1) $f \in \mathbb{P}^*_{\bar E,\epsilon}$;
  (2) A is an f-tree such that for each $\langle\nu\rangle \in A$ and each $\bar\alpha \in \operatorname{dom}\nu$,

  $$f_{|f(\bar\alpha)|-1}(\bar\alpha) < \nu(\bar\alpha).$$

  We write f^p, A^p, and $\mathrm{mc}_\xi(p)$, for f, A, and $\mathrm{mc}_\xi(f)$, respectively.
- Let $p, q \in \mathbb{P}_{\bar E,\epsilon\to}$. We say that p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_{\bar E,\epsilon\to}} q$) if
  (1) $f^p \leq^*_{\mathbb{P}^*_{\bar E,\epsilon}} f^q$;
  (2) $A^p \restriction \operatorname{dom} f^q \subseteq A^q$.
- A condition p in the forcing notion $\mathbb{P}_{\bar E,\epsilon}$ is of the form $p_\leftarrow \frown p_\to$ where $p_\to \in \mathbb{P}_{\bar E,\epsilon\to}$ and $p_\leftarrow \in \prod_{i<n} \mathbb{P}_{\bar e_i}$ ($n < \omega$), where $\bar e_i$ are extender sequences such that $\mathrm{o}(\bar e_i) \leq \mathrm{o}(\bar E)$, $\bar e_i \in V_{\mathrm{crit}(\bar e_{i+1})}$, $\bar e_{n-1} \in V_\kappa$, and for each $\langle\nu\rangle \in A^{p_\to}$, $\nu(\bar\kappa)_0 > \mathrm{crit}(\bar e_{n-1})$.
- Conditions in $\mathbb{P}_{\bar E,\epsilon}$ have lower parts $\mathbb{P}_{\bar E,\epsilon_\leftarrow}$ defined by $\mathbb{P}_{\bar E,\epsilon_\leftarrow} = \{p_\leftarrow \mid p \in \mathbb{P}_{\bar E,\epsilon}\}$.
- For $p \in \mathbb{P}_{\bar E,\epsilon}$ we define f^p recursively to be $f^{p_\leftarrow} \frown f^{p_\to}$, and we write $f^p_\leftarrow$ and $f^p_\to$ for $f^{p_\leftarrow}$ and $f^{p_\to}$, respectively.
- Let $p, q \in \mathbb{P}_{\bar E,\epsilon}$. We say that p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_{\bar E,\epsilon}} q$) if:
  (1) $p_\to \leq^* q_\to$;
  (2) $p_\leftarrow \leq^* q_\leftarrow$.
- We say that p is a strong Prikry extension of q ($p \leq^{**}_{\mathbb{P}_{\bar E,\epsilon}} q$) if $p \leq^* q$ and $f^p = f^q$.
- Assume $q \in \mathbb{P}_{\bar E,\epsilon\to}$ and $\langle\nu\rangle \in A^q$. The condition $p \in \mathbb{P}_{\bar E,\epsilon}$ is the one point extension of q by $\langle\nu\rangle$ ($p = q_{\langle\nu\rangle}$) if it is of the form $p_\leftarrow \frown p_\to$ where $p_\leftarrow \in \mathbb{P}_{\bar e\to}$ and $p_\to \in \mathbb{P}_{\bar E,\epsilon\to}$ are defined as follows.
  (1) $f^p = f^q_{\langle\nu\rangle}$.
  (2) $A^{p_\to} = A^q_{\langle\nu\rangle}$.
  (3) $A^{p_\leftarrow} = \{\langle T(\mu_0), \dots, T(\mu_n)\rangle \mid n < \omega,\ \langle\mu_0, \dots, \mu_n\rangle \in A^q,\ \mu_n < \nu\}$, where the extender sequence $T(\mu)$ is defined by:

  $$\operatorname{dom}\big(T(\mu)\big) = \{\nu(\bar\alpha) \mid \bar\alpha \in \operatorname{dom}\mu,\ \mathrm{o}(\nu(\bar\alpha)) > 0\},$$

  and

  $$\big(T(\mu)\big)(\nu(\bar\alpha)) = \mu(\bar\alpha).$$

  Define $q_{\langle\nu_0,\dots,\nu_n\rangle}$ recursively by $(q_{\langle\nu_0,\dots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$, where $\langle\nu_0, \dots, \nu_{n-1}\rangle \in A^q$.
- Assume $p \in \mathbb{P}_{\bar E,\epsilon}$ and $\langle\nu\rangle \in A^{p_\to}$. Then

  $$p_{\langle\nu\rangle} = p_\leftarrow \frown p_{\to\langle\nu\rangle} \langle\nu\rangle \in A^{p_\to}.$$

  Define $p_{\langle\nu_0,\dots,\nu_k\rangle}$ recursively by $(p_{\langle\nu_0,\dots,\nu_{k-1}\rangle})_{\langle\nu_k\rangle}$.

- Let $p, q \in \mathbb{P}_{\bar{E},\epsilon}$. Then p is stronger than q ($p \leq_{\mathbb{P}_{\bar{E},\epsilon}} q$) if $p = r \frown s$ and there is $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{q_\rightarrow}$ such that:
  (1) $s \leq^*_{\mathbb{P}_{\bar{E},\epsilon}} q_{\rightarrow \langle \nu_0, \dots, \nu_{n-1} \rangle}$.
  (2) $r \leq q_\leftarrow$.
- Assume $p \in \mathbb{P}_{\bar{E},\epsilon}$. Then $p_\rightarrow \in \mathbb{P}_{\bar{E},\epsilon}$ and we define

$$\mathbb{P}_{\bar{E},\epsilon}/p_\rightarrow = \{q \in \mathbb{P}_{\bar{E},\epsilon} \mid q \leq p_\rightarrow\},$$

  and

$$\mathbb{P}_{\bar{E},\epsilon}/p_\leftarrow = \{r \mid r \leq p_\leftarrow\}.$$

  It is obvious that $\mathbb{P}_{\bar{E},\epsilon}/p$ factors to $\mathbb{P}_{\bar{E},\epsilon}/p_\leftarrow \times \mathbb{P}_{\bar{E},\epsilon}/p_\rightarrow$.

The forcing notion $\mathbb{P}_{\bar{E},\epsilon}$ preserves all cardinals. The roadmap of the proof of this fact is as follows:

(1) The κ^{++}-cc of $\mathbb{P}_{\bar{E},\epsilon}$ is proved in 4.6.
(2) The forcing $\mathbb{P}_{\bar{E},\epsilon}$ is shown to be of Prikry type in 4.12, which together with the factoring of $\mathbb{P}_{\bar{E},\epsilon}/p$ to two Prikry type forcing notions $\mathbb{P}_{\bar{E},\epsilon}/p_\rightarrow \times \mathbb{P}_{\bar{E},\epsilon}/p_\leftarrow$, where the Prikry order on $\mathbb{P}_{\bar{E},\epsilon}/p_\rightarrow$ is closed enough and $\mathbb{P}_{\bar{E},\epsilon}/p_\leftarrow$ is small enough, yields that all cardinals up to κ are preserved. Since κ is a limit cardinal it is preserved also.
(3) A special argument is given for the preservation of κ^+ in 4.14.

Claim 4.6. *$\mathbb{P}_{\bar{E},\epsilon}$ satisfies the κ^{++}-cc.*

Proof. Take an anti-chain $\{p^\xi \mid \xi < \kappa^{++}\} \subseteq \mathbb{P}_{\bar{E},\epsilon}$. Assume without loss of generality that $p^{\xi_1}_\leftarrow = p^{\xi_2}_\leftarrow$ for each $\xi_1 < \xi_2 < \kappa^{++}$. Note that for each two conditions $p, q \in \mathbb{P}_{\bar{E},\epsilon\rightarrow}$, if $f^p \parallel_{\mathbb{P}^*_{\bar{E},\epsilon}} f^q$ then $p \parallel_{\mathbb{P}_{\bar{E},\epsilon}} q$. Necessarily the set $\langle f^{p^\xi_\rightarrow} \mid \xi < \kappa^{++} \rangle$ is of size κ^{++}, and we are done by the κ^{++}-cc of $\mathbb{P}^*_{\bar{E},\epsilon}$. □

The following claim is immediate from the definition of the forcing notion $\mathbb{P}_{\bar{E},\epsilon}$.

Claim 4.7. *Assume $p \in \mathbb{P}_{\bar{E},\epsilon}$, for each $n < \omega$, $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p_\rightarrow}$ and $r \leq p_{\langle \nu_0, \dots, \nu_{n-1} \rangle \leftarrow}$ there is an $E(p_\rightarrow)$-tree $T^r(\nu_0, \dots, \nu_{n-1}) \subseteq A^{p_\rightarrow}_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$ such that*

$$r \frown \langle f^p_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow}, T^r(\nu_0, \dots, \nu_{n-1}) \rangle \in \mathbb{P}_{\bar{E},\epsilon}.$$

Then there is a strong Prikry extension $p^* \leq^{**} p$ such that for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*_\rightarrow}$ and $r \leq p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle \leftarrow}$,

$$r \frown p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow} \leq^{**} r \frown \langle f^p_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow}, T^r(\nu_0, \dots, \nu_{n-1}) \rangle.$$

Definition 4.8.
(1) A tree $T \subseteq \mathrm{OB}(d)^{n+1}$, where $n < \omega$, is called d-fat if for each $\langle \nu_0, \dots, \nu_n \rangle \in T$, $\nu_k < \nu_{k+1}$ ($k < n$), and for each $k < n$,

$$\forall \langle \nu_0, \dots, \nu_{k-1} \rangle \in T \ \exists i < \mathrm{o}(\bar{E}) \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{k-1}) \in E_i(d).$$

As usual we call a tree f-fat if the tree is a $\operatorname{dom} f$-fat for a condition $f \in \mathbb{P}^*_{\bar{E},\epsilon}$, and p-fat if it is an f^p-fat for a condition $p \in \mathbb{P}_{\bar{E},\epsilon\rightarrow}$.
(2) d-fat trees are measure one sets for appropriate measure. To characterize such a measure a function $\iota : T \to \mathrm{o}(\bar{E})$, with domain a d-fat tree, is needed such that for each $k \leq n$,

$$\forall \langle \nu_0, \dots, \nu_{k-1} \rangle \in T \ \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{k-1}) \in E_{\iota(\nu_0, \dots, \nu_{k-1})}(d).$$

Thus define by recursion the product measure $E_\iota(d)$ on $\mathrm{OB}(d)^{n+1}$, where $\iota : T \to \mathrm{o}(Es)$ and T is a tree of height $n+1$, as follows:

$$X \in E_\iota(d) \iff \{\langle \nu_0, \dots, \nu_{n-1}\rangle \in \mathrm{Lev}_{n-1}(X) \mid \mathrm{Suc}_X(\nu_0, \dots, \nu_{n-1}) \in E_{\iota(\nu_0,\dots,\nu_{n-1})}(d)\} \in E_{\iota\restriction \mathrm{Lev}_n(T)}(d),$$

where we set $E_{()} = \{\langle\rangle\}$ and consider it a measure on $\mathrm{OB}(d)^0 = \{\langle\rangle\}$. Note that essentially $E_{\langle\iota\rangle}(d) = E_\iota(d)$. For $\iota : T \to \mathrm{o}(\bar{E})$, where for each $n < \omega$ $T \restriction \mathrm{Lev}_n(T)$ is a d-fat tree, the measure $E_\iota(d)$ on $\mathrm{OB}(d)^{<\omega}$ is defined by recursion as follows:

$$X \in E_\iota(d) \iff \forall n < \omega \ \ \mathrm{Lev}_n(X) \in E_{\iota\restriction \mathrm{Lev}_n(T)}(d).$$

The intersection of all the measures on $\mathrm{OB}(d)^n$ is named $E^{(n)}(d)$, i.e.,

$$E^{(n)}(d) = \bigcap\{E_\iota(d) \mid \iota : T \to \mathrm{o}(\bar{E}),\ T \text{ is a } d\text{-fat tree of height } n\},$$

and the intersection of all the measures on $\mathrm{OB}(d)^{<\omega}$ is named $E^{(\omega)}(d)$, i.e.,

$$E^{(\omega)}(d) = \bigcap\{E_\iota(d) \mid \iota : T \to \mathrm{o}(\bar{E}), T \restriction \mathrm{Lev}_n(T) \text{ is a } d\text{-fat tree of height } n \text{ for each } n < \omega\}.$$

Note that if T is a d-tree then $T \in E^{(\omega)}(d)$. Hence for each $n < \omega$, $\mathrm{Lev}_n(T) \in E^{(n+1)}(d)$. Note also that if T is a tree such that $T \in E^{(\omega)}(d)$, then there is a subtree $S \subseteq T$ such that S is a d-tree.

For the following parts a notation is needed converting a function with a tree domain to a sequence, thus the following definition.

Definition 4.9. Assume T is a tree and $r : T \to P$ is a function. The function $\vec{r} : T \to P^{\leq \mathrm{ht}(T)}$ is defined by recursion as follows. For each $\langle \nu_0, \dots, \nu_k\rangle \in T$,

$$\vec{r}(\nu_0, \dots, \nu_k) = \begin{cases} r(\nu_0) & k = 0, \\ \vec{r}(\nu_0, \dots, \nu_{k-1})^\frown\langle r(\nu_0, \dots, \nu_k)\rangle & 0 < k < \mathrm{ht}(T). \end{cases}$$

Observe that for $\xi < \mathrm{o}(\bar{E})$, a set $X \in E(f)$, where $f \in \mathbb{P}^*_{\bar{E},\epsilon}$, can be partititioned, modulo measure zero set, to three pairwise disjoint subsets $X_<$, $X_=$, and $X_>$, so that $X_< \in \bigcap_{\xi' < \xi} E_{\xi'}(f)$, $X_= \in E_\xi(f)$, and $X_> \in \bigcap_{\xi < \xi' < \mathrm{o}(\bar{E})} E_{\xi'}(f)$. This can be done, for example, as follows. Let Y be a set such that for each $\xi \leq \xi' < \mathrm{o}(\bar{E})$,

$$\mathrm{mc}_{\xi'}(f) \cup \{\langle j_{\xi'}(\xi), \xi\rangle\} \in j_{\xi'}(Y),$$

and $Y \restriction \mathrm{dom}\, f \subseteq X$. Then define:

$$X_< = \{\nu \restriction \mathrm{dom}\, f \mid \nu \in Y,\ \xi \notin \mathrm{dom}\,\nu \text{ or } (\xi \in \mathrm{dom}\,\nu \text{ and } \mathrm{o}(\nu(\bar{\kappa})) < \nu(\xi)\},$$
$$X_= = \{\nu \restriction \mathrm{dom}\, f \mid \nu \in Y,\ \xi \in \mathrm{dom}\,\nu,\ \mathrm{o}(\nu(\bar{\kappa})) = \nu(\xi)\},$$

and

$$X_> = \{\nu \restriction \mathrm{dom}\, f \mid \nu \in Y,\ \xi \in \mathrm{dom}\,\nu,\ \mathrm{o}(\nu(\bar{\kappa})) > \nu(\xi)\}.$$

Lemma 4.10. *Assume $p \in \mathbb{P}_{\bar{E},\epsilon}$, $T \subseteq A^{p_\rightarrow}$ is a $p_\rightarrow$-fat tree, and $r : T \to \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ is a function such that for each $\langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T$,*

$$\vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) \leq^{**} p_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \leftarrow}.$$

Then there is a strong Prikry extension $p^{**} \leq^* p$ such $p^*_\leftarrow = p_\leftarrow$ and the set

$$\{p_\leftarrow {}^\frown \vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown p_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \rightarrow} \mid \langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T\}$$

is predense below p^*.

Proof. The proof is by recursion on the height of T. We begin with the case $n = 0$, i.e., the height of T is one, and proceed to the general case $0 < n < \omega$, i.e., the height of T is $n + 1$.

The case $n = 0$: Begin by setting $S = A^{j_{E_\xi}(r)(\mathrm{mc}_\xi(f^p_\rightarrow))}$, where $\xi < \mathrm{o}(\bar{E})$ witnesses the $f^p_\rightarrow$-fatness of T. The tree satisfies S has its splitting in $\bigcap_{\xi'<\xi} E_{\xi'}(f^p_\rightarrow)$, thus it is a good candidate to be the tree in the Prikry extension p^*. However, the tree S misses measure one sets for measures E_ζ with $\zeta \geq \xi$, so our aim is to fill in the missing sets. For the duration of the proof let us use the following convention. For a set $X \in E(f^p_\rightarrow)$, the sets $X_<$, $X_=$, and $X_>$, are pairwise disjoint, $X \supseteq X_< \cup X_= \cup X_>$, $X_< \in \bigcap_{\xi'<\xi} E_{\xi'}(f^p_\rightarrow)$, $X_= \in E_\xi(f^p_\rightarrow)$, and $X_> \in \bigcap_{\xi<\xi'<\mathrm{o}(\bar{E})} E_{\xi'}(f^p_\rightarrow)$. We will define an $f^p_\rightarrow$-tree A in several steps. For each $\langle \nu_0, \dots, \nu_{k-1}\rangle \in S$ we set the successor for the lower measures as follows:

$$\begin{aligned}
&\mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1})_< = \mathrm{Suc}_S(\nu_0, \dots, \nu_{k-1}),\\
&\mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1})_= = \{\langle \nu\rangle \in T \mid \langle \nu_0, \dots, \nu_{k-1}\rangle \in A^{r(\nu)}\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1})_> = \{\mu \in \mathrm{OB}(f^p_\rightarrow) \mid \forall \nu < \mu \\
\nu \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1}) \implies \mu \in \mathrm{Suc}_{A^p}(\nu)_>\}.
\end{aligned}$$

For the higher measures we set the subtrees as follows.

$$\forall \langle \nu\rangle \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1})_= \ A_{\langle \nu_0,\dots,\nu_{k-1},\nu\rangle} = A^{p_\rightarrow}_{\langle \nu\rangle},$$

and

$$\forall \langle \mu\rangle \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{k-1})_> \ A_{\langle \nu_0,\dots,\nu_{k-1},\mu\rangle} = \bigcap \{A^p_{\langle \nu,\mu\rangle} \mid \langle \nu, \mu\rangle \in A^{p_\rightarrow}\}.$$

Set $p^* = p_\leftarrow {}^\frown \langle f^p_\rightarrow, A\rangle$. We claim that p^* is as demanded. Thus assume that $q \leq p^*_\rightarrow$. Thus there is $\langle \nu_0, \dots, \nu_{k-1}\rangle \in A^{p^*_\rightarrow}$ such that $q \leq^* p^*_{\rightarrow \langle \nu_0,\dots,\nu_{k-1}\rangle}$. We split the handling according to the whereabouts of $\langle \nu_0, \dots, \nu_{k-1}\rangle$:

Subcase $\langle \nu_0, \dots, \nu_{k-1}\rangle \in S$: Set $X = \{\langle \nu\rangle \in T \mid \langle \nu_0, \dots, \nu_{k-1}\rangle \in A^{r(\nu)}\}$. By the definition of S, $X \in E_\xi(f^p_\rightarrow)$. Choose some $\langle \nu\rangle \in X$. Then $f^q \leq^* f^{r(\nu)_{\langle \nu_0,\dots,\nu_{k-1}\rangle}} {}^\frown f^p_{\rightarrow \langle \nu\rangle \rightarrow}$.

Subcase $l < k$, $\langle \nu_0, \dots, \nu_{l-1}\rangle \in S$, and $\langle \nu_l\rangle \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{l-1})_=$: By the construction $\langle \nu_l\rangle \in T$, $\langle \nu_0, \dots, \nu_{l-1}\rangle \in A^{r(\nu_l)}$, and $\langle \nu_{l+1}, \dots, \nu_{k-1}\rangle \in A_{\langle \nu_l\rangle}$. Thus $f^q \leq^* f^{r(\nu_l)_{\langle \nu_0,\dots,\nu_{l-1}\rangle}} {}^\frown f^p_{\rightarrow \langle \nu_l\rangle \rightarrow \langle \nu_{l+1},\dots,\nu_{k-1}\rangle}$.

Subcase $l < k$, $\langle \nu_0, \dots, \nu_{l-1}\rangle \in S$, and $\langle \nu_l\rangle \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{l-1})_>$: Then for each $\nu < \nu_l$ such that $\langle \nu\rangle \in \mathrm{Suc}_A(\nu_0, \dots, \nu_{l-1})_=$, we have $\langle \nu_0, \dots, \nu_{l-1}\rangle \in A^{r(\nu)}$ and $\langle \nu, \nu_l, \dots, \nu_{k-1}\rangle \in A^{p_\rightarrow}$. Thus $f^q \leq^* f^{r(\nu)}_{\langle \nu_0,\dots,\nu_{l-1}\rangle} {}^\frown f^p_{\rightarrow \langle \nu\rangle \rightarrow \langle \nu_{l+1},\dots,\nu_{k-1}\rangle}$.

The case $0 < n < \omega$: By recursion there is a strong Prikry extension $q \leq^{**} p_{\rightarrow}$ such that the set

$$\{p_{\leftarrow} {}^\frown \vec{r}(\nu_0, \dots, \nu_{n-1}) {}^\frown p_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow} \mid \langle \nu_0, \dots, \nu_{n-1} \rangle \in T\}$$

is predense below $p_{\leftarrow} {}^\frown q$. Again by recursion, for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$ there is a strong Prikry extension $q^{\nu_0, \dots, \nu_{n-1}} \leq^{**} q$ such that the set

$$\{p_{\leftarrow} {}^\frown \vec{r}(\nu_0, \dots, \nu_{n-1}, \nu_n) {}^\frown p_{\langle \nu_0, \dots, \nu_{n-1}, \nu_n \rangle \rightarrow} \mid \langle \nu_0, \dots, \nu_{n-1}, \nu_n \rangle \in T\}$$

is pre-dense below $p_{\leftarrow} {}^\frown q^{\nu_0, \dots, \nu_{n-1}}$. Construct a strong Prikry extension $q^* \leq^{**} q$ such that $q_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow} \leq^{**} q^{\nu_0, \dots \nu_{n-1}}$ for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$. Then

$$\{p_{\leftarrow} {}^\frown \vec{r}(\nu_0, \dots, \nu_{n-1}, \nu_n) {}^\frown p_{\langle \nu_0, \dots, \nu_{n-1}, \nu_n \rangle \rightarrow} \mid \langle \nu_0, \dots, \nu_{n-1}, \nu_n \rangle \in T\}$$

is pre-dense below $p_{\leftarrow} {}^\frown q^*$. Thus by setting $p^* = p_{\leftarrow} {}^\frown q^*$ we are done. $\square$

Recall Shelah's definition of a generic condition over an elementary submodel: Assume χ is large enough, $N \prec H_\chi$ is an elementary submodel, and $P \in N$ is a forcing notion. A condition $p \in P$ is called $\langle N, P \rangle$-generic if for each dense open subset $D \in N$ of P,

$$p \Vdash_P \ulcorner \check{D} \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset \urcorner,$$

where $\underset{\sim}{G}$ is the name of the P-generic object.

Let $N \prec H_\chi$ be an elementary submodel such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $\mathbb{P}^*_{\bar{E},\epsilon} \in N$, and $f \in \mathbb{P}^*_{\bar{E},\epsilon} \cap N$. Let $\langle D_\xi \mid \xi < \kappa \rangle$ be an enumeration of the dense open subsets of $\mathbb{P}^*_{\bar{E},\epsilon}$ appearing in N. Since N is closed under $< \kappa$ sequences, and $\mathbb{P}^*_{\bar{E},\epsilon}$ is κ^+-closed, one can construct by induction a $\leq^*$-decreasing sequence below f, $\langle f^\xi \mid \xi < \kappa \rangle$, such that $f^{\xi+1} \in D_\xi \cap N$. Let $f^* = \bigcup \{f^\xi \mid \xi < \kappa\}$. It is clear that f^* is an $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-generic condition. In fact the condition f^* satisifes a stronger property than $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-genericity. I.e., it satisifes that for each $D \in N$ a dense open subset of $\mathbb{P}^*_{\bar{E},\epsilon}$ there is a weaker condition $g \geq^* f^*$ such that $g \in D \cap N$. A condition satisifying this stronger property is called $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-completely generic.

In the context of the forcing notions $\mathbb{P}_{\bar{E},\epsilon}$ and $\mathbb{P}^*_{\bar{E},\epsilon}$ we have a yet stronger property. In the forcing $\mathbb{P}^*_{\bar{E},\epsilon}$ the conditions f and $f_{\langle \nu \rangle}$ are incompatible. However, we force with $\mathbb{P}_{\bar{E},\epsilon}$, thus $\mathbb{P}_{\bar{E},\epsilon}$-conditions with Cohen parts f and $f_{\langle \nu \rangle}$ parts will appear in a $\mathbb{P}_{\bar{E},\epsilon}$-generic filter.

Thus we define a condition $f^* \in \mathbb{P}^*_{\bar{E},\epsilon}$ to be $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-fully generic if there is an f^*-tree A such that for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in A$ and each $D \in N$ a dense open subset of $\mathbb{P}^*_{\bar{E},\epsilon}$ below $f_{\langle \nu_0 \restriction \operatorname{dom} f, \dots, \nu_n \restriction \operatorname{dom} f \rangle \rightarrow}$ there is a condition $g \geq^* f^*_{\langle \nu_0, \dots, \nu_n \rangle \rightarrow}$ such that $g \in D \cap N$. The construction of an $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-fully generic condition is done like the construction of the $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-completely generic condition, one just take more dense open subsets appearing in N into the enumeration.

Lemma 4.11. *Assume $p \in \mathbb{P}_{\bar{E},\epsilon \rightarrow}$ and D is a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}$. Then there are a Prikry extension $p^* \leq^* p$, a p^*-fat tree $T \subseteq A^{p^*}$, and a function $r : T \rightarrow \mathbb{P}_{\bar{E},\epsilon \leftarrow}$, such that for each $\langle \nu_0, \dots, \nu_{\operatorname{ht}(T)-1} \rangle \in T$,*

$$\vec{r}(\nu_0, \dots, \nu_{\operatorname{ht}(T)-1}) \leq^{**} p^*_{\langle \nu_0, \dots, \nu_{\operatorname{ht}(T)-1} \rangle \leftarrow},$$

and

$$\vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown p^*_{\langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1} \rangle \rightarrow} \in D.$$

Proof. The proof is done in two stages. In the first stage we prove that given a condition $p \in \mathbb{P}_{\bar{E},\epsilon\rightarrow}$ and $n < \omega$, there is a Prikry extension $p^* \leq^* p$ such that either

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*} \ \forall q \leq^* p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \ q \notin D,$$

or there are a p^*-fat tree $T \subseteq A^{p^*}$ of height n and a function $r : T \rightarrow \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ such that for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in T$,

$$\vec{r}(\nu_0, \dots, \nu_{n-1}) \leq^{**} p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle \leftarrow},$$

and

$$\vec{r}(\nu_0, \dots, \nu_{n-1}) {}^\frown p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle \rightarrow} \in D.$$

In the second stage we pick one Prikry extension satisfying the above for all $n < \omega$, and show by contradiction that the first case above cannot hold for all $n < \omega$.

Stage I. The proof is by recursion on $\mathrm{ht}(T)$. We begin with trees of height one (the case $n = 0$) and proceed to trees of arbitrary height afterwards (the case $0 < n < \omega$).

<u>The case $n = 0$.</u> Let χ be large enough, $N \prec H_\chi$ be an elementary submodel such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $N \cap \kappa^+ \in \kappa^+$, $\mathbb{P}_{\bar{E},\epsilon}, D \in N$, and $p \in \mathbb{P}_{\bar{E},\epsilon\rightarrow} \cap N$. Let $f^* \leq^* f^p$ be an $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-fully generic condition. Let A be a tree witnessing the $\langle N, \mathbb{P}^*_{\bar{E},\epsilon} \rangle$-full genericity of f^*. For each $\langle \nu_0 \rangle \in A$ set

$$D^{\in}_{\langle \nu_0 \rangle} = \{ q \leq^* p_{\langle \nu_0 \rangle \rightarrow} \mid \operatorname{dom} f^q \supseteq \operatorname{dom} \nu_0, \ \exists s \leq^* \langle f^*, A \rangle_{\langle \nu_0 \rangle \leftarrow} \ s {}^\frown q \in D \}.$$

Since $\langle f^*, A \rangle_{\langle \nu_0 \rangle \leftarrow} \in N$, thus $D^{\in}_{\langle \nu_0 \rangle} \in N$. Now let $D^*_{\langle \nu_0 \rangle}$ be $D'^*_{\langle \nu_0 \rangle} \cup D''^*_{\langle \nu_0 \rangle}$, where

$$D'^*_{\langle \nu_0 \rangle} = \{ f^q \leq^* f^p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow} \mid q \in D^{\in}_{\langle \nu_0 \rangle} \},$$

and

$$D''^*_{\langle \nu_0 \rangle} = \{ g \leq^* f^p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow} \mid \operatorname{dom} g \supseteq \operatorname{dom} \nu_0, \ \forall g' \in D'^*_{\langle \nu_0 \rangle} \ g \perp g' \}.$$

Observe that the sets $D'^*_{\langle \nu_0 \rangle}, D''^*_{\langle \nu_0 \rangle}$ are open subsets of $\mathbb{P}^*_{\bar{E},\epsilon}$, and the sets $D^*_{\langle \nu_0 \rangle}$ are dense open subsets of $\mathbb{P}^*_{\bar{E},\epsilon}$ below $f^p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow}$. Moreover, all of these sets are in N. Thus in particular $f^*_{\langle \nu_0 \rangle \rightarrow} \in \bigcap \{ D^*_{\langle \nu_0 \rangle} \mid \langle \nu_0 \rangle \in A \}$. We split the handling according to the whereabouts of the condition f^*:

(1) By removing a measure zero set from the tree A we have that for each $\langle \nu_0 \rangle \in A$, $f^*_{\langle \nu_0 \rangle \rightarrow} \in D''^*_{\langle \nu_0 \rangle}$: Set $p^* = \langle f^*, A \rangle$. Consider the condition $q \leq^* p^*_{\langle \nu_0 \rangle}$. By the definition of the order $\leq^*$, $q_\leftarrow \leq^* \langle f^*, A \rangle_{\langle \nu_0 \rangle \leftarrow}$ and $q_\rightarrow \leq^* p_{\langle \nu_0 \rangle \rightarrow}$. If $q \in D$ then we have $f^q_\rightarrow \in D'^*_{\langle \nu_0 \rangle}$. Thus $f^q_\rightarrow \perp f^*_{\langle \nu_0 \rangle \rightarrow}$. Contradiction. Thus we get that for each $\langle \nu_0 \rangle \in A^{p^*}$ and $q \leq^* p^*_{\langle \nu_0 \rangle}$, $q \notin D$.

(2) There exists an f^*-fat tree $T' \subseteq A$ of height one such that for each $\langle \nu_0 \rangle \in T'$, $f^*_{\langle \nu_0 \rangle \rightarrow} \in D'^*_{\langle \nu_0 \rangle}$: First let $q : T' \rightarrow \mathbb{P}_{\bar{E},\epsilon_\rightarrow}$ be a witnessing function for the satisfcation of the formula. That is for each $\langle \nu_0 \rangle \in T'$, $f^{q(\nu_0)} = f^*_{\langle \nu_0 \rangle \rightarrow}$ and $q(\nu_0) \in D_{\langle \nu_0 \rangle}$. Then let $s : T' \rightarrow \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ be a witnessing function for the definition of the set $D^{\in}_{\langle \nu_0 \rangle}$, i.e., for each $\langle \nu_0 \rangle \in T'$, $s(\nu_0) \leq^* \langle f^*, A \rangle_{\langle \nu_0 \rangle \leftarrow}$ and

$s(\nu_0) \frown q(\nu_0) \in D$. Set $g' = f^{j_{E_\xi}(s)(\mathrm{mc}_\xi(f^*))}$, where $\xi < \mathrm{o}(\bar{E})$ witnesses that T' is an f^*-fat tree of height one. Then set $g = \{\langle \bar{\alpha}, g'(\bar{\alpha} \restriction \xi)\rangle \mid \bar{\alpha} \in \mathfrak{D},\ \bar{\alpha} \restriction \xi \in \operatorname{dom} g'\}$. Observe that $g \leq^* f^*$. Construct a condition $p^* \leq^* \langle f^*, A\rangle$ such that $f^{p^*} = g$, and for each $\langle \nu_0 \rangle \in A^{p^*}$ such that $\langle \nu_0 \restriction \operatorname{dom} f^* \rangle \in T'$, $p^*_{\langle \nu_0 \rangle \rightarrow} \leq^* q(\nu_0 \restriction \operatorname{dom} f^*)$. Set $T = \{\langle \nu_0 \rangle \in A^{p^*} \mid \langle \nu_0 \restriction \operatorname{dom} f^* \rangle \in T',\ f^{s(\nu_0 \restriction \operatorname{dom} f^*)} = f_{\leftarrow}^{p^*_{\langle \nu_0 \rangle}}\}$, and observe that T is a p^*-fat tree of height one with the same witness ξ as the tree T'. Define a function $r : T \to \mathbb{P}_{\bar{E},\epsilon_{\leftarrow}}$ by choosing a condition $r(\nu_0)$ such that $r(\nu_0) \leq^{**} s(\nu_0 \restriction \operatorname{dom} f^*), p^*_{\langle \nu_0 \rangle \leftarrow}$. All in all we have that for each $\langle \nu_0 \rangle \in T$,

$$r(\nu_0) \frown p^*_{\langle \nu_0 \rangle \rightarrow} \leq^* s(\nu_0 \restriction \operatorname{dom} f^*) \frown q(\nu_0 \restriction \operatorname{dom} f^*) \in D,$$

thus by the openess of D,

$$r(\nu_0) \frown p^*_{\langle \nu_0 \rangle \rightarrow} \in D,$$

by which we are done.

<u>The case $0 < n < \omega$</u>: Let χ be large enough, $N \prec H_\chi$ be an elementary submodel such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $N \cap \kappa^+ \in \kappa^+$, $\mathbb{P}_{\bar{E},\epsilon}, D \in N$, and $p \in \mathbb{P}_{\bar{E},\epsilon \rightarrow} \cap N$. Let $f^* \leq^* f^p$ be an $\langle N, \mathbb{P}^*_{\bar{E},\epsilon}\rangle$-fully generic condition. Let A be a tree witnessing the $\langle N, \mathbb{P}^*_{\bar{E},\epsilon}\rangle$-full genericity of $\mathbb{P}^*_{\bar{E},\epsilon}$. For each $\langle \nu_0 \rangle \in A$ the set

$$D_{\langle \nu_0 \rangle} = \{q \leq p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow} \mid \operatorname{dom} f^q \supseteq \operatorname{dom} \nu_0,\ \exists s \leq \langle f^*, A\rangle_{\langle \nu_0 \rangle \leftarrow}\ s \frown q \in D\}$$

is a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow}$. By recursion the set

$$\begin{aligned} D^{\in}_{\langle \nu_0 \rangle} = \{q \leq^* p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow} \mid{}& \operatorname{dom} f^q \supseteq \operatorname{dom} \nu_0, \\ &\big(\forall \langle \nu_1, \dots, \nu_n \rangle \in A^q\ \forall q^* \leq^* q_{\langle \nu_1, \dots, \nu_n \rangle}\ q^* \notin D_{\langle \nu_0 \rangle}\big) \text{ or} \\ &\big(\exists T \subseteq A^q \text{ a } q\text{-fat tree of height } n\ \exists r : T \to \mathbb{P}_{\bar{E},\epsilon \leftarrow} \\ \vec{r}(\nu_1, \dots, \nu_n) \leq^{**} q_{\langle \nu_1, \dots, \nu_n \rangle \leftarrow},&\ \vec{r}(\nu_1, \dots, \nu_n) \frown q_{\langle \nu_1, \dots, \nu_n \rangle \rightarrow} \in D_{\langle \nu_0 \rangle}\big)\} \end{aligned}$$

is a dense open subset of $\langle \mathbb{P}_{\bar{E},\epsilon}/p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle \rightarrow}, \leq^* \rangle$. Thus the set $D^*_{\langle \nu_0 \rangle} = \{f^q \mid q \in D^{\in}_{\langle \nu_0 \rangle}\}$ is a dense open subset of $\mathbb{P}^*_{\bar{E},\epsilon}$ below $f_{\rightarrow}^{p_{\langle \nu_0 \restriction \operatorname{dom} f^p \rangle}}$. Thus for each $\langle \nu_0 \rangle \in A$, $f^*_{\langle \nu_0 \rangle} \in D^*_{\langle \nu_0 \rangle}$. By the definition of the sets $D_{\langle \nu_0 \rangle}$ and $D^{\in}_{\langle \nu_0 \rangle}$, there is a function $q : A \to \mathbb{P}_{\bar{E},\epsilon \rightarrow}$ such that for each $\langle \nu_0 \rangle \in A$, we have $f^{q(\nu_0)} = f^*_{\langle \nu_0 \rangle \rightarrow}$ and either

$$\forall \langle \nu_1, \dots, \nu_n \rangle \in A^{q(\nu_0)}\ \forall q^* \leq^* q(\nu_0)_{\langle \nu_1, \dots, \nu_n \rangle}\ q^* \notin D,$$

or there are is a condition $s \leq \langle f^*, A\rangle_{\langle \nu_0 \rangle \leftarrow}$, an f^*-fat tree T of height n, and a function $r : T \to \mathbb{P}_{\bar{E},\epsilon \leftarrow}$ such that for each $\langle \nu_1, \dots, \nu_n \rangle \in T$,

$$\vec{r}(\nu_1, \dots, \nu_n) \leq^{**} q(\nu_0)_{\langle \nu_1, \dots, \nu_n \rangle \leftarrow},$$

and

$$s \frown \vec{r}(\nu_1, \dots, \nu_n) \frown q(\nu_0)_{\langle \nu_1, \dots, \nu_n \rangle \rightarrow} \in D.$$

The crucial point is the possibility to choose a Prikry extension $s \leq^* \langle f^*, A\rangle_{\langle \nu_0 \rangle \leftarrow}$ on a measure one set. Thus one and only one of the following can hold.

(1) For each $\langle \nu_0 \rangle \in A$,

$$\forall \langle \nu_1, \dots, \nu_n \rangle \in A^{q(\nu_0)}\ \forall s^* \leq^* \langle f^*, A\rangle_{\langle \nu_0 \rangle \leftarrow}\ \forall q^* \leq^* q(\nu_0)_{\langle \nu_1, \dots, \nu_n \rangle}\ s^* \frown q^* \notin D.$$

In this case we define the condition p^* so as to satisfy $p^* \leq^* p$ and $p^*_{\langle\nu_0\rangle\to} \leq^{**} q(\nu_0)$ for each $\langle\nu_0\rangle \in A$. We get that for each $\langle\nu_0,\nu_1,\dots,\nu_n\rangle \in A^{p^*}$ and each $q \leq^* p^*_{\langle\nu_0,\nu_1,\dots,\nu_n\rangle}$, $q \notin D$, by which the current case was proved.

(2) There are an f^*-fat tree $S \subseteq A$ of height one, a function $s : S \to \mathbb{P}_{\bar{E},\epsilon\leftarrow}$, for each $\langle\nu_0\rangle \in S$ there are an f^*-fat tree T^{ν_0} of height n, and a function $r^{\nu_0} : T^{\nu_0} \to \mathbb{P}_{\bar{E},\epsilon\leftarrow}$ such that for each $\langle\nu_1,\dots,\nu_n\rangle \in T^{\nu_0}$,

$$s(\nu_0) \leq^* \langle f^*, A\rangle_{\langle\nu_0\rangle\leftarrow},$$
$$\vec{r}^{\nu_0}(\nu_1,\dots,\nu_n) \leq^{**} q(\nu_0)_{\langle\nu_1,\dots,\nu_n\rangle\leftarrow},$$

and

$$s(\nu_0) \frown \vec{r}^{\nu_0}(\nu_1,\dots,\nu_n) \frown q(\nu_0)_{\langle\nu_1,\dots,\nu_n\rangle\to} \in D.$$

Set $g' = f^{j_{E_\xi}(s)(\mathrm{mc}_\xi(f^*))}$, where $\xi < \mathrm{o}(\bar{E})$ witnesses that S is an f^*-fat tree of height one. Then set $g = \{\langle\bar{\alpha}, g'(\bar{\alpha} \restriction \xi)\rangle \mid \bar{\alpha} \in \mathfrak{D},\ \bar{\alpha} \restriction \xi \in \operatorname{dom} g'\}$. Observe that $g \leq^* f^*$. Construct a condition $p^* \leq^* \langle f^*, A\rangle$ such that $f^{p^*} = g$, and for each $\langle\nu_0\rangle \in A^{p^*}$ such that $\langle\nu_0 \restriction \operatorname{dom} f^*\rangle \in S$, $p^*_{\langle\nu_0\rangle\to} \leq^* q(\nu_0 \restriction \operatorname{dom} f^*)$. Now construct an f^*-fat tree of height $n+1$ as follows.

$$\mathrm{Lev}_0(T) = \{\langle\nu_0\rangle \in A^{p^*} \mid \langle\nu_0 \restriction \operatorname{dom} f^*\rangle \in S,\ f^{s(\nu_0 \restriction \operatorname{dom} f^*)} = f^{p^*_{\langle\nu_0\rangle}}_{\leftarrow}\}$$

and

$$T_{\langle\nu_0\rangle} = \{\langle\nu_1,\dots,\nu_k\rangle \in A^{p^*}_{\langle\nu_0\rangle} \mid 1 \leq k \leq n,$$
$$\langle\nu_1 \restriction \operatorname{dom} f^*,\dots,\nu_k \restriction \operatorname{dom} f^*\rangle \in T^{\nu_0 \restriction \operatorname{dom} f^*}\}.$$

Define a function $r : T \to \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ so as to satisfy

$$r(\nu_0) \leq^{**} s(\nu_0 \restriction \operatorname{dom} f^*), p^*_{\langle\nu_0\rangle\leftarrow},$$

and

$$\vec{r}(\nu_0,\nu_1,\dots,\nu_n) \leq^* \vec{r}^{\nu_0 \restriction \operatorname{dom} f^*}(\nu_1 \restriction \operatorname{dom} f^*,\dots \nu_n \restriction \operatorname{dom} f^*).$$

All in all we have that for each $\langle\nu_0,\nu_1,\dots,\nu_n\rangle \in T$,

$$\vec{r}(\nu_0,\nu_1,\dots,\nu_n) \frown p^*_{\langle\nu_0,\nu_1,\dots,\nu_n\rangle\to} \leq^*$$
$$s(\nu_0 \restriction \operatorname{dom} f^*) \frown \vec{r}^{\nu_0 \restriction \operatorname{dom} f^*}(\nu_1 \restriction \operatorname{dom} f^*,\dots \nu_n \restriction \operatorname{dom} f^*) \frown$$
$$q(\nu_0 \restriction \operatorname{dom} f^*)_{\langle\nu_1 \restriction \operatorname{dom} f^*,\dots\nu_n \restriction \operatorname{dom} f^*)\rangle\to} \in D,$$

thus by the openess of D,

$$\vec{r}(\nu_0,\dots\nu_n) \frown p^*_{\langle\nu_0,\dots,\nu_n\rangle\to} \in D,$$

by which we are done.

Stage II. Assume $p \in \mathbb{P}_{\bar{E},\epsilon\to}$. By invocation of stage I for ω-many times construct a condition $p^* \leq^* p$ such that for each $n < \omega$ either

$$\forall\langle\nu_0,\dots,\nu_{n-1}\rangle \in A^{p^*}\ \forall q \leq^* p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\ q \notin D,$$

or there are a p^*-fat tree T of height n, and a function $r : T \to \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ such that for each $\langle\nu_0,\dots,\nu_{n-1}\rangle \in T$,

$$\vec{r}(\nu_0,\dots,\nu_{n-1}) \leq^{**} p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle\leftarrow},$$

and

$$\vec{r}(\nu_0, \dots, \nu_{n-1}) {}^\frown p^*_{\langle \nu_0,\dots,\nu_{n-1}\rangle \to} \in D.$$

Towards a contradiction let us assume that for each $n < \omega$ we have

$$\forall \langle \nu_0, \dots, \nu_{n-1}\rangle \in A^{p^*}\ \forall q \leq^* p^*_{\langle \nu_0,\dots,\nu_{n-1}\rangle}\ q \notin D.$$

This means that for each $q \leq p^*$, $q \notin D$, contradiction to the density of D. Thus we must have a p^*-fat tree T and a function $r : T \to \mathbb{P}_{\bar{E},\epsilon_\leftarrow}$ of height $n < \omega$ such that for each $\langle \nu_0, \dots, \nu_{n-1}\rangle \in T$,

$$\vec{r}(\nu_0, \dots, \nu_{n-1}) \leq^{**} p^*_{\langle \nu_0,\dots,\nu_{n-1}\rangle \leftarrow},$$

and

$$\vec{r}(\nu_0, \dots, \nu_{n-1}) {}^\frown p^*_{\langle \nu_0,\dots,\nu_{n-1}\rangle \to} \in D,$$

by which we are done. □

Claim 4.12. *The forcing $\langle \mathbb{P}_{\bar{E},\epsilon}, \leq, \leq^*\rangle$ is of Prikry type.*

Proof. Assume $p \in \mathbb{P}_{\bar{E},\epsilon}$ and let σ be a formula in the $\mathbb{P}_{\bar{E},\epsilon}$-forcing language. We will construct by induction the sequence $\langle s_\xi, q_\xi \mid \xi < \lambda\rangle$ where $\{s_\xi \mid \xi < \lambda\}$ is a maximal anti-chain below $p_\leftarrow$ (and thus $\lambda < \kappa$), and $\langle q_\xi \mid \xi < \lambda\rangle$ is a $\leq^*$-decreasing sequence below $p_\to$ such that $s_\xi {}^\frown q_\xi \parallel \sigma$ as follows.

Assume $\langle s_{\xi'}, q_{\xi'} \mid \xi' < \xi\rangle$ were defined. If $\{s_{\xi'} \mid \xi' < \xi\}$ is a maximal anti-chain below $p_\leftarrow$ then we are done and we set λ to be ξ. Otherwise choose a condition $s' \leq p_\leftarrow$ such that s' is incompatible with each of the conditions $s_{\xi'}$ ($\xi' < \xi$). Then choose a condition q' such that $q' \leq^* q_{\xi'}$ for each $\xi' < \xi$. The set $D^\in = \{q \leq q' \mid \exists s \leq s'\ s {}^\frown q \parallel \sigma\}$ is a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}$ below q'. By 4.11 there are a condition $q'' \leq^* q'$, a q''-fat tree $T \subseteq A^{q''}$, and a function r with domain T, such that for each $\langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T$,

$$\vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) \leq^{**} q''_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \leftarrow},$$

and

$$\vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown q''_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \to} \in D^\in.$$

By removing a measure zero set from T we get that there is a condition $s_\xi \leq s'$ such that for each $\langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T$,

$$s_\xi {}^\frown \vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown q''_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \to} \parallel \sigma.$$

Remvoing another measure zero set from T yields that for each $\langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T$,

$$s_\xi {}^\frown \vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown q''_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \to} \Vdash \phi,$$

where ϕ is $\neg\sigma$ or σ. Invocation of 4.10 yields a condition $q_\xi \leq^{**} q''$ such that

$$\{s_\xi {}^\frown \vec{r}(\nu_0, \dots, \nu_{\mathrm{ht}(T)-1}) {}^\frown q''_{\langle \nu_0,\dots,\nu_{\mathrm{ht}(T)-1}\rangle \to} \mid \langle \nu_0, \dots, \nu_{\mathrm{ht}(T)-1}\rangle \in T\}$$

is pre-dense below $s_\xi {}^\frown q_\xi$. Thus we get $s_\xi {}^\frown q_\xi \parallel \sigma$.

When the induction terminates choose a condition q^* such that $q^* \leq^* q_\xi$ for each $\xi < \lambda$. Of course we have $s_\xi \frown q^* \parallel \sigma$ for each $\xi < \lambda$. Let X_0 and X_1 be a partition of $\{s_\xi \mid \xi < \lambda\}$ defined as follows:

$$X_0 = \{s_\xi \mid \xi < \lambda,\ s_\xi \frown q^* \Vdash \neg\sigma\},$$

and

$$X_1 = \{s_\xi \mid \xi < \lambda, s_\xi \frown q^* \Vdash \sigma\}.$$

By recursion there is a direct extension $s^* \leq^* p_\leftarrow$ such that $s^* \parallel \ulcorner \check{X}_0 \cap \underset{\sim}{G}_\leftarrow \neq \emptyset \urcorner$, where $\underset{\sim}{G}_\leftarrow$ is the name of the $\mathbb{P}_{\bar{E},\epsilon}/p_\leftarrow$-generic filter. Now, if $s^* \Vdash \ulcorner \check{X}_0 \cap \underset{\sim}{G}_\leftarrow \neq \emptyset \urcorner$ then $s^* \frown q^* \Vdash \neg\sigma$. If $s^* \Vdash \ulcorner \check{X}_0 \cap \underset{\sim}{G}_\leftarrow = \emptyset \urcorner$ then (since $\{s_\xi \mid \xi < \lambda\}$ is a maximal anti-chain below $p_\leftarrow$) $s^* \Vdash \ulcorner \check{X}_1 \cap \underset{\sim}{G}_\leftarrow \neq \emptyset \urcorner$, thus $s^* \frown q^* \Vdash \sigma$. Either way by setting $p^* = s^* \frown q^*$ we get $p^* \parallel \sigma$. □

Lemma 4.13. *Assume χ is large enough, and $N \prec H_\chi$ is an elementary submodel such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $N \cap \kappa^+ \in \kappa^+$, $\mathbb{P}_{\bar{E},\epsilon} \in N$, and $p \in \mathbb{P}_{\bar{E},\epsilon} \cap N$. Then there is a Prikry extension $p^* \leq^* p$, satisfying $p^*_\leftarrow = p_\leftarrow$, which is $\langle N, \mathbb{P}_{\bar{E},\epsilon}\rangle$-generic.*

Proof. Let f^* be an $\langle N, \mathbb{P}^*_{\bar{E},\epsilon}\rangle$-generic condition, and A be an f^*-tree witnessing the $\langle N, \mathbb{P}^*_{\bar{E},\epsilon}\rangle$-genericity of f^*. We will get an $\langle N, \mathbb{P}_{\bar{E},\epsilon}\rangle$-generic condition by removing a measure zero set from A. We do this as follows. Let $\langle D_\xi \mid \xi < \kappa\rangle$ be an enumeration of the dense open subsets of $\mathbb{P}_{\bar{E},\epsilon}$ appearing in N. For each $n < \omega$ and $\langle \nu_0, \ldots, \nu_{n-1}\rangle \in A$ set

$$\begin{aligned} D^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} = \{f \leq^* f^{p_\rightarrow} \mid \exists q \in \mathbb{P}_{\bar{E},\epsilon}\ f^q = f_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow} \\ \forall \xi < \nu_{n-1}(\kappa)\ q \Vdash_{\mathbb{P}_{\bar{E},\epsilon}/p_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow}} \ulcorner D_{\xi\langle \nu_0,\ldots,\nu_{n-1}\rangle} \cap \underset{\sim}{G} \neq \emptyset \urcorner\}, \end{aligned}$$

(in the above formula $\underset{\sim}{G}$ is used as the name of a $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow}$-generic filter) where for each $\xi < \kappa$,

$$D_{\xi\langle \nu_0,\ldots,\nu_{n-1}\rangle} = \{s \leq p_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow} \mid \exists r \leq p_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\leftarrow}\ r \frown s \in D_\xi\}.$$

Since the set $D_{\xi\langle \nu_0,\ldots,\nu_{n-1}\rangle}$ is a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow}$, the set $D^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle}$ is a dense open subset of $\mathbb{P}^*_{\bar{E},\epsilon}$ below $f^{p_\rightarrow}$. Since $A \subset N$, both sets $D_{\xi\langle \nu_0,\ldots,\nu_{n-1}\rangle}$ and $D^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle}$ are in N. Thus by the construction of f^* we can choose for each $\langle \nu_0, \ldots, \nu_{n-1}\rangle \in A$ a condition $q(\nu_0, \ldots, \nu_{n-1})$ such that for some $f \in D^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \cap N$ we have $f \geq^* f^*$ and $f^{q(\nu_0,\ldots,\nu_{n-1})} = f_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow}$.

Use 4.7 to construct a strong Prikry extension $p^* \leq^{**} \langle f^*, A\rangle$ satisfying for each $\langle \nu_0, \ldots, \nu_{n-1}\rangle \in A^{p^*}$, $p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow} \leq^* q(\nu_0, \ldots, \nu_{n-1})$. We claim that p^* is $\langle N, \mathbb{P}_{\bar{E},\epsilon}\rangle$-generic. To show this let $D \in N$ be a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}$, and G be $\mathbb{P}_{\bar{E},\epsilon}$-generic with $p^* \in G$. Let $\zeta < \kappa$ be such that $D = D_\zeta$. The set

$$\{p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \mid n < \omega,\ \langle \nu_0, \ldots, \nu_{n-1}\rangle \in A^{p^*},\ \nu_{n-1}(\kappa) > \zeta\}$$

is a predense subset of $\mathbb{P}_{\bar{E},\epsilon}$ below p^*. Thus we will be done if we show that $p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \Vdash \ulcorner \check{D}_\zeta \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset \urcorner$ when $\nu_{n-1}(\kappa) > \zeta$. So assume that $\langle \nu_0, \ldots, \nu_{n-1}\rangle \in A^{p^*}$ and $\nu_{n-1}(\kappa) > \zeta$. By the construction of p^* we know that

$$p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle\rightarrow} \leq^* q(\nu_0, \ldots, \nu_{n-1})$$

while

$$q(\nu_0,\dots,\nu_{n-1}) \Vdash_{\mathbb{P}_{\bar{E},\epsilon}/p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\to}} \ulcorner \check{D}_{\zeta\langle\nu_0,\dots,\nu_{n-1}\rangle} \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset \urcorner.$$

Thus $p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle\to} \Vdash_{\mathbb{P}_{\bar{E},\epsilon}/p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\to}} \ulcorner \check{D}_{\zeta\langle\nu_0,\dots,\nu_{n-1}\rangle} \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset\urcorner$. Factor G to $G_\leftarrow * G_\to$ where $G_\to$ is a $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\to}$-generic filter over V, and $G_\leftarrow$ is $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\leftarrow}$-generic filter over $V[G_\to]$. In $V[G]$ define the set $D' = \{r \leq p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\leftarrow} \mid \exists s \in D_{\zeta\langle\nu_0,\dots,\nu_{n-1}\rangle} \cap G_\to,\ r \frown s \in D_\zeta\}$. Observe that D' is a dense open subset of $\mathbb{P}_{\bar{E},\epsilon}/p_{\langle\nu_0,\dots,\nu_{n-1}\rangle\leftarrow}$ and trivially $D' \subset N$. Moreover, $D' \in N[G]$ where $N[G] = \{\dot{a}[G] \mid \dot{a} \in N \text{ is a } \mathbb{P}_{\bar{E},\epsilon}\text{-name}\} \prec H^{V[G]}_\chi$. Thus $D' \cap G_\leftarrow \cap N \neq \emptyset$. That is, there are $r \in G_\leftarrow \cap N$ and $s \in G_\to \cap N$ such that $r \frown s \in D_\zeta \cap G \cap N$. $\square$

A properness type argument using the above lemma yields:

Corollary 4.14. *In a $\mathbb{P}_{\bar{E},\epsilon}$-generic extension $(\kappa^+)_V$ is preserved.*

Following the roadmap appearing after 4.5 one gets:

Corollary 4.15. *The forcing $\mathbb{P}_{\bar{E},\epsilon}$ preserves all cardinals.*

Proof. By 4.6 and 4.14 the cardinals above κ are preserved. Let $\lambda < \kappa$ be cardinal. Choose a condition $p \in \mathbb{P}_{\bar{E},\epsilon}$ such that $p_{\leftarrow\to}$ is above λ. Factor $\mathbb{P}_{\bar{E},\epsilon}$ to $\mathbb{P}_{\bar{E},\epsilon}/p_{\leftarrow\leftarrow} \times \mathbb{P}_{\bar{E},\epsilon}/(p_{\leftarrow\to} \frown p_\to)$. The λ-closure of $\langle \mathbb{P}_{\bar{E},\epsilon}/(p_{\leftarrow\to} \frown p_\to), \leq^*\rangle$ together with the Prikry property yield that a witness to a possible collpase of λ should be in $V^{\mathbb{P}_{\bar{E},\epsilon}/p_{\leftarrow\leftarrow}}$ while by recursion there is no such witness. $\square$

Definition 4.16. Let G be $\mathbb{P}_{\bar{E},\epsilon}$-generic. For each $\kappa \leq \alpha < \epsilon$,

$$G^{\bar{\alpha}} = \bigcup \{f^p_\to(\bar{\alpha}) \mid p \in G,\ \bar{\alpha} \in \operatorname{dom} f^p_\to\},$$
$$C^{\bar{\alpha}} = \{\bar{\nu}_0 \mid \bar{\nu} \in G^{\bar{\alpha}}\}.$$

It is evident that $C^{\bar{\kappa}}$ is a club. Density arguments shows that for $\bar{\alpha} \neq \bar{\beta}$, $C^{\bar{\alpha}} \neq C^{\bar{\beta}}$. Taking into consideration the number of anti-chains we get:

Corollary 4.17. $\Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner 2^\kappa = |\epsilon| \urcorner$.

Claim 4.18. *Assume $\kappa \leq \alpha < \epsilon$. Then*

$$\operatorname{ot}(C^{\bar{\alpha}}) = \begin{cases} \omega^\beta (\textit{in ordinal exponentation}) & \operatorname{o}(\bar{\alpha}) = \beta < \kappa, \\ \kappa & \operatorname{o}(\bar{\alpha}) \geq \kappa. \end{cases}$$

5. Properties of κ in $V^{\mathbb{P}_{\bar{E},\epsilon}}$ related to $\operatorname{cf}(\operatorname{o}(\bar{E}))$

Claim 5.1. *If $\operatorname{cf}(\operatorname{o}(\bar{E})) = 1$, then $\Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \operatorname{cf}\kappa = \omega \urcorner$.*

Proof. Let $\zeta < \operatorname{o}(\bar{E})$ be such that $\operatorname{o}(\bar{E}) = \zeta + 1$. Choose a set $X \in E_\zeta(\{\bar{\kappa}\}) \setminus \bigcup_{\zeta'<\zeta} E_{\zeta'}(\{\bar{\kappa}\})$. In $V[G]$ define by induction $\bar{\nu}^0 = \min G^{\bar{\kappa}}$, and $\bar{\nu}^{n+1} = \min\{\bar{\nu} \in G^{\bar{\kappa}} \mid \bar{\nu} > \bar{\nu}^n,\ \bar{\nu} \in X\}$. Then $\langle \bar{\nu}^n_0 \mid n < \omega\rangle$ is a cofinal sequence in κ. $\square$

The following lemma shows that short enough new sequences into κ are, in general, bounded. This means that they are generated in forcing smaller than $\mathbb{P}_{\bar{E},\epsilon}$.

Lemma 5.2. *Assume η is a cardinal, $\omega \leq \eta < \min\big(\kappa, \operatorname{cf}(\operatorname{o}(\bar{E}))\big)$, $\operatorname{cf}(\operatorname{o}(\bar{E})) \neq \kappa$, and $p \Vdash \ulcorner \dot{f} : \check{\eta} \to \check{\kappa} \urcorner$. Then there is a Prikry extension $p^* \leq^* p$ such that $p^*_\leftarrow = p_\leftarrow$ and $p^* \Vdash \ulcorner \dot{f}$ is bounded in $\check{\kappa} \urcorner$.*

Proof. We will construct by induction the sequence $\langle p^\xi, F^\xi : T^\xi \to \kappa, I^\xi \mid \xi < \eta\rangle$ such that for each $\xi < \eta$:

(1) $\forall \xi_1 < \xi_2 < \eta\ p^{\xi_2} \leq^* p^{\xi_1} \leq^* p_{\to}$;
(2) T^ξ is a $p^{\xi+1}$-fat tree of characteristic I^ξ, i.e., for each $k < \operatorname{ht}(T^\xi)$ and $\langle \nu_0, \dots, \nu_{k-1}\rangle \in T^\xi$,
$$\operatorname{Suc}_{T^\xi}(\nu_0, \dots, \nu_{k-1}) \in E_{I^\xi(\nu_0,\dots,\nu_{k-1})}(p^{\xi+1});$$
(3) For each $\langle \nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}\rangle \in T^\xi$ there is a condition
$$r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) \leq^{**} p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T^\xi)-1}\rangle \leftarrow}$$
such that
$$r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) {}^\frown p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T^\xi)-1}\rangle \to} \Vdash \ulcorner \dot f(\xi) < F^\xi(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1})\urcorner;$$
(4) $\operatorname{ran}(I^\xi)$ is bounded in $\operatorname{o}(\bar E)$.

The induction is done as follows.

$\xi = 0$: Set $p^0 = p_{\to}$.

$\xi \leq \eta$ is limit: Thus $\langle p^{\xi'} \mid \xi' < \xi\rangle$ is defined to be a $\leq^*$-decreasing sequence. Since $\langle \mathbb{P}_{\bar E,\epsilon}, \leq^*\rangle$ is κ-closed there is a condition $p^\xi \in \mathbb{P}_{\bar E,\epsilon}$ such that $\forall \xi' < \xi\ p^\xi \leq^* p^{\xi'}$.

$\xi + 1 \leq \eta$: Thus p^ξ was constructed. Set
$$D = \{q \leq p^\xi \mid \exists \zeta < \kappa\ p_{\leftarrow} {}^\frown q \Vdash \ulcorner \dot f(\check\xi) < \check\zeta\urcorner\}.$$

First we note that D is a dense open subset of $\mathbb{P}_{\bar E,\epsilon}/p_{\to}$ below p^ξ. To show the density take a condition $q \in \mathbb{P}_{\bar E,\epsilon}/p_{\to}$. By factoring $\mathbb{P}_{\bar E,\epsilon}/p$ to $\mathbb{P}_{\bar E,\epsilon}/p_{\to} \times \mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}$ we get a condition $r \leq_{\mathbb{P}_{\bar E,\epsilon}/p_{\to}} q$ and a $\mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}$-name $\dot\tau$ such that $r \Vdash_{\mathbb{P}_{\bar E,\epsilon}/p_{\to}} \ulcorner \dot f(\xi) = \dot\tau\urcorner$ and $\Vdash_{\mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}} \ulcorner \dot\tau < \kappa\urcorner$. Since $|\mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}| < \kappa$ there is a set $A \in \mathcal{P}_\kappa \kappa$ such that $\Vdash_{\mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}} \ulcorner \dot\tau \in \check A\urcorner$. By the regularity of κ, $\zeta = \sup A < \kappa$. Thus we have $p_{\leftarrow} {}^\frown r \Vdash \ulcorner \dot f(\check\xi) < \check\zeta\urcorner$.

By 4.11 and 4.10 there are a condition $p^{\xi+1} \leq^* p^\xi$ and a $p^{\xi+1}$-fat tree T^ξ such that for each $\langle \nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}\rangle \in T^\xi$ there is a condition
$$r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) \leq^{**}_{\mathbb{P}_{\bar E,\epsilon}/p_{\leftarrow}} p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T^\xi)-1}\rangle \leftarrow}$$
such that
$$(*) \qquad \forall \langle \nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}\rangle \in T^\xi\ r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) {}^\frown p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T^\xi)-1}\rangle \to} \in D,$$
and
$$\{r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) {}^\frown p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T'^\xi)-1}\rangle \to} \mid \langle \nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}\rangle \in T^\xi\}$$
is predense below $p^{\xi+1}$. Define a function $F^\xi : T^\xi \to \kappa$ witnessing (*), i.e.,
$$\forall \langle \nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}\rangle \in T^\xi\ p_{\leftarrow} {}^\frown r(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1}) {}^\frown p^{\xi+1}_{\langle \nu_0,\dots,\nu_{\operatorname{ht}(T^\xi)-1}\rangle \to} \Vdash \ulcorner \dot f(\check\xi) < F^\xi(\nu_0, \dots, \nu_{\operatorname{ht}(T^\xi)-1})\urcorner.$$

Let I^ξ be the characteristic function of T^ξ. If $\operatorname{cf}(\operatorname{o}(\bar E)) > \kappa$ then $\operatorname{ran} I^\xi$ is trivially bounded in $\operatorname{o}(\bar E)$. If $\operatorname{cf}(\operatorname{o}(\bar E)) < \kappa$ then using the κ-completeness of the measures at hand we can remove a measure zero set from T^ξ so that $\operatorname{ran} I^\xi$ will be bounded in $\operatorname{o}(\bar E)$.

At this point the induction has terminated. Choose an ordinal $\zeta < \mathrm{o}(\bar{E})$ such that $\mathrm{ran}\, I^\xi \subset \zeta$ for each $\xi < \eta$. Choose a set $A \in E_\zeta(p^\eta) \setminus \bigcap_{\zeta'<\zeta} E_{\zeta'}(p^\eta)$. Let us fix some $\nu \in A$. Then $p^\eta_{\langle \nu \rangle} \perp p^\eta_{\langle \nu_0, \dots, \nu_{\mathrm{ht}(T^\xi)-1} \rangle}$ whenever $\langle \nu_0, \dots, \nu_{\mathrm{ht}(T^\xi)-1} \rangle \not\leq \nu$. This means that

$$p_\leftarrow {}^\frown p^\eta_{\langle \nu \rangle} \Vdash \ulcorner \mathrm{ran}\, \dot{f} \subseteq \check{X} \urcorner,$$

where

$$X = \{ F^\xi(\nu_0, \dots, \nu_{\mathrm{ht}(T^\xi)-1}) \mid \xi < \eta,\ \langle \nu_0, \dots, \nu_{\mathrm{ht}(T^\xi)-1} \rangle \in T^\xi,\ \nu_{\mathrm{ht}(T^\xi)-1} < \nu \}.$$

Since $|X| < \kappa$, $\sup X < \kappa$. Thus $p_\leftarrow {}^\frown p^\eta_{\langle \nu \rangle} \Vdash \ulcorner \dot{f}$ is bounded in $\kappa \urcorner$. This is true for arbitrary $\nu \in A$. That is

$$\forall \nu \in A\ p_\leftarrow {}^\frown p^\eta_{\langle \nu \rangle} \Vdash \ulcorner \dot{f} \text{ is bounded in } \kappa \urcorner.$$

Use 4.10 to get a Prikry extension $p^* \leq^{**} p^\eta$ such that $\{ p^\eta_{\langle \nu \rangle} \mid \nu \in A \}$ is predense below p^*. $p_\leftarrow {}^\frown p^* \Vdash \ulcorner \dot{f}$ is bounded in $\kappa \urcorner$. □

Corollary 5.3. *If* $\omega \leq \mathrm{cf}(\mathrm{o}(\bar{E})) < \kappa$ *then* $\Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \mathrm{cf}\, \kappa = \mathrm{cf}(\mathrm{o}(\bar{E})) \urcorner$.

Proof. Since $\mathrm{cf}(\mathrm{o}(\bar{E})) < \kappa$, lemma 5.2 implies $\Vdash \ulcorner \mathrm{cf}(\kappa) \geq \mathrm{cf}(\mathrm{o}(\bar{E})) \urcorner$. Thus we are left to exhibit a sequence witnessing $\Vdash \ulcorner \mathrm{cf}(\kappa) = \mathrm{cf}(\mathrm{o}(\bar{E})) \urcorner$. Fix an increasing continuous sequence $\langle \zeta_\xi \mid \xi < \mathrm{cf}(\mathrm{o}(\bar{E})) \rangle$ cofinal in $\mathrm{o}(\bar{E})$. Choose a family of pairwise disjoint sets $\{ A_\xi \mid \xi < \mathrm{cf}(\mathrm{o}(\bar{E})) \}$ such that for each $\xi < \mathrm{cf}(\mathrm{o}(\bar{E}))$, $A_\xi \in \bigcap_{\zeta_\xi \leq \zeta < \zeta_{\xi+1}} E_\zeta(\{\bar{\kappa}\})$. In $V[G]$ set for each $\xi < \mathrm{cf}(\mathrm{o}(\bar{E}))$, $\bar{\nu}^\xi = \min\{ \bar{\nu} \in G^{\bar{\kappa}} \mid \bar{\nu} \in A_\xi \}$. Then $\langle \bar{\nu}^\xi_0 \mid \xi < \mathrm{cf}(\mathrm{o}(\bar{E})) \rangle$ is cofinal in κ. □

Corollary 5.4. *If* $\mathrm{cf}(\mathrm{o}(\bar{E})) > \kappa$ *then* $\Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \kappa$ *is regular*$\urcorner$.

Proof. Since $\mathrm{cf}(\mathrm{o}(\bar{E})) > \kappa$, lemma 5.2 implies the corollary at once. □

We deal now with the case not covered by lemma 5.2, that is, when $\mathrm{cf}(\mathrm{o}(\bar{E})) = \kappa$.

Claim 5.5. *If* $\mathrm{cf}(\mathrm{o}(\bar{E})) = \kappa$ *then* $\Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \mathrm{cf}\, \kappa = \omega \urcorner$.

Proof. Fix an increasing continuous sequence $\langle \zeta_\xi \mid \xi < \kappa \rangle$ cofinal in $\mathrm{o}(\bar{E})$. Choose a family of pairwise disjoint sets $\{ A_\xi \mid \xi < \kappa \}$ such that for each $\xi < \kappa$, $A_\xi \in \bigcap_{\zeta_\xi \leq \zeta < \zeta_{\xi+1}} E_\zeta(\{\kappa\})$. In $V[G]$ construct by induction the sequence $\langle \bar{\nu}^n \mid n < \omega \rangle$ as follows: $\bar{\nu}^0 = \min G^{\bar{\kappa}}$, and for each $n < \omega$, $\bar{\nu}^{n+1} = \min\{ \bar{\nu} \in G^{\bar{\kappa}} \mid \bar{\nu} \in A_{\bar{\nu}^n_0} \}$. Then $\langle \bar{\nu}^n_0 \mid n < \omega \rangle$ is cofinal in κ. □

As is usual with Radin forcing, some form of repeat point is needed in order to preserve measurability. The following definition seems to be enough for this. It is followed by a technical lemma containing the machinery needed in order to construct a measure.

Definition 5.6. An ordinal $\zeta < \mathrm{o}(\bar{E})$ is called a repeat point of $\bar{E}$ if for each $d \in \mathcal{P}_{\kappa^+} \mathfrak{D}$ and set $X \in \bigcap_{\xi < \zeta} E_\xi(d)$, X is in $E(d)$.

Lemma 5.7. *Assume* $\zeta < \mathrm{o}(\bar{E})$ *is a repeat point of* $\bar{E}$, $p \in \mathbb{P}_{\bar{E},\epsilon}$ *is a condition, and* $\dot{X}$ *is a* $\mathbb{P}_{\bar{E},\epsilon}$*-name such that* $p \Vdash \ulcorner \dot{X} \subseteq \check{\kappa} \urcorner$. *Then there is an extension* $q \leq p$ *such that*

$$q_\leftarrow \leq p_\leftarrow,$$
$$q_\rightarrow \leq^* p_\rightarrow,$$

and

$$j_{E_\zeta}(q)_{\langle \mathrm{mc}_\zeta(q_\to)\rangle} \parallel_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner.$$

Proof. Assume χ is large enough. Let $N \prec H_\chi$ be an elementary submodel such that $|N| = \kappa$, $N \supset N^{<\kappa}$, $N \cap \kappa^+ \in \mathrm{On}$, and $p, \mathbb{P}_{\bar{E},\epsilon}, \dot{X} \in N$. Let $f^* \leq^* f^p$ be a fully $\langle N, \mathbb{P}^*_{\bar{E},\epsilon}\rangle$-generic condition. Let A be a tree such that $\langle f^*, A\rangle \leq^* p$. For each $\langle \nu\rangle \in A$ set

$$D_{\langle\nu\rangle} = \{f \leq^* f^p_{\langle\nu\rangle\to} \mid \exists r \leq p_\leftarrow \ \exists s \leq \langle f^*, A\rangle_{\langle\nu\rangle\leftarrow} \ \exists B \ r \frown s \frown \langle f, B\rangle \parallel \ulcorner \nu(\check{\kappa})_0 \in \dot{X}\urcorner\}$$

Observe that by Prikry's condition the set $D_{\langle\nu\rangle}$ ($\langle\nu\rangle \in A$) is dense open below $f^p_{\langle\nu\rangle\to}$. Thus for each $\langle\nu\rangle \in A$, $f^*_{\langle\nu\rangle} \in D_{\langle\nu\rangle}$. In particular there is a condition $r \leq p_\leftarrow$ and functions $s : \mathrm{Lev}_0(A) \to \mathbb{P}_{\bar{E},\epsilon\leftarrow}$ and $B : \mathrm{Lev}_0(A) \to$ such that

$$\{\langle\nu\rangle \in A \mid r \frown s(\nu) \frown \langle f^*_{\langle\nu\rangle\to}, B(\nu)\rangle \parallel \ulcorner \nu(\check{\kappa})_0 \in \dot{X}\urcorner\} \in E_\zeta(f^*).$$

Let $g' = f^{j_{E_\zeta}(s)(\mathrm{mc}_\zeta(f^*))}$. Then set $g = \{\langle \bar{\alpha}, g'(\bar{\alpha} \upharpoonright \zeta)\rangle \mid \bar{\alpha} \upharpoonright \zeta \in \operatorname{dom} g'\}$. Observe that $g \leq^* f^*$. Let p^* be a condition such that $f^{p^*} = g$, $p^*_\leftarrow = r$, and

$$\{\langle\nu\rangle \in A^{p^*} \mid s(\bar{\nu} \upharpoonright \operatorname{dom} f^*) = p^*_{\to\langle\bar{\nu}\rangle\leftarrow},\ p^*_{\langle\nu\rangle\to} \leq^* \langle f^*_{\langle\nu\upharpoonright\operatorname{dom} f^*\rangle\to}, B(\nu \upharpoonright \operatorname{dom} f^*)\rangle\} \in E_\zeta(g).$$

Thus $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle} \parallel_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$. ☐

Observe that if p and q are compatible conditions, and $\zeta < \mathrm{o}(\bar{E})$ is a repeat point of $\bar{E}$, then $j_{E_\zeta}(p)_{\langle \mathrm{mc}_\zeta(p)\rangle} \parallel_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} j_{E_\zeta}(q)_{\langle \mathrm{mc}_\zeta(q)\rangle}$. Thus it makes sense to define a subset U of $\mathcal{P}(\kappa)$ in a $\mathbb{P}_{\bar{E},\epsilon}$-generic extension by:

$$j_{E_\zeta}(p)_{\langle \mathrm{mc}_\zeta(p_\to)\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner \implies p \Vdash \ulcorner \dot{X} \in \dot{U}\urcorner.$$

Let us fix through the end of the section the repeat point ζ and the name $\dot{U}$.

Corollary 5.8. *Assume $p \Vdash \ulcorner \dot{X} \in \dot{U}\urcorner$. Then there is a Prikry extension $p^* \leq^* p$ such that $p^*_\leftarrow = p_\leftarrow$, and*

$$j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner.$$

Proof. Construct by induction the sequence $\langle p^\xi \mid \xi < \lambda\rangle$ ($\lambda < \kappa$) where $\{p^\xi_\leftarrow \mid \xi < \lambda\}$ is a maximal anti-chain below $p_\leftarrow$, $\langle p^\xi_\to \mid \xi < \lambda\rangle$ is $\leq^*$-decreasing below $p_\to$, and $j_{E_\zeta}(p^\xi)_{\langle \mathrm{mc}_\zeta(p^\xi_\to)\rangle} \parallel_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$ as follows.

Assume $\langle p^{\xi'} \mid \xi' < \xi\rangle$ was constructed. If $\{p^{\xi'}_\leftarrow \mid \xi' < \xi\}$ is a maximal anti-chain below $p_\leftarrow$ then the induction terminates with $\lambda = \xi$. Otherwise choose $q' \leq p_\leftarrow$ such that $q' \perp p^{\xi'}_\leftarrow$ for each $\xi' < \xi$. Let p' be a Prikry extension of $p^{\xi'}_\to$ for each $\xi' < \xi$. By 5.7 there is an extension $q \leq q' \frown p'$ such that $q_\leftarrow \leq q'$, $q_\to \leq^* p'$, and $j_{E_\zeta}(q)_{\langle \mathrm{mc}_\zeta(q_\to)\rangle} \parallel_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$. The inductive step is finished by setting $p^\xi = q$.

When the induction terminates let p^* be a condition such that $p^*_\leftarrow = p_\leftarrow$ and $p^*_\to \leq^* p^\xi_\to$ for each $\xi < \lambda$. We claim that $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$. To show this take an arbitrary condition $s \frown r \leq_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle}$ satisfying $s \leq j_{E_\zeta}(p^*_\leftarrow)$ and $r \leq j_{E_\zeta}(p^*_\to)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle}$. Since $j_{E_\zeta}(p^*_\leftarrow) = p_\leftarrow$, there is $\xi < \lambda$ such that $s \parallel p^\xi_\leftarrow$. Since $j_{E_\zeta}(p^*_\to)_{\langle \mathrm{mc}_\zeta(p^*_\to)\rangle} \leq^*_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} j_{E_\zeta}(p^\xi_\to)_{\langle \mathrm{mc}_\zeta(p^\xi_\to)\rangle}$, $r \leq_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})}$

$j_{E_\zeta}(p^\xi_{\rightarrow})_{\langle \mathrm{mc}_\zeta(p^\xi_{\rightarrow})\rangle}$. Thus $s \frown r \parallel j_{E_\zeta}(p^\xi)_{\langle \mathrm{mc}_\zeta(p^\xi_{\rightarrow})\rangle}$. Since $j_{E_\zeta}(p^\xi)_{\langle \mathrm{mc}_\zeta(p^\xi_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})}$ $\ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$, $r \frown s \not\Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \notin \dot{X}\urcorner$. □

The following claim connects the preservation of measurability with $|\epsilon|$. In the case of Radin forcing, a sequence of measures of length of cofinality greater than κ^+ is enough in order to preserve measurability, so one might suspect a technical weakness in the proof of the claim. However, we give an example showing the dependency on $|e|$ cannot be removed. Thus let $\bar{E}$ be an extender sequence of length κ^{+4}, where each extender has only κ^{+3}-generator, and the set of all generators is unbounded in κ^{+4}. Let ϵ be $\sup\{j_{E_\xi}(\kappa) \mid \xi < \kappa^{+4}\}$. Then in the generic extension we have $2^\kappa = \kappa^{+4}$, and there is a club of κ on which $2^\mu = \mu^{+3}$ holds. Now, κ remains regular but it cannot be measurable. Incidently, this example demonstrates that the claim on preservation of measurability appearing in [5] is incorrect.

Claim 5.9. *If $\mathrm{cf}(\mathrm{o}(\bar{E})) > |\epsilon|$, then κ is measurable in a $\mathbb{P}_{\bar{E},\epsilon}$-generic extension.*

Proof. Fix some $d \in \mathcal{P}_{\kappa^+}\mathfrak{D}$. For each $\rho < \mathrm{o}(\bar{E})$ define $F_\rho = \bigcap_{\rho' < \rho} E_{\rho'}(d)$. Since the sequence $\langle F_\rho \mid \rho < \mathrm{o}(\bar{E})\rangle$ is $\subseteq$-decreasing, each filter has at most κ^+ elements, and $\mathrm{cf}(\mathrm{o}(\bar{E})) > \kappa^+$, there is $\zeta_d < \mathrm{o}(\bar{E})$ such that the sequence stablizes, i.e., for each $\zeta_d \leq \rho, \rho' < \mathrm{o}(\bar{E})$, $F_\rho = F_{\rho'}$.

Since $|\mathfrak{D}| = |\epsilon|$, $|\{\zeta_d \mid d \in \mathcal{P}_{\kappa^+}\mathfrak{D}\}| \leq |\epsilon|$. Since $\mathrm{cf}(\mathrm{o}(\bar{E})) > |\epsilon|$, $\zeta = \sup\{\zeta_d \mid d \in \mathcal{P}_{\kappa^+}(\mathfrak{D})\} < \mathrm{o}(\bar{E})$. By the construction, ζ is a repeat point of $\bar{E}$. Thus, one can consider a condition $p \in \mathbb{P}_{\bar{E}\restriction\zeta,\epsilon}$ to be also in $\mathbb{P}_{\bar{E},\epsilon}$.

We prove that U is a normal measure on κ. We use the common convention that the $\mathbb{P}_{\bar{E},\epsilon}$-names of the sets X, X_ξ, Y, and U, in the generic extension are $\dot{X}$, $\dot{X}_\xi$, $\dot{Y}$, and $\dot{U}$, respectively.

$\underline{X \in U,\ X \subseteq Y \implies Y \in U}$: Assume $p \Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \dot{X} \in \dot{U},\ \dot{X} \subseteq \dot{Y}\urcorner$. By lemma 5.7 there is a Prikry extension $p^* \leq^* p$ such that $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{X})\urcorner$. It is immediate that $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner j_{E_\zeta}(\dot{X}) \subseteq j_{E_\zeta}(\dot{Y})\urcorner$, thus $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\dot{Y})\urcorner$. Hence, $p^* \Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \dot{Y} \in \dot{U}\urcorner$.

$\underline{X \notin U \implies \kappa \setminus X \in U}$: Assume $p \Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \dot{X} \notin \dot{U}\urcorner$. By lemma 5.7 there is a Prikry extension $p^* \leq^* p$ such that $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \notin j_{E_\zeta}(\dot{X})\urcorner$. Thus $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \check{\kappa} \in j_{E_\zeta}(\check{\kappa} \setminus \dot{X})\urcorner$. Hence $p^* \Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \check{\kappa} \setminus \dot{X} \in \dot{U}\urcorner$.

$\underline{\forall \xi < \kappa\ X_\xi \in U \implies \triangle_{\xi<\kappa} X_\xi \in U}$: Assume $p \Vdash_{\mathbb{P}_{\bar{E},\epsilon}} \ulcorner \forall \xi < \check{\kappa}\ \dot{X}_\xi \in \dot{U}\urcorner$. Using lemma 5.7 construct by induction the sequence $\langle p^\xi \mid \xi < \kappa\rangle$ such that $\langle p^\xi_{\rightarrow} \mid \xi < \kappa\rangle$ is $\leq^*$-decreasing below $p_{\rightarrow}$, and for each $\xi < \kappa$, $p^\xi_{\leftarrow} = p_{\leftarrow}$. Let $f^* = \bigcup_{\xi<\kappa} f^{p^\xi_{\rightarrow}}$. Construct an f^*-tree A as follows:

$$\mathrm{Lev}_0(A) = \{\nu \in \mathrm{OB}(f) \mid \forall \xi < \nu(\bar{\kappa})_0\ \langle \nu \restriction \mathrm{dom}\, f^{p^\xi_{\rightarrow}}\rangle \in A^{p^\xi_{\rightarrow}}\},$$

and

$$A_{\langle\nu\rangle} = \bigcap_{\xi<\nu(\bar{\kappa})_0} A^{p^\xi_{\rightarrow}}.$$

Set $p^* = p \frown \langle f^*, A\rangle$. By the construction we get that for each $\xi < \kappa$,

$$j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \leq^*_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} j_{E_\zeta}(p^\xi)_{\langle \mathrm{mc}_\zeta(p^\xi_{\rightarrow})\rangle}.$$

Thus $j_{E_\zeta}(p^*)_{\langle \mathrm{mc}_\zeta(p^*_{\rightarrow})\rangle} \Vdash_{j_{E_\zeta}(\mathbb{P}_{\bar{E},\epsilon})} \ulcorner \forall \xi < \check{\kappa}\ \check{\kappa} \in j_{E_\zeta}(\dot{X}_\xi)\urcorner$. Thus $p^* \Vdash \ulcorner \triangle_{\xi<\kappa} \dot{X}_\xi \in \dot{U}\urcorner$. □

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