

EXTENDER BASED RADIN FORCING

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ABSTRACT. We define extender sequences, generalizing measure sequences of Radin forcing.

Using the extender sequences, we show how to combine the Gitik-Magidor forcing for adding many Prikry sequences with Radin forcing.

We show that this forcing satisfies a Prikry like condition destroys no cardinals, and has a kind of properness.

Depending on the large cardinals we start with this forcing can blow the power of a cardinal together with changing its cofinality to a prescribe value. It can even blow the power of a cardinal while keeping it regular or measurable.

1. INTRODUCTION

We give some background on previous work relating directly to the present work. The first forcing which changed the cofinality of a cardinal without changing the cardinal structure was Prikry forcing [6]. In this forcing a measurable cardinal, κ , was ‘invested’ in order to get $\text{cf}(\kappa) = \omega$ without collapsing any cardinal. Developing that idea, Magidor [2] used a coherent sequence of measures of length $\lambda < \kappa$ in order to get $\text{cf}(\kappa) = \lambda$ without collapsing any cardinals. In [7] Radin, introducing the notion of measure sequence, showed that it is useful to continue the coherent sequence to $\lambda > \kappa$. For example, κ remains regular when $\lambda = \kappa^+$. In general the longer the measure sequence the more resemblance there is between κ in the generic extension and the ground model.

As is well known, and unlike regular cardinals, blowing the power of a singular cardinal is not an easy task. A natural approach to try was to blow the power of a cardinal while it was regular and after that make it singular by one of the above methods. A crucial idea of Gitik and Magidor [3] was to combine the power set blowing and the cofinality change in one forcing. They introduced a forcing notion which added many Prikry sequences at once and still collapsed no cardinals. The ‘investment’ they needed for this was an extender of length which is the size of power they wanted. Building on the idea of Gitik and Magidor, Segal [8] implemented the idea of adding many sequences to Magidor forcing. So by investing a coherent sequence of extenders of length $\lambda < \kappa$ she was able to get a singular cardinal of cofinality λ together with power as large as the length of the extenders in question.

Date: October 10, 1998.

1991 Mathematics Subject Classification. Primary 03E35, 03E55, 04A30.

Key words and phrases. Forcing, Radin forcing, Extender, Extender based forcing, Generalized continuum hypothesis, Singular Cardinal hypothesis.

This work is a part of research which, hopefully, will become the author’s Ph.D. thesis. It was done at Tel-Aviv university under the supervision of M. Gitik. The author thanks M. Gitik for his help with this work, in other works and just in general.

We thank Asaf Sharon for reading and pointing out some problems in a preliminary version of this work.

Our work also builds on the idea of Gitik and Magidor. However, we implement the idea of adding many sequences to Radin forcing. So we introduce the notion of extender sequence and show that it makes sense to deal with quite long extender sequences. As in Radin forcing, for long enough sequences we are left with κ which is regular and even measurable. The power size will be the length of the extenders we start with.

The structure of this work is as follows. In section 2 we define extender sequences. In section 3 we define $P_{\bar{E}}$, the forcing notion which is the purpose of this work. In section 4 we define Radin forcing as a means to prove lemmas we need later. The definition is not the usual one. The definition we give here is more in the spirit of the forcing we defined in section 3. In section 5 we show the chain-conditions satisfied by $P_{\bar{E}}$ and how ‘locally’ it resembles Radin forcing. We also show here that there are many new subsets in the generic extension. In section 6 we investigate the structure of dense open subsets of $P_{\bar{E}}$. We show that they satisfy a strong homogeneity property. In section 7 we prove Prikry’s condition for $P_{\bar{E}}$. The proof is a simple corollary of the strong homogeneity of dense open subsets. In section 8 we show that $P_{\bar{E}}$ satisfies a kind of properness. In section 9 we combine the machinery developed so far in order to show that no cardinals are collapsed. In section 10 we show how the length of the extender sequences affect the properties of κ . Section 11 summarizes what the forcing $P_{\bar{E}}$ does. In section 12 we have a result concerning $P_{\bar{E}}$ when $l(\bar{E}) = 1$. We show that there is, in V , a generic filter over an elementary submodel in an ω -iterate of V . We were not able to prove something equivalent (or weaker) for the general case. Section 13 contains a list of missing or unknown points to check. The last point in this list is in preparation.

Our notation is standard. We assume fluency with forcing and extenders. Some basic properties of Radin forcing are taken for granted.

2. EXTENDER SEQUENCES

2.1. Constructing from elementary embedding. Suppose we have an elementary embedding $j:V \rightarrow M \supset V_\lambda$, $\text{crit}(j) = \kappa$. The value of λ is determined later, according to the different applications we have.

Construct from j a nice extender like in [3]:

$$E(0) = \langle \langle E_\alpha(0) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha, \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

We recall the properties of this extender:

- (1) $\mathcal{A} \subseteq |V_\lambda| \setminus \kappa$,
- (2) $|\mathcal{A}| = |V_\lambda|$,
- (3) $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ is a κ^+ -directed partial order,
- (4) $\forall \alpha, \beta \in \mathcal{A} \beta \geq_{\mathcal{A}} \alpha \implies \pi_{\beta,\alpha}: V_\kappa \rightarrow V_\kappa$,
- (5) $\kappa \in \mathcal{A}$,
- (6) $\forall \alpha \in \mathcal{A} \kappa \leq_{\mathcal{A}} \alpha$. We write $\pi_{\alpha,0}$ instead of $\pi_{\alpha,\kappa}$,
- (7) $\forall \alpha, \beta \in \mathcal{A} \forall \nu < \kappa \nu^0 = \pi_{\alpha,0}(\nu) = \pi_{\beta,0}(\nu)$,
- (8) $\forall \alpha, \beta \in \mathcal{A} \beta \geq_{\mathcal{A}} \alpha \implies \forall \nu < \kappa \pi_{\beta,0}(\nu) = \pi_{\alpha,0}(\pi_{\beta,\alpha}(\nu))$,
- (9) $\forall \alpha, \beta, \gamma \in \mathcal{A} \gamma \geq_{\mathcal{A}} \beta \geq_{\mathcal{A}} \alpha \implies$

$$\exists A \in E_\gamma(0) \forall \nu \in A \pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu)).$$

If, for example, we need $|E(0)| = \kappa^{+3}$ then, under GCH, we require $\lambda = \kappa + 3$. A typical large set in this extender concentrates on singletons.

If j is not sufficiently closed, then $E(0) \notin M$ and the construction stops. We set

$$\forall \alpha \in \mathcal{A} \bar{E}_\alpha = \langle \alpha, E(0) \rangle.$$

We say that \bar{E}_α is an extender sequence of length 1. ($l(\bar{E}_\alpha) = 1$)

If, on the other hand, $E(0) \in M$ we can construct for each $\alpha \in \text{dom } E(0)$ the following ultrafilter

$$A \in E_{\langle \alpha, E(0) \rangle}(1) \iff \langle \alpha, E(0) \rangle \in j(A).$$

Such an A concentrates on elements of the form $\langle \xi, e(0) \rangle$ where $e(0)$ is an extender on ξ^0 and $\xi \in \text{dom } e(0)$. Note that $e(0)$ concentrates on singletons below ξ^0 . If, for example, $|E(0)| = \kappa^{+3}$ then on a large set we have $|e(0)| = (\xi^0)^{+3}$.

We define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ as

$$\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}(\langle \xi, e(0) \rangle) = \langle \pi_{\beta, \alpha}(\xi), e(0) \rangle.$$

From this definition we get

$$j(\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle})(\langle \beta, E(0) \rangle) = \langle \alpha, E(0) \rangle.$$

Hence we have here an extender

$$E(1) = \langle \langle E_{\langle \alpha, E(0) \rangle}(1) \mid \alpha \in \mathcal{A} \rangle, \langle \pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle} \mid \beta \geq_{\mathcal{A}} \alpha, \beta \in \mathcal{A} \rangle \rangle.$$

Note that the difference between $\pi_{\beta, \alpha}$ and $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ is quite superficial. We can define $\pi_{\langle \beta, E(0) \rangle, \langle \alpha, E(0) \rangle}$ in a uniform way for both extenders. Just project the first element of the argument using $\pi_{\beta, \alpha}$.

If $\langle E(0), E(1) \rangle \notin M$ then the construction stops. In this case we set

$$\forall \alpha \in \mathcal{A} \bar{E}_\alpha = \langle \alpha, E(0), E(1) \rangle.$$

We say that \bar{E}_α is an extender sequence of length 2. ($l(\bar{E}_\alpha) = 2$)

If $\langle E(0), E(1) \rangle \in M$ then we construct the extender $E(2)$ in the same way as we constructed $E(1)$ from $E(0)$.

The above special case being worked out we continue with the general case. Assume we have constructed

$$\langle E(\tau') \mid \tau' < \tau \rangle.$$

If $\langle E(\tau') \mid \tau' < \tau \rangle \notin M$ then the construction stops here. We set

$$\forall \alpha \in \mathcal{A} \bar{E}_\alpha = \langle \alpha, E(\tau') \mid \tau' < \tau \rangle,$$

and we say that \bar{E}_α is an extender sequence of length τ . ($l(\bar{E}_\alpha) = \tau$)

If, on the other hand, $\langle E(\tau') \mid \tau' < \tau \rangle \in M$ then we construct

$$A \in E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}(\tau) \iff \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle \in j(A).$$

Defining $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}$ using the first coordinate as before gives the needed projection.

We are quite casual in writing the indices of the projections and ultrafilters. By this we mean that we sometimes write $\pi_{\beta, \alpha}$ when we should have written $\pi_{\langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle, \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}$ and $E_\alpha(\tau)$ when we should have written $E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau \rangle}(\tau)$.

With this abuse of notation the projection we just defined satisfies

$$j(\pi_{\beta,\alpha})(\langle\beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle) = \langle\alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle,$$

and we have the extender

$$E(\tau) = \langle\langle E_\alpha(\tau) \mid \alpha \in \mathcal{A}\rangle, \langle\pi_{\beta,\alpha} \mid \beta \geq_{\mathcal{A}} \alpha, \beta \in \mathcal{A}\rangle\rangle.$$

We let the construction run until it stops due to the extender sequence not being in M .

Definition 2.1. We call $\bar{\mu}$ an extender sequence if there is an elementary embedding $j:V \rightarrow M$ such that $\bar{\nu}$ is an extender sequence generated as above and $\bar{\mu} = \bar{\nu} \upharpoonright \tau$ for $\tau \leq l(\bar{\nu})$. $\kappa(\bar{\mu})$ is the ordinal at the beginning of the sequence. (i.e. $\kappa(\bar{E}_\alpha) = \alpha$). $\kappa^0(\bar{\mu})$ is $(\kappa(\bar{\mu}))^0$. (i.e. $\kappa^0(\bar{E}_\alpha) = \kappa$).

That is, we do not have to construct the extender sequence until it is not in M . We can stop anywhere on the way.

The generalization of the measure on the α coordinate in Gitik-Magidor forcing [3] is \bar{E}_α .

Definition 2.2. A sequence of extender sequences $\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ is called $^\circ$ -increasing if $\kappa^0(\bar{\mu}_1) < \dots < \kappa^0(\bar{\mu}_n)$.

Definition 2.3. Let $\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ be $^\circ$ -increasing. An extender sequences $\bar{\mu}$ is called permitted to $\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ if $\kappa(\bar{\mu}_n) < \kappa^0(\bar{\mu})$.

Definition 2.4. We say $A \in \bar{E}_\alpha$ if $\forall \xi < l(\bar{E}_\alpha) A \in E_\alpha(\xi)$.

Definition 2.5. $\bar{E} = \langle\bar{E}_\alpha \mid \alpha \in \mathcal{A}\rangle$ is an extender sequence system if there is an elementary embedding $j:V \rightarrow M$ such that all \bar{E}_α are extender sequences generated from j as prescribed above and $\forall \alpha, \beta \in \mathcal{A} l(\bar{E}_\alpha) = l(\bar{E}_\beta)$. This common length is called the length of the system, $l(\bar{E})$. We write $\bar{E}(\bar{\mu})$ for the extender sequence system to which $\bar{\mu}$ belongs (i.e. $\bar{E}(\bar{E}_\alpha) = \bar{E}$).

We point out that there is a κ^+ -directed partial order on \bar{E} inherited from \mathcal{A} . That is $\bar{E}_\beta \geq_{\bar{E}} \bar{E}_\alpha \iff \beta \geq_{\mathcal{A}} \alpha$. Of course, this implies that there is $\min \bar{E}$, namely \bar{E}_κ . From now on we use only the order $\geq_{\bar{E}}$, even for \mathcal{A} , and we write $\text{dom } \bar{E}$ for \mathcal{A} .

2.2. \bar{E}_α -tree.

Definition 2.6. A tree T is an \bar{E}_α -tree if its elements are of the form

$$\langle\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle, S\rangle$$

where

(1) Set $\text{dom } T = \{\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle \mid \langle\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle, S\rangle \in T\}$. Then the function

$$\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle \mapsto \langle\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle, S\rangle$$

from $\text{dom } T$ to T is 1 – 1 and onto,

(2) $t \in \text{Lev}_n(\text{dom } T) \implies |t| = n + 1$,

(3) $\langle\bar{\mu}_1, \dots, \bar{\mu}_n\rangle$ is a $^\circ$ -increasing sequence,

(4) $\text{Lev}_0(\text{dom } T) \in \bar{E}_\alpha$ and for each $t \in \text{dom } T \text{SuC}_{\text{dom } T}(t) \in \bar{E}_\alpha$,

(5) S is a $\bar{\mu}_n$ -tree. When $l(\bar{\mu}_n) = 0$ we set $S = \emptyset$.

Note that this clause is recursive.

Note 2.7. While we call T a tree, formally speaking, it is a function with domain a tree (in the usual sense) of finite sequences.

Note 2.8. Later on, we abuse notation and use T instead of $\text{dom } T$. I.e., $\text{Suc}_T(t)$ instead of $\text{Suc}_{\text{dom } T}(t)$.

Definition 2.9. Assume T is a \bar{E}_α -tree and $t \in T$, then:

- (1) $T_t = \{\langle s, S \rangle \mid \langle t \frown s, S \rangle \in T\}$.
- (2) $T(\bar{\mu})$ is the tree, S , satisfying $\langle \bar{\mu}, S \rangle \in \text{Lev}_0(T)$.
- (3) $T_t(\bar{\mu})$ is the tree, S , satisfying $\langle \bar{\mu}, S \rangle \in \text{Suc}_T(t)$.

Definition 2.10. Let T, S be \bar{E}_α -trees, where $l(\bar{E}) = 1$. We say that $T \leq S$ if

- (1) $\text{Lev}_0(T) \subseteq \text{Lev}_0(S)$,
- (2) $\forall t \in T \text{ Suc}_T(t) \subseteq \text{Suc}_S(t)$.

Definition 2.11. Let T, S be \bar{E}_α -trees. We say that $T \leq S$ if

- (1) $\text{Lev}_0(T) \subseteq \text{Lev}_0(S)$,
- (2) $\forall t \in T \text{ Suc}_T(t) \subseteq \text{Suc}_S(t)$,
- (3) $\forall \langle \bar{\mu} \rangle \in \text{Lev}_0(T) T(\bar{\mu}) \leq S(\bar{\mu})$,
- (4) $\forall t \in T \forall \langle \bar{\mu} \rangle \in T_t T_t(\bar{\mu}) \leq S_t(\bar{\mu})$.

Note that the last 2 conditions are recursive.

Definition 2.12. Let S be \bar{E}_α -tree and $\beta > \alpha$. Define $T = \pi_{\beta, \alpha}^{-1}(S)$ by

- (1) $\text{dom } T = \pi_{\beta, \alpha}^{-1}(\text{dom } S)$,
- (2) $T_{\langle \bar{\mu}_1, \dots, \bar{\mu}_{n-1} \rangle}(\bar{\mu}_n) = \pi_{\bar{\mu}_n, \pi_{\beta, \alpha}(\bar{\mu}_n)}^{-1} S_{\langle \pi_{\beta, \alpha}(\bar{\mu}_1), \dots, \pi_{\beta, \alpha}(\bar{\mu}_{n-1}) \rangle}(\pi_{\beta, \alpha}(\bar{\mu}_n))$.

Definition 2.13. Let T, S be $\bar{E}_\beta, \bar{E}_\alpha$ -trees respectively, where $\beta \geq \bar{E} \alpha$. We say that $T \leq S$ if

- (1) $T \leq \pi_{\beta, \alpha}^{-1}(S)$.

Definition 2.14. Assume we have $A_{\langle \bar{\nu}, R \rangle}$ where $\bar{\nu}$ is an extender sequence such that each element in $A_{\langle \bar{\nu}, R \rangle}$ is of the form $\langle \bar{\mu}, S \rangle$ where $\bar{\mu}$ is an extender sequence and S is a tree. (in this work we always have S is $\bar{\mu}$ -tree). We define $\Delta_{\langle \bar{\nu}, R \rangle}^0 A_{\langle \bar{\nu}, R \rangle}$ as

$$\langle \bar{\mu}, S \rangle \in \Delta_{\langle \bar{\nu}, R \rangle}^0 A_{\langle \bar{\nu}, R \rangle} \iff \forall \langle \bar{\nu}, R \rangle \kappa(\bar{\nu}) < \kappa^0(\bar{\mu}) \rightarrow \langle \bar{\mu}, S_{\langle \bar{\nu}, R \rangle} \rangle \in A_{\langle \bar{\nu}, R \rangle}.$$

3. $P_{\bar{E}}$ -FORCING

The definition of the forcing is done by induction on $l(\bar{E})$. Originally we started directly giving the general case. At the suggestion of the referee we have added concrete definitions also for the cases $l(\bar{E}) = 1$, $l(\bar{E}) = 2$ and several figures we hope will enhance the intuition standing behind the definitions.

Definition 3.1. Assume $l(\bar{E}) = 0$. A condition in $P_{\bar{E}}^*$ is of the form

$$\left\{ \left\langle \langle \kappa \rangle, p^{(\kappa)} \right\rangle \right\}$$

where

- (1) $p^{(\kappa)} \in V_\kappa$ is an extender sequence. ($p^{(\kappa)} = \emptyset$ is allowed.)

When $l(\bar{E}) = 0$ the partial order \leq^* degenerates into $=$:

Definition 3.2. Assume $l(\bar{E}) = 0$. Let $p, q \in P_{\bar{E}}^*$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if

$$(1) \ p^0 = q^0.$$

Definition 3.3. Assume $l(\bar{E}) = 0$. We set $\langle P_{\bar{E}}, \leq \rangle = \langle P_{\bar{E}}^*, \leq^* \rangle$.

Hence, when $l(\bar{E}) = 0$, $P_{\bar{E}}$ is not really a forcing. We use it more as a placeholder in order not to convolute later proofs with 2 cases everywhere.

Definition 3.4. Assume $l(\bar{E}) = 1$. A condition in $P_{\bar{E}}^*$ is of the form

$$\{\langle \langle \alpha, E(0) \rangle, p^{\langle \alpha, E(0) \rangle} \mid \langle \alpha, E(0) \rangle \in g\} \cup \{T\}$$

where

- (1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
- (2) $\min \bar{E} = \langle \kappa, E(0) \rangle \in g$ and g has a maximal element,
- (3) $p^{\langle \kappa, E(0) \rangle} \in V_{\kappa^0(\bar{E})}$ is an extender sequence which we call p^0 , (We allow $p^0 = \emptyset$),
- (4) $\forall \langle \alpha, E(0) \rangle \in g \setminus \{\langle \kappa, E(0) \rangle\}$ $p^{\langle \alpha, E(0) \rangle} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a $^\circ$ -increasing sequence. We allow $p^{\langle \alpha, E(0) \rangle} = \emptyset$,
- (5) T is a max g -tree such that for all $t \in T$ $p^{\max g} \cap t$ is $^\circ$ -increasing. If $\langle \bar{\nu} \rangle \in T$ then $\bar{\nu} = \langle \xi \rangle$ for some $\xi < \kappa$. (Meaning that $\bar{\nu}$ is an extender sequence of length 0).
- (6) For all $\langle \alpha, E(0) \rangle \in g$, p^0 is not permitted to $p^{\langle \alpha, E(0) \rangle}$,
- (7) $\forall \langle \bar{\nu} \rangle \in T$ $|\{\langle \alpha, E(0) \rangle \in g \mid \bar{\nu} \text{ is permitted to } p^{\langle \alpha, E(0) \rangle}\}| \leq \kappa^0(\bar{\nu})$,
- (8) $\forall \langle \bar{\nu} \rangle \in T$ if $\bar{\nu}$ is permitted to $p^{\langle \beta, E(0) \rangle}, p^{\langle \alpha, E(0) \rangle}$ then

$$\pi_{\max g, \langle \beta, E(0) \rangle}(\bar{\nu}) \neq \pi_{\max g, \langle \alpha, E(0) \rangle}(\bar{\nu}).$$

We write $\text{mc}(p)$, p^{mc} , T^p , $\bar{E}(p)$, $\text{supp } p$ for $\max g$, $p^{\max g}$, T , \bar{E} , g respectively.

Definition 3.5. Assume $l(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}^*$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if

- (1) $\text{supp } p \supseteq \text{supp } q$,
- (2) $\forall \langle \alpha, E(0) \rangle \in \text{supp } q$ $p^{\langle \alpha, E(0) \rangle} = q^{\langle \alpha, E(0) \rangle}$,
- (3) $T^p \leq \pi_{\text{mc}(p), \text{mc}(q)}^{-1} T^q$,
- (4) $\forall \langle \alpha, E(0) \rangle \in \text{supp } q$ $\forall \langle \bar{\nu} \rangle \in T^p$

$$\max \kappa(p^{\langle \alpha, E(0) \rangle}) < \kappa^0(\bar{\nu}) \implies$$

$$\pi_{\text{mc}(p), \alpha}(\bar{\nu}) = \pi_{\text{mc}(q), \alpha}(\pi_{\text{mc}(p), \text{mc}(q)}(\bar{\nu})).$$

Definition 3.6. Assume $l(\bar{E}) = 1$. A condition in $P_{\bar{E}}$ is of the form

$$p_n \hat{\ } \cdots \hat{\ } p_0$$

where

- $p_0 \in P_{\bar{E}}^*$,
- $p_1 \in P_{\bar{\mu}_1}^*$,
- \vdots ,
- $p_n \in P_{\bar{\mu}_n}^*$,

where $\bar{E}, \bar{\mu}_1, \dots, \bar{\mu}_n$ are extender sequence systems such that $l(\bar{\mu}_1) = 0, \dots, l(\bar{\mu}_n) = 0$ satisfying

$$\forall i \leq n-1 \ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

Definition 3.7. Assume $l(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$\begin{aligned} p &= p_n \hat{\ } \cdots \hat{\ } p_0, \\ q &= q_n \hat{\ } \cdots \hat{\ } q_0, \end{aligned}$$

and

- $p_0, q_0 \in P_{\bar{E}}^*$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in P_{\bar{\mu}_1}^*$, $p_1 \leq^* q_1$,
- \vdots ,
- $p_n, q_n \in P_{\bar{\mu}_n}^*$, $p_n \leq^* q_n$.

$p_{0\langle \bar{v} \rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition p_0 , which adds the extender sequence $\bar{v} \in T^{p_0}$ to the finite sequence.

Definition 3.8. Assume $l(\bar{E}) = 1$. Let $p \in P_{\bar{E}}^*$ and $\langle \bar{v} \rangle \in T^p$. In this case $\bar{v} = \langle \xi \rangle$ for some $\xi < \kappa$. We define $(p)_{\langle \bar{v} \rangle}$ to be $p'_1 \hat{\ } p'_0$ where

- (1) $\text{supp } p'_0 = \text{supp } p$,
- (2) $\forall \langle \alpha, E(0) \rangle \in \text{supp } p'_0 \ p'_0{}^{\langle \alpha, E(0) \rangle} = \begin{cases} \langle \pi_{\text{mc}(p), \alpha}(\xi) \rangle & \max \kappa(p^{\langle \alpha, E(0) \rangle}) < \kappa^0(\xi), \ \alpha = \kappa, \\ p^{\langle \alpha, E(0) \rangle} \hat{\ } \langle \pi_{\text{mc}(p), \alpha}(\xi) \rangle & \max \kappa(p^{\langle \alpha, E(0) \rangle}) < \kappa^0(\xi), \ \alpha \neq \kappa, \\ p^{\langle \alpha, E(0) \rangle} & \text{Otherwise.} \end{cases}$
- (3) $T^{p'_0} = T_{\langle \bar{v} \rangle}^p$.
- (4) $\text{supp } p'_1 = \{ \langle \pi_{\text{mc}(p), \kappa}(\xi) \rangle \}$,
- (5) $p'_1{}^{\langle \pi_{\text{mc}(p), \kappa}(\xi) \rangle} = p^{\langle \kappa, E(0) \rangle}$. (Note that $p'_1 \in P_{\langle \pi_{\text{mc}(p), \kappa} \rangle}^*$.)

Definition 3.9. Assume $l(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that p is a 1-point extension of q ($p \leq^1 q$) if p, q are of the form

$$\begin{aligned} p &= p_{n+1} \hat{\ } p_n \hat{\ } \cdots \hat{\ } p_0, \\ q &= q_n \hat{\ } \cdots \hat{\ } q_0, \end{aligned}$$

and

- $p_{i+1}, q_i \in P_{\bar{\mu}_i}^*$, $p_{i+1} \leq^* q_i$ for $i = 1, \dots, n$,
- There is $\langle \bar{v} \rangle \in T^{q_0}$ such that $p_1 \hat{\ } p_0 \leq^* (q_0)_{\langle \bar{v} \rangle}$.

Definition 3.10. Assume $l(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that p is an n -point extension of q ($p \leq^n q$) if there are p^n, \dots, p^0 such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 3.11. Assume $l(\bar{E}) = 1$. Let $p, q \in P_{\bar{E}}$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

It is quite clear that when $l(\bar{E}) = 1$, $\langle P_{\bar{E}}, \leq \rangle$ is the Gitik-Magidor forcing from [3] with a slight change in

- (1) The support elements are $\langle \alpha, E(0) \rangle$ instead of α .

(2) $p^{\langle \kappa, E(0) \rangle}$ has at most one element and not a finite sequence.

We give some figures in order to enhance the intuition behind these definitions. In figure 1 we show a typical condition $p_0 \in P_{\bar{E}}^*$. In figure 2 we show a 1-point extension of p_0 using $\langle \nu \rangle \in T^{p_0}$, assuming $\xi_{1,2}^0 \geq \nu^0$. In figure 3 we show a 1-point extension of $p_{0\langle \bar{\nu} \rangle}$ using $\langle \mu \rangle \in T_{\langle \bar{\nu} \rangle}^{p_0}$, again assuming $\xi_{2,1} \geq \mu^0$.

If G is $P_{\bar{E}}$ -generic and we set $C^\alpha = \cup \{p^{\langle \alpha, E(0) \rangle} \mid p \in G, \langle \alpha, E(0) \rangle \in \text{supp } p\}$ then C^α is an ω -sequence unbounded in κ .

$$\frac{\langle \xi_{0,0} \rangle}{\langle \kappa, E(0) \rangle} \quad \frac{\begin{matrix} \langle \xi_{1,2} \rangle \\ \langle \xi_{1,1} \rangle \\ \langle \xi_{1,0} \rangle \end{matrix}}{\langle \alpha_1, E(0) \rangle} \quad \frac{\begin{matrix} \langle \xi_{2,1} \rangle \\ \langle \xi_{2,0} \rangle \end{matrix}}{\langle \alpha_2, E(0) \rangle} \quad T^{p_0}$$

FIGURE 1. An example of $p_0 \in P_{\bar{E}}^*$, $l(\bar{E}) = 1$.

$$\frac{\langle \xi_{0,0} \rangle}{\langle \nu^0 \rangle} \quad \frac{\langle \nu^0 \rangle}{\langle \kappa, E(0) \rangle} \quad \frac{\begin{matrix} \langle \xi_{1,2} \rangle \\ \langle \xi_{1,1} \rangle \\ \langle \xi_{1,0} \rangle \end{matrix}}{\langle \alpha_1, E(0) \rangle} \quad \frac{\begin{matrix} \langle \nu \rangle \\ \langle \xi_{2,1} \rangle \\ \langle \xi_{2,0} \rangle \end{matrix}}{\langle \alpha_2, E(0) \rangle} \quad T_{\langle \nu \rangle}^{p_0}$$

FIGURE 2. $p_{0\langle \nu \rangle}$, a 1-point extension of p_0 using $\langle \nu \rangle$.

$$\frac{\langle \xi_{0,0} \rangle}{\langle \nu^0 \rangle} \quad \frac{\langle \nu^0 \rangle}{\langle \mu^0 \rangle} \quad \frac{\langle \mu^0 \rangle}{\langle \kappa, E(0) \rangle} \quad \frac{\begin{matrix} \langle \xi_{1,2} \rangle \\ \langle \xi_{1,1} \rangle \\ \langle \xi_{1,0} \rangle \end{matrix}}{\langle \alpha_1, E(0) \rangle} \quad \frac{\begin{matrix} \langle \mu \rangle \\ \langle \nu \rangle \\ \langle \xi_{2,1} \rangle \\ \langle \xi_{2,0} \rangle \end{matrix}}{\langle \alpha_2, E(0) \rangle} \quad T_{\langle \nu \rangle, \langle \mu \rangle}^{p_0}$$

FIGURE 3. $p_{0\langle \nu \rangle, \langle \mu \rangle}$, a 1-point extension of $p_{0\langle \nu \rangle}$ using $\langle \mu \rangle$.

We continue with the definition of $P_{\bar{E}}$ for $l(\bar{E}) = 2$.

Definition 3.12. Assume $l(\bar{E}) = 2$. A condition in $P_{\bar{E}}^*$ is of the form

$$\{\langle \langle \alpha, E(0), E(1) \rangle, p^{\langle \alpha, E(0), E(1) \rangle} \mid \langle \alpha, E(0), E(1) \rangle \in g\} \cup \{T\}$$

where

- (1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
- (2) $\min \bar{E} = \langle \kappa, E(0), E(1) \rangle \in g$ and g has a maximal element,
- (3) $p^{\langle \kappa, E(0), E(1) \rangle} \in V_{\kappa^0(\bar{E})}$ is an extender sequence, we call it p^0 . (We allow $p^0 = \emptyset$),
- (4) $\forall \langle \alpha, E(0), E(1) \rangle \in g \setminus \{\langle \kappa, E(0), E(1) \rangle\}$ $p^{\langle \alpha, E(0), E(1) \rangle} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a \circ -increasing sequence. We allow $p^{\langle \alpha, E(0), E(1) \rangle} = \emptyset$,
- (5) T is a max g -tree such that for all $t \in T$ $p^{\max g} \frown t$ is \circ -increasing. If $\langle \bar{\nu} \rangle \in T$ then either $\bar{\nu} = \langle \xi \rangle$ or $\bar{\nu} = \langle \xi, e(0) \rangle$, where $e(0)$ is extender and $\xi \in \text{dom } e(0)$.
- (6) For all $\langle \alpha, E(0), E(1) \rangle \in g$, p^0 is not permitted to $p^{\langle \alpha, E(0), E(1) \rangle}$,
- (7) $\forall \langle \bar{\nu} \rangle \in T$ $|\{\langle \alpha, E(0), E(1) \rangle \in g \mid \bar{\nu} \text{ is permitted to } p^{\langle \alpha, E(0), E(1) \rangle}\}| \leq \kappa^0(\bar{\nu})$,
- (8) $\forall \langle \bar{\nu} \rangle \in T$ if $\bar{\nu}$ is permitted to $p^{\langle \beta, E(0), E(1) \rangle}, p^{\langle \alpha, E(0), E(1) \rangle}$ then

$$\pi_{\max g, \langle \beta, E(0), E(1) \rangle}(\bar{\nu}) \neq \pi_{\max g, \langle \alpha, E(0), E(1) \rangle}(\bar{\nu}).$$

We write $\text{mc}(p)$, p^{mc} , T^p , $\bar{E}(p)$, $\text{supp } p$ for $\max g$, $p^{\max g}$, T , \bar{E} , g respectively.

Definition 3.13. Assume $l(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}^*$. We say that p is a Prikrý extension of q ($p \leq^* q$ or $p \leq^0 q$) if

- (1) $\text{supp } p \supseteq \text{supp } q$,
- (2) $\forall \langle \alpha, E(0), E(1) \rangle \in \text{supp } q$ $p^{\langle \alpha, E(0), E(1) \rangle} = q^{\langle \alpha, E(0), E(1) \rangle}$,
- (3) $T^p \leq \pi_{\text{mc}(p), \text{mc}(q)}^{-1} T^q$,
- (4) $\forall \langle \alpha, E(0), E(1) \rangle \in \text{supp } q$ $\forall \langle \bar{\nu} \rangle \in T^p$

$$\max \kappa(p^{\langle \alpha, E(0), E(1) \rangle}) < \kappa^0(\bar{\nu}) \implies$$

$$\pi_{\text{mc}(p), \alpha}(\bar{\nu}) = \pi_{\text{mc}(q), \alpha}(\pi_{\text{mc}(p), \text{mc}(q)}(\bar{\nu})).$$

Definition 3.14. Assume $l(\bar{E}) = 2$. A condition in $P_{\bar{E}}$ is of the form

$$p_n \frown \cdots \frown p_0$$

where

- $p_0 \in P_{\bar{E}}^*$,
- $p_1 \in P_{\bar{\mu}_1}^*$,
- \vdots ,
- $p_n \in P_{\bar{\mu}_n}^*$,

where $\bar{E}, \bar{\mu}_1, \dots, \bar{\mu}_n$ are extender sequence systems such that $l(\bar{\mu}_1) < 2, \dots, l(\bar{\mu}_n) < 2$ satisfying

$$\forall i \leq n-1 \ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

Definition 3.15. Assume $l(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that p is a Prikrý extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$p = p_n \frown \cdots \frown p_0,$$

$$q = q_n \frown \cdots \frown q_0,$$

and

- $p_0, q_0 \in P_{\bar{E}}^*$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in P_{\bar{\mu}_1}^*$, $p_1 \leq^* q_1$,
- \vdots ,
- $p_n, q_n \in P_{\bar{\mu}_n}^*$, $p_n \leq^* q_n$.

$p_{0\langle\bar{\nu}\rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition p_0 , which adds the extender sequence $\bar{\nu} \in T^{p_0}$ to the finite sequence. If $\bar{\nu} = \langle \xi, e(0) \rangle$ then a condition $p'_1 \in P_{e(0)}$ is added.

Definition 3.16. Assume $l(\bar{E}) = 2$. Let $p \in P_{\bar{E}}^*$ and $\langle \bar{\nu} \rangle \in T^p$. We define $p_{\langle\bar{\nu}\rangle}$ to be $p'_1 \frown p'_0$ where

- (1) $\text{supp } p'_0 = \text{supp } p$,
- (2) $\forall \langle \alpha, E(0), E(1) \rangle \in \text{supp } p'_0 \quad p'_0{}^{\langle \alpha, E(0), E(1) \rangle} =$

$\langle \pi_{\text{mc}(p), \alpha}(\xi), e(0) \rangle$	$\max \kappa(p^{\langle \alpha, E(0), E(1) \rangle}) < \kappa^0(\xi)$, $\bar{\nu} = \langle \xi, e(0) \rangle$.
$p^{\langle \alpha, E(0), E(1) \rangle} \frown \langle \pi_{\text{mc}(p), \alpha}(\xi), e(0) \rangle$	$\max \kappa(p^{\langle \alpha, E(0), E(1) \rangle}) < \kappa^0(\xi)$, $\bar{\nu} = \langle \xi \rangle$, $\alpha \neq \kappa$.
$\langle \pi_{\text{mc}(p), \alpha}(\xi) \rangle$	$\max \kappa(p^{\langle \alpha, E(0), E(1) \rangle}) < \kappa^0(\xi)$, $\bar{\nu} = \langle \xi \rangle$, $\alpha = \kappa$.
$p^{\langle \alpha, E(0), E(1) \rangle}$	Otherwise.

- (3) $T^{p'_0} = T_{\langle\bar{\nu}\rangle}^p$.

- (4) If $\bar{\nu} = \langle \xi \rangle$ then

- (4.1) $\text{supp } p'_1 = \{ \langle \pi_{\text{mc}(p), \kappa}(\xi) \rangle \}$,

- (4.2) $p'_1{}^{\langle \pi_{\text{mc}(p), \kappa}(\xi) \rangle} = p^{\langle \kappa, E(0), E(1) \rangle}$,

- (4.3) $T^{p'_1} = \emptyset$.

- (5) If $\bar{\nu} = \langle \xi, e(0) \rangle$ then

- (5.1) $\text{supp } p'_1 = \{ \langle \pi_{\text{mc}(p), \alpha}(\xi), e(0) \rangle \mid \langle \alpha, E(0), E(1) \rangle \in \text{supp } p, \max \kappa(p^{\langle \alpha, E(0), E(1) \rangle}) < \kappa^0(\xi) \}$,

- (5.2) $\forall \langle \pi_{\text{mc}(p), \alpha}(\xi), e(0) \rangle \in \text{supp } p'_1 \quad p'_1{}^{\langle \pi_{\text{mc}(p), \alpha}(\xi), e(0) \rangle} = p^{\langle \alpha, E(0), E(1) \rangle}$,

- (5.3) $T^{p'_1} = T^p(\langle \xi, e(0) \rangle)$. (note that $p'_1 \in P_{e(0)}^*$.)

Definition 3.17. Assume $l(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that p is a 1-point extension of q ($p \leq^1 q$) if p, q are of the form

$$p = p_{n+1} \frown p_n \frown \cdots \frown p_0,$$

$$q = q_n \frown \cdots \frown q_0,$$

and there is $0 \leq k \leq n$ such that

- $p_i, q_i \in P_{\bar{\mu}_i}^*$, $p_i \leq^* q_i$ for $i = 0, \dots, k-1$,
- $p_{i+1}, q_i \in P_{\bar{\mu}_i}^*$, $p_{i+1} \leq^* q_i$ for $i = k+1, \dots, n$,
- There is $\langle \bar{\nu} \rangle \in T^{q_k}$ such that $p_{k+1} \frown p_k \leq^* (q_k)_{\langle\bar{\nu}\rangle}$.

Definition 3.18. Assume $l(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that p is an n -point extension of q ($p \leq^n q$) if there are p^n, \dots, p^0 such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 3.19. Assume $l(\bar{E}) = 2$. Let $p, q \in P_{\bar{E}}$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

We leave now the special cases and define the forcing notion for arbitrary $l(\bar{E})$.

Definition 3.20. A condition in $P_{\bar{E}}^*$ is of the form

$$\{\langle \bar{\gamma}, p^{\bar{\gamma}} \rangle \mid \bar{\gamma} \in g\} \cup \{T\}$$

where

- (1) $g \subseteq \bar{E}$, $|g| \leq \kappa$,
- (2) $\min \bar{E} = \bar{E}_\kappa \in g$ and g has a maximal element,
- (3) $p^{\min \bar{E}} \in V_{\kappa^0(\bar{E})}$ is an extender sequence, we call it p^0 . (We allow $p^0 = \emptyset$),
- (4) $\forall \bar{\gamma} \in g \setminus \{\min \bar{E}\}$ $p^{\bar{\gamma}} \in [V_{\kappa^0(\bar{E})}]^{<\omega}$ is a $^\circ$ -increasing sequence. We allow $p^{\bar{\gamma}} = \emptyset$,
- (5) T is a max g -tree such that for all $t \in T$ $p^{\max g} \cap t$ is $^\circ$ -increasing,
- (6) For all $\bar{\gamma} \in g$, p^0 is not permitted to $p^{\bar{\gamma}}$,
- (7) $\forall \bar{\nu} \in T$ $|\{\bar{\gamma} \in g \mid \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}\}| \leq \kappa^0(\bar{\nu})$,
- (8) $\forall \bar{\nu} \in T$ if $\bar{\nu}$ is permitted to $p^{\bar{\beta}}, p^{\bar{\gamma}}$ then $\pi_{\max g, \bar{\beta}}(\bar{\nu}) \neq \pi_{\max g, \bar{\gamma}}(\bar{\nu})$.

We write $\text{mc}(p)$, p^{mc} , T^p , $\bar{E}(p)$, $\text{supp } p$ for $\max g$, $p^{\max g}$, T , \bar{E} , g respectively.

Definition 3.21. Let $p, q \in P_{\bar{E}}^*$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if

- (1) $\text{supp } p \supseteq \text{supp } q$,
- (2) $\forall \gamma \in \text{supp } q$ $p^\gamma = q^\gamma$,
- (3) $T^p \leq \pi_{\text{mc}(p), \text{mc}(q)}^{-1} T^q$,
- (4) $\forall \bar{\gamma} \in \text{supp } q$ $\forall \langle \bar{\nu} \rangle \in T^p$

$$\max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}) \implies \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) = \pi_{\text{mc}(q), \bar{\gamma}}(\pi_{\text{mc}(p), \text{mc}(q)}(\bar{\nu})).$$

Note that if $p, q \in P_{\bar{E}}$ satisfies all details of the above definition except 4, then by just shrinking T^p we can get 4 also. The proof is the same as the one given in [3].

In many cases in the sequel we use $\pi_{\bar{\beta}, \bar{\alpha}}^{-1} T$. Whenever we do so we assume automatically the above shrinkage, which of course depends on the conditions used.

Definition 3.22. A condition in $P_{\bar{E}}$ is of the form

$$p_n \hat{\ } \cdots \hat{\ } p_0$$

where

- $p_0 \in P_{\bar{E}}^*$,
- $p_1 \in P_{\bar{\mu}_1}^*$,
- \vdots ,
- $p_n \in P_{\bar{\mu}_n}^*$,

where $\bar{E}, \bar{\mu}_1, \dots, \bar{\mu}_n$ are extender sequence systems such that $l(\bar{\mu}_1) < l(\bar{E}), \dots, l(\bar{\mu}_n) < l(\bar{E})$ satisfying

$$\forall i \leq n-1 \ \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i) \text{ where } \bar{\mu}_0 = \bar{E}.$$

Definition 3.23. Let $p, q \in P_{\bar{E}}$. We say that p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$\begin{aligned} p &= p_n \hat{\ } \cdots \hat{\ } p_0, \\ q &= q_n \hat{\ } \cdots \hat{\ } q_0, \end{aligned}$$

and

- $p_0, q_0 \in P_{\bar{E}}^*$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in P_{\bar{\mu}_1}^*$, $p_1 \leq^* q_1$,
- \vdots ,
- $p_n, q_n \in P_{\bar{\mu}_n}^*$, $p_n \leq^* q_n$.

$p_{0\langle\bar{\nu}\rangle}$, defined now, is the basic non-direct extension in $P_{\bar{E}}$ of the condition p_0 , which adds the extender sequence $\bar{\nu} \in T^{p_0}$ to the finite sequence. A condition $p'_1 \in P_{\bar{E}\langle\bar{\nu}\rangle}$ is added.

Definition 3.24. Let $p \in P_{\bar{E}}^*$ and $\langle\bar{\nu}\rangle \in T^p$. We define $(p)_{\langle\bar{\nu}\rangle}$ to be $p'_1 \hat{\wedge} p'_0$ where

$$(1) \text{ supp } p'_0 = \text{supp } p,$$

$$(2) \forall \bar{\gamma} \in \text{supp } p'_0 \quad p'_0 \upharpoonright \bar{\gamma} = \begin{cases} \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) & \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}), \text{ l}(\bar{\nu}) > 0. \\ p^{\bar{\gamma}} \hat{\wedge} \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) & \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}), \text{ l}(\bar{\nu}) = 0, \\ & \bar{\gamma} \neq \bar{E}_{\kappa}. \\ \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) & \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu}), \text{ l}(\bar{\nu}) = 0, \\ & \bar{\gamma} = \bar{E}_{\kappa}. \\ p^{\bar{\gamma}} & \text{Otherwise.} \end{cases}$$

$$(3) T^{p'_0} = T_{\langle\bar{\nu}\rangle}^p.$$

(4) If $\text{l}(\bar{\nu}) = 0$ then

$$(4.1) \text{ supp } p'_1 = \{\pi_{\text{mc}(p), \bar{E}_{\kappa}}(\bar{\nu})\},$$

$$(4.2) p'_1 \upharpoonright \pi_{\text{mc}(p), \bar{E}_{\kappa}}(\bar{\nu}) = p^{\bar{E}_{\kappa}},$$

$$(4.3) T^{p'_1} = \emptyset,$$

(5) If $\text{l}(\bar{\nu}) > 0$ then

$$(5.1) \text{ supp } p'_1 = \{\pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) \mid \bar{\gamma} \in \text{supp } p, \max \kappa(p^{\bar{\gamma}}) < \kappa^0(\bar{\nu})\},$$

$$(5.2) \forall \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) \in \text{supp } p'_1 \quad p'_1 \upharpoonright \pi_{\text{mc}(p), \bar{\gamma}}(\bar{\nu}) = p^{\bar{\gamma}},$$

$$(5.3) T^{p'_1} = T^p(\bar{\nu}), \text{ (note that } p'_1 \in P_{\bar{\epsilon}}^* \text{ for } \bar{\epsilon} \text{ associated with } \bar{\nu}),$$

Definition 3.25. Let $p, q \in P_{\bar{E}}$. We say that p is a 1-point extension of q ($p \leq^1 q$) if p, q are of the form

$$p = p_{n+1} \hat{\wedge} p_n \hat{\wedge} \cdots \hat{\wedge} p_0,$$

$$q = q_n \hat{\wedge} \cdots \hat{\wedge} q_0,$$

and there is $0 \leq k \leq n$ such that

- $p_i, q_i \in P_{\bar{\mu}_i}^*$, $p_i \leq^* q_i$ for $i = 0, \dots, k-1$,
- $p_{i+1}, q_i \in P_{\bar{\mu}_i}^*$, $p_{i+1} \leq^* q_i$ for $i = k+1, \dots, n$,
- There is $\langle\bar{\nu}\rangle \in T^{q_k}$ such that $p_{k+1} \hat{\wedge} p_k \leq^* (q_k)_{\langle\bar{\nu}\rangle}$.

Definition 3.26. Let $p, q \in P_{\bar{E}}$. We say that p is an n -point extension of q ($p \leq^n q$) if there are p^n, \dots, p^0 such that

$$p = p^n \leq^1 \cdots \leq^1 p^0 = q.$$

Definition 3.27. Let $p, q \in P_{\bar{E}}$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

Later on by $P_{\bar{E}}$ we mean $\langle P_{\bar{E}}, \leq \rangle$.

Note 3.28. When $\text{l}(\bar{E}) = 1$ the forcing $P_{\bar{E}}$ is the Gitik-Magidor forcing from section 1 of [3]. When $\text{l}(\bar{E}) < \kappa$ the forcing $P_{\bar{E}}$ is similar to the forcing defined in [8].

$$\begin{array}{cccc}
\bar{\xi}_{0,0} & \bar{\xi}_{1,2} & \bar{\xi}_{2,1} & T^{p_0} \\
\bar{\xi}_{1,1} & \bar{\xi}_{1,1} & \bar{\xi}_{2,0} & \\
\bar{\xi}_{1,0} & \bar{\xi}_{1,0} & & \\
\hline
\bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} &
\end{array}$$

FIGURE 7. An example of $p_0 \in P_{\bar{E}}^*$.

$$\begin{array}{cccc}
\bar{\xi}_{0,0} & & & \\
\hline
\langle \nu^0 \rangle & & & \\
\langle \nu^0 \rangle & \bar{\xi}_{1,2} & \langle \nu \rangle & T^{p_0} \\
\langle \nu^0 \rangle & \bar{\xi}_{1,1} & \bar{\xi}_{2,1} & \\
\langle \nu^0 \rangle & \bar{\xi}_{1,0} & \bar{\xi}_{2,0} & \\
\hline
\bar{E}_\kappa & \bar{E}_{\alpha_1} & \bar{E}_{\alpha_2} & T_{\langle \langle \nu \rangle \rangle}^{p_0}
\end{array}$$

FIGURE 8. $p_{0\langle \bar{\nu} \rangle}$, a 1-point extension of p_0 using $\langle \nu \rangle$.

$$\begin{array}{cccc}
\bar{\xi}_{0,0} & & & \\
\hline
\langle \nu^0 \rangle & & & \\
\langle \nu^0 \rangle & \bar{\xi}_{1,2} & \langle \nu \rangle & T_{\langle \langle \nu \rangle \rangle}^{p_0} \\
\langle \nu^0 \rangle & \bar{\xi}_{1,1} & \bar{\xi}_{2,1} & \\
\langle \nu^0 \rangle & \bar{\xi}_{1,0} & \bar{\xi}_{2,0} & \\
\hline
\bar{\mu}^0 & \bar{\mu} & & \\
\bar{\mu}^0 & \bar{\xi}_{1,2} & \bar{\mu} & T_{\langle \langle \nu \rangle, \bar{\mu} \rangle}^{p_0} \\
\bar{\mu}^0 & \bar{\xi}_{1,1} & & \\
\bar{\mu}^0 & \bar{\xi}_{1,0} & \bar{E}_{\alpha_2} &
\end{array}$$

FIGURE 9. $p_{0\langle \langle \nu \rangle, \bar{\mu} \rangle}$, a 1-point extension of $p_{0\langle \nu \rangle}$ using $\bar{\mu}$.

In several places we want to prevent enlargement of the support of a condition. This makes all the conditions which are stronger than some condition but with the same support resemble Radin forcing. The following definition catches the meaning of not enlarging the support. The ‘resemblance’ we look for is 5.3.

Definition 3.29. Let $p, q \in P_{\bar{E}}^*$. We say that $p \leq_R^* q$ if

- (1) $p \leq^* q$,
- (2) $\text{supp } p = \text{supp } q$.

Definition 3.30. Let $p, q \in P_{\bar{E}}$. We say that $p \leq_R^1 q$ if

- (1) $p \leq^1 q$,

(2) In the definition of \leq^1 we can substitute \leq^* by \leq_R^* .

Definition 3.31. Let $p, q \in P_{\bar{E}}$. We say that $p \leq_R^n q$ if there are p^n, \dots, p^0 such that

$$p = p^n \leq_R^1 \dots \leq_R^1 p^0 = q.$$

Definition 3.32. Let $p, q \in P_{\bar{E}}$. We say that $p \leq_R q$ if there is n such that $p \leq_R^n q$.

Note 3.33. The above definitions imply that if $p \leq q$ then there is r such that $p \leq^* r \leq_R q$.

Note 3.34. When $l(\bar{E}) = 1$ and we force with $\langle P_{\bar{E}}, \leq_R \rangle$ below some p with maximal coordinate α , we are forcing just the tree Prikry forcing for the measure $E_\alpha(0)$.

Definition 3.35. Let $\bar{\epsilon}$ be an extender sequence such that $\kappa^0(\bar{\epsilon}) < \kappa^0(\bar{E})$.

$$P_{\bar{E}}/P_{\bar{\epsilon}} = \{p \mid \exists q \in P_{\bar{\epsilon}}, q \hat{\wedge} p \in P_{\bar{E}}\}.$$

4. RADIN FORCING

The main aims of this section are 4.8, 4.10. As the simplest way we found to formulate them was with Radin forcing [7, 5, 10] we took the opportunity to depart from usual formulation in order to work in the spirit of the extender based forcing we defined in section 3.

The main point is that possible extensions of a condition are stored in \bar{E}_α -tree and not in a set. The α is fixed, so practically we deal here with a measure sequence and not an extender sequence.

Definition 4.1. A condition in R_α is of the form

$$\langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle$$

where

- (1) $\bar{\mu}_0, \dots, \bar{\mu}_n$ are extender sequences,
- (2) $\forall i \leq n-1 \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i)$,
- (3) $\bar{\mu}_0 = \bar{E}_\alpha$,
- (4) $\forall i \leq n S^i$ is $\bar{\mu}_i$ -tree,
- (5) $\forall i \leq n s^i \in V_{\kappa^0(\bar{\mu}_i)}$ is an extender sequences.

Definition 4.2. Let $p, q \in R_\alpha$. We say that p is Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) if p, q are of the form

$$\begin{aligned} p &= \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{E}_\alpha, s^0 \rangle, S^0 \rangle, \\ q &= \langle \langle \bar{\mu}_n, t^n \rangle, T^n, \dots, \langle \bar{\mu}_1, t^1 \rangle, T^1, \langle \bar{E}_\alpha, t^0 \rangle, T^0 \rangle, \end{aligned}$$

and

- $\forall i \leq n S^i \leq T^i$,
- $\forall i \leq n s^i = t^i$.

Definition 4.3. Let $p = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle$ where $\bar{\mu}_0 = \bar{E}_\alpha$. Let $\langle \bar{\nu} \rangle \in S^i$. We define $(p)_{\langle \bar{\nu} \rangle}$ to be

$$\begin{aligned} (p)_{\langle \bar{\nu} \rangle} &= \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_{i+1}, s^{i+1} \rangle, S^{i+1}, \\ &\quad \langle \bar{\nu}, s^i \rangle, S^i(\bar{\nu}), \langle \bar{\mu}_i, \bar{\nu} \rangle, S^i_{\langle \bar{\nu} \rangle}, \\ &\quad \langle \bar{\mu}_{i-1}, s^{i-1} \rangle, S^{i-1}, \dots, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle. \end{aligned}$$

Note the degenerate case in this definition when $l(\bar{\nu}) = 0$. In this case $S^i(\bar{\nu}) = \emptyset$.

Definition 4.4. Let $p, q \in R_\alpha$ where

$$q = \langle \langle \bar{\mu}_n, s^n \rangle, S^n, \dots, \langle \bar{\mu}_1, s^1 \rangle, S^1, \langle \bar{\mu}_0, s^0 \rangle, S^0 \rangle.$$

We say that p is 1-point extension of q ($p \leq^1 q$) if there is $\langle \bar{\nu} \rangle \in S^i$ such that $p \leq^* (q)_{\langle \bar{\nu} \rangle}$.

Definition 4.5. Let $p, q \in R_\alpha$. We say that p is n -point extension of q ($p \leq^n q$) if there are p^n, \dots, p^0 such that

$$p = p^n \leq^1 \dots \leq^1 p^0 = q.$$

Definition 4.6. Let $p, q \in R_\alpha$. We say that p is an extension of q ($p \leq q$) if there is n such that $p \leq^n q$.

4.8 is needed in the proof of 7.1. Very loosely speaking 4.8 means that if ‘‘something’’ happens on a measure one set for one of the measures, that ‘‘something’’ is happening on a measure one set for all the measures.

4.8 is proved by induction and 4.7 is the first case of the induction.

Lemma 4.7. *Suppose $l(\bar{E}_\alpha) = 2$, $i < 2$, there is a tree T (not an \bar{E}_α tree) such that $\text{Lev}_0(T) \in E_\alpha(i)$ and $\forall \langle \bar{\nu} \rangle \in T$ $T_{\langle \bar{\nu} \rangle}$ is an \bar{E}_α -tree. Then there is an \bar{E}_α -tree, T^* , satisfying*

- (1) $\forall \langle \bar{\nu} \rangle \in T \cap T^* \ T_{\langle \bar{\nu} \rangle}^* \leq T_{\langle \bar{\nu} \rangle}, T^*(\bar{\nu}) \leq T(\bar{\nu})$,
- (2) If $p \leq \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle$ then there is $\langle \bar{\mu} \rangle \in T^* \cap T$ such that

$$p \parallel \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle_{\langle \bar{\mu} \rangle}.$$

Proof. There are 2 cases which to deal with:

- $A_0 = \text{Lev}_0(T) \in E_\alpha(0)$: If $A_0 \in E_\alpha(1)$ we set $T^* = T$ and the proof is finished. So suppose $A_0 \notin E_\alpha(1)$. We would like to build $A_1 \in E_\alpha(1)$. Set $\forall \langle \bar{\mu}_0 \rangle \in T \ A_{\langle \bar{\mu}_0 \rangle, 1} = \{ \langle \bar{\mu}_1, S \rangle \in T_{\langle \bar{\mu}_0 \rangle} \mid A_0 \cap \kappa^0(\bar{\mu}_1) \in \bar{\mu}_1(0)_{\kappa(\bar{\mu}_1)} \}$. As $A_0 \in E_\alpha(0)$ and $\text{Suc}_T(\langle \bar{\mu}_0 \rangle) \in E_\alpha(1)$ we get that $A_{\langle \bar{\mu}_0 \rangle, 1} \in E_\alpha(1)$. Let

$$A_1 = \bigcup_{\langle \bar{\mu}_0 \rangle \in T} \Delta^0 A_{\langle \bar{\mu}_0 \rangle, 1}.$$

We can construct now T^* :

$$\begin{aligned} \text{Lev}_0(T^*) &= A_0 \cup A_1, \\ \forall \bar{\mu}_0 \in A_0 \ T_{\langle \bar{\mu}_0 \rangle}^* &= T_{\langle \bar{\mu}_0 \rangle}, \\ \forall \bar{\mu}_1 \in A_1 \ T_{\langle \bar{\mu}_1 \rangle}^* &= \bigcap_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T} T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}. \end{aligned}$$

- $A_1 = \text{Lev}_0(T) \in E_\alpha(1)$: If $A_1 \in E_\alpha(0)$ we set $T^* = T$ and finish the proof. So assume $A_1 \notin E_\alpha(0)$. We would like to build $A_0 \in E_\alpha(0)$. Set

$$\begin{aligned} S &= j(T)(\langle \alpha, E(0) \rangle), \\ A_0 &= \text{Lev}_0(S) \setminus A_1. \end{aligned}$$

We construct T^* :

$$\begin{aligned} \text{Lev}_0(T^*) &= A_0 \cup A_1, \\ \forall \bar{\mu}_1 \in A_1 \ T_{\langle \bar{\mu}_1 \rangle}^* &= T_{\langle \bar{\mu}_1 \rangle}. \end{aligned}$$

We are left with the construction of $T_{\langle \bar{\mu}_0 \rangle}^*$ for $\bar{\mu}_0 \in A_0$. For all $\bar{\mu} \in A_0$ set

$$\begin{aligned} A_{\langle \bar{\mu}_0 \rangle, 0} &= \text{Suc}_S(\langle \bar{\mu}_0 \rangle), \\ A_{\langle \bar{\mu}_0 \rangle, 1} &= \{\langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \mid \langle \bar{\mu}_1 \rangle \in T, \langle \bar{\mu}_0 \rangle \in T(\bar{\mu}_1)\}, \\ \text{Suc}_{T^*}(\langle \bar{\mu}_0 \rangle) &= A_{\langle \bar{\mu}_0 \rangle, 0} \cup A_{\langle \bar{\mu}_0 \rangle, 1}, \\ \forall \bar{\mu}_1 \in A_{\langle \bar{\mu}_0 \rangle, 1} \quad T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}^* &= T_{\langle \bar{\mu}_1 \rangle}. \end{aligned}$$

We continue one more level and hope this will convince the reader we indeed can complete T^* . We are left with the construction of $T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}^*$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$. For all $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0}$ set

$$\begin{aligned} A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} &= \text{Suc}_S(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle), \\ A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} &= \{\langle \bar{\mu}_2, T(\bar{\mu}_2)_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle} \mid \langle \bar{\mu}_2 \rangle \in T, \langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in T(\bar{\mu}_2)\}, \\ \text{Suc}_{T^*}(\langle \bar{\mu}_0, \bar{\mu}_1 \rangle) &= A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0} \cup A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1}, \\ \forall \bar{\mu}_2 \in A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 1} \quad T_{\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle}^* &= T_{\langle \bar{\mu}_2 \rangle}. \end{aligned}$$

We are left with the construction of $T_{\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle}^*$ for $\langle \bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2 \rangle \in A_0 \times A_{\langle \bar{\mu}_0 \rangle, 0} \times A_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle, 0}$ and we hope that by now the continuation is clear. \square

Lemma 4.8. *Let $\xi_0 < l(\bar{E}_\alpha)$, T a tree such that $\text{Lev}_0(T) \in E_\alpha(\xi_0)$ and $\forall \langle \bar{\mu} \rangle \in T$ $T_{\langle \bar{\mu} \rangle}$ is an \bar{E}_α -tree. Then there is an \bar{E}_α -tree, T^* , satisfying*

- (1) $\forall \langle \bar{\nu} \rangle \in T \cap T^* \quad T_{\langle \bar{\nu} \rangle}^* \leq T_{\langle \bar{\nu} \rangle}, T^*(\bar{\nu}) \leq T(\bar{\nu})$,
- (2) *If $p \leq \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle$ then there is $\langle \bar{\mu} \rangle \in T^* \cap T$ such that*

$$p \parallel \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle_{\langle \bar{\mu} \rangle}.$$

Proof. Our induction hypothesis is that this lemma is true for $\bar{\mu}$'s with $l(\bar{\mu}) < l(\bar{E}_\alpha)$. The previous lemma is the case for $l(\bar{\mu}) = 2$.

Let $S = j(T)(\bar{E}_\alpha \upharpoonright \xi_0)$. The tree S is an $\bar{E}_\alpha \upharpoonright \xi_0$ -tree. We extend it step by step to a full \bar{E}_α -tree as required.

Let

$$\begin{aligned} A_{\xi_0} &= \text{Lev}_0(T), \\ A_{< \xi_0} &= \text{Lev}_0(S) \setminus A_{\xi_0}. \end{aligned}$$

For $\xi_0 < \xi < l(\bar{E}_\alpha)$ do the following:

$$\begin{aligned} N_\xi &= \text{Ult}(V, E_\alpha(\xi)), \\ k_\xi([h]_{E_\alpha(\xi)}) &= j(h)(\bar{E}_\alpha \upharpoonright \xi). \end{aligned}$$

$$\begin{array}{ccc} V & \xrightarrow{j} & M \\ \downarrow & \nearrow k_\xi & \\ N_\xi = \text{Ult}(V, E_\alpha(\xi)) & & \end{array} .$$

As

$$\langle \alpha, E(0), \dots, E(\tau), \dots \mid \tau < \xi \rangle = k_\xi([id]_{E_\alpha(\xi)}) \in \text{ran}(k_\xi)$$

there is in N_ξ a preimage for it:

$$\langle \alpha', E'(0), \dots, E'(\tau') \dots \mid \tau' < \xi' \rangle.$$

As $A_{\xi_0} \in E_\alpha(\xi_0)$, $\xi_0 < \xi$ we have $\tau' < \xi'$ such that $A_{\xi_0} \in E'_{\alpha'}(\tau')$, (where $\alpha' = [\kappa(id)]_{E_\alpha(\xi)}$). Taking a function h_ξ such that $[h_\xi]_{E_\alpha(\xi)} = \tau'$ we get

$$\{\bar{\mu} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}) \in \bar{\mu}(h_\xi(\bar{\mu}))_{\kappa(\bar{\mu})}\} \in E_\alpha(\xi).$$

For each $\bar{\mu}_0 \in A_{\xi_0}$ we set

$$\begin{aligned} A_{\xi, \langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle} &= \{\langle \bar{\mu}_1, R \rangle \in T_{\langle \bar{\mu}_0 \rangle} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}_1) \in \bar{\mu}_1(h_\xi(\bar{\mu}_1))_{\kappa(\bar{\mu}_1)}\}, \\ A'_\xi &= \Delta^0_{\langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle} A_{\xi, \langle \bar{\mu}_0, T(\bar{\mu}_0) \rangle}. \end{aligned}$$

For any tree R which appears in a pair $\langle \bar{\mu}_1, R \rangle \in A'_\xi$ we can invoke by induction our lemma and generate R^* which is a $\bar{\mu}_1$ -tree. Define now A_ξ as:

$$\langle \bar{\mu}_1, R^* \rangle \in A_\xi \iff \langle \bar{\mu}_1, R \rangle \in A'_\xi.$$

When we have $\{A_\xi \mid \xi_0 < \xi < l(\bar{E}_\alpha)\}$ we set

$$\begin{aligned} A_{>\xi_0} &= \bigcup_{\xi_0 < \xi < l(\bar{E}_\alpha)} A_\xi \setminus (A_{<\xi_0} \cup A_{\xi_0}), \\ \text{Lev}_0(T^*) &= A_{<\xi_0} \cup A_{\xi_0} \cup A_{>\xi_0}, \\ \forall \langle \bar{\mu}_0 \rangle \in A_{\xi_0} \quad T_{\langle \bar{\mu}_0 \rangle}^* &= T_{\langle \bar{\mu}_0 \rangle}, \\ \forall \langle \bar{\mu}_1 \rangle \in A_{>\xi_0} \quad T_{\langle \bar{\mu}_1 \rangle}^* &= \bigcap T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}. \end{aligned}$$

We are left to define $T_{\langle \bar{\mu}_0 \rangle}$ for $\langle \bar{\mu}_0 \rangle \in A_{<\xi_0}$. For each $\bar{\mu}_0 \in A_{<\xi_0}$ set:

$$\begin{aligned} A_{\langle \bar{\mu}_0 \rangle, \xi_0} &= \{\langle \bar{\mu}_1, R_{\langle \bar{\mu}_0 \rangle} \rangle \mid \langle \bar{\mu}_1, R \rangle \in A_{\xi_0}, \langle \bar{\mu}_0 \rangle \in R\}, \\ A_{\langle \bar{\mu}_0 \rangle, <\xi_0} &= \text{Suc}_S(\langle \bar{\mu}_0 \rangle) \setminus A_{\langle \bar{\mu}_0 \rangle, \xi_0}. \end{aligned}$$

For each $\bar{\mu}_1 \in A_{\xi_0}$ we set

$$\begin{aligned} A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle} &= \{\langle \bar{\mu}_2, R \rangle \in T_{\langle \bar{\mu}_1 \rangle} \mid A_{\xi_0} \cap \kappa^0(\bar{\mu}_2) \in \bar{\mu}_2(h_\xi(\bar{\mu}_2))_{\kappa(\bar{\mu}_2)}\}, \\ A'_{\langle \bar{\mu}_0 \rangle, \xi} &= \Delta^0_{\langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle} A_{\langle \bar{\mu}_0 \rangle, \xi, \langle \bar{\mu}_1, T(\bar{\mu}_1)_{\langle \bar{\mu}_0 \rangle} \rangle}. \end{aligned}$$

We define

$$\langle \bar{\mu}_0, R^* \rangle \in A_{\langle \bar{\mu}_0 \rangle, \xi} \iff \langle \bar{\mu}_0, R \rangle \in A'_{\langle \bar{\mu}_0 \rangle, \xi}$$

where R^* is generated from R using the current lemma by induction. Now we set

$$\begin{aligned} A_{\langle \bar{\mu}_0 \rangle, >\xi_0} &= \bigcup_{\xi_0 < \xi < l(\bar{E}_\alpha)} A_{\langle \bar{\mu}_0 \rangle, \xi} \setminus (A_{\langle \bar{\mu}_0 \rangle, <\xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0}), \\ \text{Suc}_{T^*}(\langle \bar{\mu}_0 \rangle) &= A_{\langle \bar{\mu}_0 \rangle, <\xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, \xi_0} \cup A_{\langle \bar{\mu}_0 \rangle, >\xi_0}, \\ \forall \langle \bar{\mu}_1 \rangle \in A_{\bar{\mu}_0, \xi_0} \quad T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}^* &= T_{\langle \bar{\mu}_1 \rangle}, \\ \forall \langle \bar{\mu}_2 \rangle \in A_{\bar{\mu}_0, >\xi_0} \quad T_{\langle \bar{\mu}_0, \bar{\mu}_2 \rangle}^* &= \bigcap T_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}. \end{aligned}$$

This leaves us with the definition of $T_{\langle \bar{\mu}_0, \bar{\mu}_1 \rangle}$ for $\langle \bar{\mu}_0, \bar{\mu}_1 \rangle \in A_{<\xi_0} \times A_{\langle \bar{\mu}_0 \rangle, <\xi_0}$ which is done exactly as in this step. \square

4.10 is needed in the proof of 6.1. Loosely speaking it says that if “something” happens on all extensions which are taken from $\text{dom } T$, then that “something” happens on all extensions from T .

4.10 is proved by induction where 4.9 is the first case.

Lemma 4.9. *Assume $l(\bar{E}_\alpha) = 2$ and let T be \bar{E}_α -tree. Then there is $T^* \leq T$ such that if*

$$p \leq \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle$$

then there is $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T$ such that

$$p \leq^* \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T \rangle_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}.$$

Proof. As is usual in this section the proof is done level by level. Let us set $T^1 = T$ and it is trivially true that if

$$p \leq^1 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^1 \rangle$$

then there is $\langle \bar{\nu}_1 \rangle \in T^1 = T$ such that

$$p \leq^* \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^1 \rangle_{\langle \bar{\nu}_1 \rangle}.$$

We continue to the second level. Let us set

$$\begin{aligned} A_{\langle \bar{\nu}_1, S^1 \rangle} &= \text{Suc}_{T^1}(\langle \bar{\nu}_1 \rangle), \\ A_{\langle \rangle} &= \Delta_{\langle \bar{\nu}_1, S^1 \rangle}^0 A_{\langle \bar{\nu}_1, S^1 \rangle}, \\ B_{\langle \rangle} &= \{ \langle \bar{\nu}_2 \rangle \mid l(\bar{\nu}_2) = 0 \text{ or } \text{Lev}_0(T^1) \cap \kappa^0(\bar{\nu}_2) \in \bar{\nu}_2 \}, \\ \text{Lev}_0(T^{(0)}) &= A_{\langle \rangle} \cap B_{\langle \rangle}, \\ T_{\langle \bar{\nu}_2 \rangle}^{(0)} &= \bigcap_{\langle \bar{\nu}_1 \rangle \in T^{(0)}(\bar{\nu}_2)} T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}^1, \\ T^2 &= T^1 \cap T^{(0)}. \end{aligned}$$

Let us assume that

$$p \leq^2 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \rangle.$$

There are 2 cases to consider here:

- (1) $p \leq^2 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \rangle_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^2$: At once we have $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^1 \leq T$.
- (2) $p \leq^2 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^2 \rangle_{\langle \bar{\nu}_2, \bar{\nu}_1 \rangle}$ where $\langle \bar{\nu}_2 \rangle \in T^2$, $\langle \bar{\nu}_1 \rangle \in T^2(\bar{\nu}_2)$: By construction $\forall \langle \bar{\mu} \rangle \in T^2(\bar{\nu}_2)$ $\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\mu} \rangle}^1$. As $\langle \bar{\nu}_1 \rangle \in T^2(\bar{\nu}_2)$ we get $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^1 \leq T$.

We show how to continue to the third level. Let us set

$$\begin{aligned}
A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle} &= \text{Suc}_{T^2}(\langle \bar{\nu}_1, \bar{\nu}_2 \rangle), \\
A_{\langle \bar{\nu}_1, S^1 \rangle} &= \Delta_{\langle \bar{\nu}_2, S^2 \rangle}^0 A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle}, \\
B_{\langle \bar{\nu}_1, S^1 \rangle} &= \{ \langle \bar{\nu}_3 \rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \text{Suc}_{T^2}(\langle \bar{\nu}_1, S^1 \rangle) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \}, \\
\text{Lev}_0(T^{(0)}) &= \text{Lev}_0(T^2), \\
\text{Suc}_{T^{(0)}}(\langle \bar{\nu}_1 \rangle) &= A_{\langle \bar{\nu}_1, S^1 \rangle} \cap B_{\langle \bar{\nu}_1, S^1 \rangle}, \\
T_{\langle \bar{\nu}_1, \bar{\nu}_3 \rangle}^{(0)} &= \bigcap_{\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^{(0)}(\bar{\nu}_3)} T_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}^2, \\
A_{\langle \rangle} &= \Delta_{\langle \bar{\nu}_1, S^1 \rangle}^0 A_{\langle \bar{\nu}_1, S^1 \rangle}, \\
B_{\langle \rangle} &= \{ \langle \bar{\nu}_3 \rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \text{Lev}_0(T^2) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \}, \\
\text{Lev}_0(T^{(1)}) &= A_{\langle \rangle} \cap B_{\langle \rangle}, \\
\text{Suc}_{T^{(1)}}(\langle \bar{\nu}_3 \rangle) &= \bigcap_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^{(0)}(\bar{\nu}_3)} T_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}^2, \\
T^3 &= T^2 \cap T^{(0)} \cap T^{(1)}.
\end{aligned}$$

Let us assume that

$$p \leq^3 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle.$$

There are 3 cases to consider here:

- (1) $p \leq^* (\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle)_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T^3$: At once we get $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T$.
- (2) $p \leq^* (\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle)_{\langle \bar{\nu}_1, \bar{\nu}_3, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$: In this case $\forall \langle \bar{\mu} \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$ $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\mu} \rangle}^3$. As $\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$ we get $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.
- (3) $p \leq^* (\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle)_{\langle \bar{\nu}_3, \bar{\nu}_1, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^3(\bar{\nu}_3)$: $\forall \langle \bar{\mu}_1, \bar{\mu}_2 \rangle \in T^3(\bar{\nu}_3)$ $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}^3$ hence $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

In this way we continue to all levels. \square

Lemma 4.10. *Let T be \bar{E}_α tree. Then there is $T^* \leq T$ such that if*

$$p \leq \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^* \rangle$$

then there is $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T$ such that

$$p \leq^* (\langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T \rangle)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}.$$

Proof. The proof is by induction on $l(\bar{E}_\alpha)$. The first case was done in 4.9. The proof is almost the same. We just make sure to invoke the induction hypothesis while repeating the construction.

Construction of T^1 and T^2 is exactly like in 4.9. We show the construction at the 3rd level.

Let us set

$$\begin{aligned}
A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle} &= \text{Suc}_{T^2}(\langle \bar{\nu}_1, \bar{\nu}_2 \rangle), \\
A_{\langle \bar{\nu}_1, S^1 \rangle} &= \Delta_{\langle \bar{\nu}_2, S^2 \rangle}^0 A_{\langle \bar{\nu}_1, S^1, \bar{\nu}_2, S^2 \rangle}, \\
B_{\langle \bar{\nu}_1, S^1 \rangle} &= \{ \langle \bar{\nu}_3 \rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \text{Suc}_{T^2}(\langle \bar{\nu}_1, S^1 \rangle) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \}, \\
\text{Lev}_0(T^{(0)}) &= \text{Lev}_0(T^2), \\
\text{Suc}_{T^{(0)}}(\langle \bar{\nu}_1 \rangle) &= A_{\langle \bar{\nu}_1, S^1 \rangle} \cap B_{\langle \bar{\nu}_1, S^1 \rangle}, \\
T_{\langle \bar{\nu}_1, \bar{\nu}_3 \rangle}^{(0)} &= \bigcap_{\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^{(0)}(\bar{\nu}_3)} T_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}^2, \\
A'_{\langle \rangle} &= \Delta_{\langle \bar{\nu}_1, S^1 \rangle}^0 A_{\langle \bar{\nu}_1, S^1 \rangle}, \\
A_{\langle \rangle} &= \{ \langle \bar{\nu}_3, S^3 \rangle \mid \langle \bar{\nu}_3, S^{3'} \rangle \in A'_{\langle \rangle} \text{ } S \text{ is generated from } S^{3'} \text{ by induction} \}, \\
B_{\langle \rangle} &= \{ \langle \bar{\nu}_3 \rangle \mid l(\bar{\nu}_3) = 0 \text{ or } \text{Lev}_0(T^2) \cap \kappa^0(\bar{\nu}_3) \in \bar{\nu}_3 \}, \\
\text{Lev}_0(T^{(1)}) &= A_{\langle \rangle} \cap B_{\langle \rangle}, \\
\text{Suc}_{T^{(1)}}(\langle \bar{\nu}_3 \rangle) &= \bigcap_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T^{(0)}(\bar{\nu}_3)} T_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}^2, \\
T^3 &= T^2 \cap T^{(0)} \cap T^{(1)}.
\end{aligned}$$

Let us assume that

$$p_2 \frown p_1 \frown p_0 = p \leq^3 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle.$$

There are 3 cases to consider here:

- (1) $p \leq^* \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle_{\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T^3$: At once we get $\langle \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \rangle \in T$.
- (2) $p \leq^* \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle_{\langle \bar{\nu}_1, \bar{\nu}_3, \bar{\nu}_2 \rangle}$ where $\langle \bar{\nu}_1, \bar{\nu}_3 \rangle \in T^3$, $\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$: In this case $\forall \langle \bar{\mu} \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$ $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\mu} \rangle}^3$. As $\langle \bar{\nu}_2 \rangle \in T_{\langle \bar{\nu}_1 \rangle}^3(\bar{\nu}_3)$ we get $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.
- (3) $p \leq^2 \langle \langle \bar{E}_\alpha, \langle \rangle \rangle, T^3 \rangle_{\langle \bar{\nu}_3 \rangle}$ where $\langle \bar{\nu}_3 \rangle \in T^3$ and $p_2 \frown p_1 \leq^2 \langle \langle \bar{\nu}_3, \langle \rangle \rangle, T^3(\bar{\nu}_3) \rangle$
By induction there is $\langle \bar{\nu}_1, \bar{\nu}_2 \rangle \in T(\bar{\nu}_3)$ such that

$$p_2 \frown p_1 \leq^* \langle \langle \bar{\nu}_3, \langle \rangle \rangle, T^2(\bar{\nu}_3) \rangle_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}.$$

By construction $\forall \langle \bar{\mu}_1, \bar{\mu}_2 \rangle \in T^2(\bar{\nu}_3)$ $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\mu}_1, \bar{\mu}_2 \rangle}^2$ hence $\langle \bar{\nu}_3 \rangle \in T_{\langle \bar{\nu}_1, \bar{\nu}_2 \rangle}$.

In this way we continue to all levels. \square

5. BASIC PROPERTIES OF $P_{\bar{E}}$

Claim 5.1. $P_{\bar{E}}$ satisfies κ^{++} -c.c.

Proof. The usual Δ -lemma argument on the support will do. \square

Claim 5.2. Let $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. Then

- (1) $\langle P^*, \leq_R \rangle$ satisfies κ^+ -c.c.,
- (2) $\langle P^*, \leq_R \rangle$ is sub-forcing of $\langle P_{\bar{E}}/p, \leq \rangle$.

Proof. Showing κ^+ -c.c. is trivial.

Showing that P^* is sub-forcing of $P_{\bar{E}}/p$ amounts to showing that any maximal anti-chain of P^* is also a maximal anti-chain of $P_{\bar{E}}/p$.

Let A be a maximal anti-chain of P^* . Let $q \in P_{\bar{E}}/p$. As $q \leq p$, there is $r' \in P^*$ such that $q \leq^* r' \leq_R p$. Assume that $r' = r'_n \hat{\wedge} \dots \hat{\wedge} r'_0$. Then also $q = q_n \hat{\wedge} \dots \hat{\wedge} q_0$. Let r_i be r'_i with $T^{r'_i}$ substituted by $T^{r'_i} \cap \pi_{\text{mc}(q_i), \text{mc}(r'_i)}(T^{q_i})$ and $r = r_n \hat{\wedge} \dots \hat{\wedge} r_0$. As $r \in P^*$ and A is a maximal anti-chain there is $a \in A$ such that $a \parallel r$. Take $s \leq_R a, r$. Considering how we constructed r from r' we must have $t \leq^* s$ such that $t \leq q$. Hence $q \parallel a$. So we get that A is a maximal anti-chain of $P_{\bar{E}}/p$. \square

Claim 5.3. *Let $p \in P_{\bar{E}}$, $P^* = \{q \leq_R p \mid p \in P_{\bar{E}}\}$. Then there is $r \in R_{\text{mc}(p)}$ such that $P^* \simeq R_{\text{mc}(p)}/r$*

Proof. For simplicity assume that $p = p_0$. Then we set $r = \langle \langle \text{mc}(p_0), p_0^{\text{mc}} \rangle, T^p \rangle$.

We give the isomorphism: The image of $q \in P^*$ is $s \in \langle R_{\text{mc}(p)}/r, \leq \rangle$ such that

- (1) $q \leq^* s$,
- (2) $T^{s_i} = T^{q_i}$ where $s = s_n \hat{\wedge} \dots \hat{\wedge} s_0$, $q = q_n \hat{\wedge} \dots \hat{\wedge} q_0$.

\square

Let G be $P_{\bar{E}}$ -generic.

Definition 5.4. \bar{E}_G is the enumeration of $\{\bar{E}(p_k) \mid p_n \hat{\wedge} \dots \hat{\wedge} p_0 \in G\}$ ordered increasingly by $\kappa^0(\bar{E}(p_k))$.

Definition 5.5. Let $\zeta < \text{otp}(\bar{E}_G)$. Then

- (1) $G \upharpoonright \zeta = \{p_n \hat{\wedge} \dots \hat{\wedge} p_k \mid p_n \hat{\wedge} \dots \hat{\wedge} p_k \hat{\wedge} \dots \hat{\wedge} p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta)\}$,
- (2) $G \setminus \zeta = \{p_{k-1} \hat{\wedge} \dots \hat{\wedge} p_0 \mid p_n \hat{\wedge} \dots \hat{\wedge} p_k \hat{\wedge} \dots \hat{\wedge} p_0 \in G, \bar{E}(p_k) = \bar{E}_G(\zeta)\}$.

Definition 5.6.

$$M_G^\alpha = \bigcup \{p^{\bar{E}_\alpha} \mid p \in G, \bar{E}_\alpha \in \text{supp } p\}$$

$$C_G^\alpha = \{\kappa(\bar{\mu}) \mid \bar{\mu} \in M_G^\alpha\}$$

Proposition 5.7. (1) C_G^κ is a club in κ ,

(2) C_G^α is unbounded in κ ,

(3) $\alpha \neq \beta \implies C_G^\alpha \neq C_G^\beta$.

Proof. The first two claims are immediate as these are sequences which are generated by Radin forcing.

The last is by density and noticing that for $p \in P_{\bar{E}}$ when $\bar{\nu} \in T^p$ is permitted for $p^{\bar{E}_\alpha}, p^{\bar{E}_\beta}$ we required $\pi_{\text{mc}(p), \bar{E}_\alpha}(\bar{\nu}) \neq \pi_{\text{mc}(p), \bar{E}_\beta}(\bar{\nu})$. \square

6. HOMOGENEITY IN DENSE OPEN SUBSETS

Our aim in this section is to prove the following

Theorem 6.1. *Let $D \subseteq P_{\bar{E}}$ be dense open and $p = p_k \hat{\wedge} \dots \hat{\wedge} p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\exists S^k \exists n_k \forall \langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle \in S^k \dots \exists S^0 \exists n_0 \forall \langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle \in S^0$$

$$(p_k^*)_{\langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle} \hat{\wedge} \dots \hat{\wedge} (p_0^*)_{\langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle} \in D$$

where

- (1) $S^i \subseteq T^{p_i^*} \upharpoonright [V_\kappa]^{n_i}$,
- (2) $\forall l < n_i \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle \in S^i \exists \xi \text{ Suc}_{S^i}(\langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle) \in E_{\text{mc}(p_i^*)}(\xi)$.

The proof is done by a series of lemmas.

Definition 6.2. Let $p \in P_{\bar{E}}^*$. Let s be a function such that $\text{dom } s \subseteq \bar{E}$. and for all $\bar{\alpha}, \bar{\beta} \in \text{dom } s$, $\bar{\alpha} \neq \bar{\beta}$

- (1) $s(\bar{\alpha})$ is an extender sequence,
- (2) $l(s(\bar{\alpha})) = l(s(\bar{\beta}))$,
- (3) $\kappa^0(s(\bar{\alpha})) = \kappa^0(s(\bar{\beta}))$,
- (4) $s(\bar{\alpha}) \neq s(\bar{\beta})$.

We define $(p)_{\langle s \rangle}$ to be $p'_1 \frown p'_0$ where

- (1) $\text{supp } p'_0 = \text{supp } p$,
- (2) $\forall \bar{\alpha} \in \text{supp } p'_0$

$$p'_0{}^{\bar{\alpha}} = \begin{cases} s(\bar{\alpha}) & \bar{\alpha} \in \text{dom } s, \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})), l(s(\bar{\alpha})) > 0, \\ p_0^{\bar{\alpha}} \frown s(\bar{\alpha}) & \bar{\alpha} \in \text{dom } s, \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha})), l(s(\bar{\alpha})) = 0, \\ p^{\bar{\alpha}} & \text{Otherwise,} \end{cases}$$

- (3) If $s(\text{mc}(p)) \in T^p$ then $T^{p'_0} = T^p_{\langle s(\text{mc}(p)) \rangle}$. Otherwise we leave $T^{p'_0}$ undefined.
- (4) If $\forall \bar{\alpha} \in \text{dom } s \ l(s(\bar{\alpha})) = 0$ then $p'_1 = \emptyset$,
- (5) if $\forall \bar{\alpha} \in \text{dom } s \ l(s(\bar{\alpha})) > 0$ then
 - (5.1) $\text{supp } p'_1 = \{s(\bar{\alpha}) \mid \bar{\alpha} \in \text{supp } p \cap \text{dom } s, \max \kappa(p^{\bar{\alpha}}) < \kappa^0(s(\bar{\alpha}))\}$,
 - (5.2) $\forall \bar{\alpha} \in \text{supp } p'_0 \ \forall s(\bar{\alpha}) \in \text{supp } p'_1 \ p_1^{s(\bar{\alpha})} = p^{\bar{\alpha}}$,
 - (5.3) If $s(\text{mc}(p)) \in T^p$ then $T^{p'_1} = T^p(s(\text{mc}(p)))$. Otherwise we leave $T^{p'_1}$ undefined,

Definition 6.3. Let $p \in P_{\bar{E}}^*$. Let s be a function with $\text{dom } s = 1, \dots, n$ such that for all $i \ s(i)$ satisfies definition 6.2. Then we define $(p)_{\langle s \rangle}$ as p^n where p^n is defined by induction as follows:

$$p^0 = p, \\ p^{i+1} = p_i^i \frown \dots \frown p_1^i \frown (p_0^i)_{\langle s(i+1) \rangle}.$$

We note the following: If $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^p$ and we set for all $1 \leq i \leq n$

$$s(i) = \{ \langle \bar{\alpha}, \pi_{\text{mc}(p), \bar{\alpha}}(\bar{\nu}_i) \rangle \mid \bar{\alpha} \in \text{supp } p \}$$

then

$$(p)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} = (p)_{\langle s \rangle}.$$

We use this operation also in cases where p is not strictly a condition. That is if $p \cup \{T\} \in P_{\bar{E}}$ we also use $(p)_{\langle s \rangle}$. In this case we ignore the trees in the definition.

This definition is used in the proof of the homogeneity for the following reason: Beforehand we do not know what a legitimate extension is. By checking all possible μ 's we check all legitimate conditions which *might* be extensions.

Claim 6.4. Let D be dense open in $P_{\bar{E}}/P_{\bar{E}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{E}}$, $0 < n < \omega$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:

- (1) There is $S \subseteq T^{p^*} \upharpoonright [V_\kappa]^n$ such that
 - (1.1) $\forall k < n \exists \xi < l(\bar{E}) \text{ Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\text{mc}(p^*)}(\xi)$,
 - (1.2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \ (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D$.

$$(2) \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} q \notin D.$$

Proof. We give the proof for $n = 1$. Adapting the proof for higher n 's require that whenever we enumerate singletons we should enumerate n -tuples and when we use j we should use j_n .

We start an induction on ξ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa \rangle.$$

We start by setting

$$\begin{aligned} u^0 &= p_0 \setminus \{T^{p_0}\}, \\ \bar{\alpha}^0 &= \text{mc}(p_0), \\ T^0 &= T^{p_0} \upharpoonright \pi_{\bar{\alpha}^0, 0}^{-1} \{\bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible}\}, \end{aligned}$$

and taking an increasing enumeration

$$\{\kappa^0(\bar{\nu}) \mid \langle \bar{\nu} \rangle \in T^0\} = \langle \tau_\xi \mid \xi < \kappa \rangle.$$

Assume that we have constructed

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle.$$

We have 2 cases: ξ_0 is limit: Choose $\bar{\alpha}^{\xi_0} >_{\mathbb{E}} \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi < \xi_0} u^\xi \cup \{\langle \bar{\alpha}^{\xi_0}, t \rangle \mid \text{where } \kappa^0(t) = \tau_{\xi_0}\}.$$

$\xi_0 = \xi + 1$: For each $\bar{\nu}$ such that $\kappa^0(\bar{\nu}) = \tau_\xi$ we set

$$S(\bar{\nu}) = \left(\prod_{\substack{\bar{\alpha} \in \text{supp } u^\xi \\ \kappa((u^\xi)^{\bar{\alpha}}) < \tau_\xi}} \{\bar{\mu} \mid \kappa^0(\bar{\mu}) = \kappa^0(\bar{\nu})\} \right) \times \{\langle \bar{\nu} \rangle\}.$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu}) = \tau_\xi} S(\bar{\nu})$$

and fix an enumeration of S

$$S = \langle s^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle.$$

There are fewer than τ_{ξ_0} elements in S . We use τ_{ξ_0} as this is the maximum size S can have which is not ‘killing’ the induction.

We do induction on ρ which builds

$$\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho}, u_1^{\xi_0, \rho}, T_1^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle$$

from which we build $\langle \bar{\alpha}^{\xi_0}, u^{\xi_0} \rangle$. Set

$$\begin{aligned} \bar{\alpha}^{\xi_0, 0} &= \bar{\alpha}^\xi, \\ u_0^{\xi_0, 0} &= u_0^\xi. \end{aligned}$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho} \mid \rho < \rho_0 \rangle$.

We have 2 cases:

ρ_0 is limit: Set

$$\begin{aligned} \forall \rho < \rho_0 \quad \bar{\alpha}^{\xi_0, \rho_0} >_{\bar{E}} \bar{\alpha}^{\xi_0, \rho}, \\ u^{\xi_0, \rho_0} = \bigcup_{\rho < \rho_0} u^{\xi_0, \rho} \cup \{ \langle \bar{\alpha}^{\xi_0, \rho_0}, t \rangle \} \text{ where } \kappa^0(t) = \tau_\xi. \end{aligned}$$

We set $T_0^{\xi_0, \rho_0}, T_1^{\xi_0, \rho_0}$ to anything we like as we do not use them later.
 $\rho_0 = \rho + 1$: Let $\langle \bar{\nu} \rangle = s^{\xi_0, \rho}(2)$. Set

$$\begin{aligned} u'' &= u''_1 \frown u''_0 = (u_0^{\xi_0, \rho})_{\langle s^{\xi_0, \rho} \rangle}, \\ T_0'' &= \pi_{\bar{\alpha}^\xi, \bar{\alpha}^0}^{-1}(T_{\langle \bar{\nu} \rangle}^0), \\ T_1'' &= \pi_{\text{mc}(u''_1), \bar{\nu}}^{-1}(T^0(\bar{\nu})). \end{aligned}$$

If there are

$$\begin{aligned} q'_1 &\leq^* u''_1 \cup \{T_1''\}, \\ q'_0 &\leq^* u''_0 \cup \{T_0''\}, \end{aligned}$$

such that

$$q'_1 \frown q'_0 \in D$$

then set

$$\begin{aligned} \bar{\alpha}^{\xi_0, \rho_0} &= \text{mc}(q'_0), \\ u_0^{\xi_0, \rho_0} &= u^{\xi_0, \rho} \cup (q'_0 \setminus (u''_0 \cup \{T^{q'_0}\})), \\ T_0^{\xi_0, \rho_0} &= T^{q'_0}, \\ u_1^{\xi_0, \rho_0} &= q'_1 \setminus (u''_1 \cup \{T^{q'_1}\}), \\ \bar{\alpha}_1^{\xi_0, \rho_0} &= \text{mc}(q'_1), \\ T_1^{\xi_0, \rho_0} &= T^{q'_1}, \end{aligned}$$

otherwise set

$$\begin{aligned} \bar{\alpha}^{\xi_0, \rho_0} &= \bar{\alpha}^{\xi_0, \rho}, \\ u_0^{\xi_0, \rho_0} &= u^{\xi_0, \rho}, \\ T_0^{\xi_0, \rho_0} &= T_0'', \\ u_1^{\xi_0, \rho_0} &= \emptyset, \\ \bar{\alpha}_1^{\xi_0, \rho_0} &= \text{mc}(u''), \\ T_1^{\xi_0, \rho_0} &= T_1''. \end{aligned}$$

When the induction on ρ terminates we have $\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho}, u_1^{\xi_0, \rho}, T_1^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle$. We continue with the induction on ξ . We set

$$\begin{aligned} \forall \rho < \tau_{\xi_0} \quad \bar{\alpha}^{\xi_0} >_{\bar{E}} \bar{\alpha}^{\xi_0, \rho}, \\ u_0^{\xi_0} = \bigcup_{\rho < \tau_{\xi_0}} u_0^{\xi_0, \rho} \cup \{ \langle \bar{\alpha}^{\xi_0}, t \rangle \} \text{ where } \max \kappa^0(t) = \tau_\xi. \end{aligned}$$

When the induction on ξ terminates we have $\langle \bar{\alpha}^\xi, u_0^\xi \mid \xi < \kappa \rangle$. Let

$$\begin{aligned} \forall \xi < \kappa \quad \bar{\alpha}^{*\prime} >_{\mathbb{E}} \bar{\alpha}^\xi, \\ p_0^{*\prime} = \bigcup_{\xi < \kappa} u_0^\xi \cup \{ \langle \bar{\alpha}^{*\prime}, t \rangle \} \text{ where } \max \kappa^0(t) = \max p_0^0. \end{aligned}$$

We set

$$\text{Lev}_0(T^{p_0^{*\prime}}) = \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^0}^{-1} \text{Lev}_0(T^0).$$

Let us consider $\langle \bar{\nu} \rangle \in \text{Lev}_0(T^{p_0^{*\prime}})$. There is ξ such that $\kappa^0(\bar{\nu}) = \tau_\xi$. We set

$$\begin{aligned} s(1) &= \{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}}(\bar{\nu}) \rangle \mid \bar{\alpha} \in \text{supp } p_0^* \}, \\ s(2) &= \{ \langle \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^0}(\bar{\nu}) \rangle \}. \end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0, \rho_0})_{\langle s \rangle} = (u_0^{\xi_0, \rho_0})_{\langle s^{\xi_0, \rho} \rangle}$$

where $\rho_0 = \rho + 1$. We set

$$\begin{aligned} T_{\langle \bar{\nu} \rangle}^{p_0^{*\prime}} &= \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^{\xi_0, \rho_0}}^{-1} (T^{\xi_0, \rho_0}) \cap \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^0}^{-1} (T_{\langle \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^0}(\bar{\nu}) \rangle}^0), \\ T^{p_0^{*\prime}}(\bar{\nu}) &= T_1^{\xi_0, \rho_0}, \\ \bar{\alpha}'_1(\bar{\nu}) &= \bar{\alpha}_1^{\xi_0, \rho_0}, \\ p'_1(\bar{\nu}) &= u_1^{\xi_0, \rho_0}. \end{aligned}$$

Note that $T^{p_0^{*\prime}}$ is not legal as tree in a condition because if $p'_1(\bar{\nu}) \neq \emptyset$ then $T^{p_0^{*\prime}}(\bar{\nu})$ might be defined on a too high coordinate. We abuse the notation because wherever $T^{p_0^{*\prime}}(\bar{\nu})$ appears so does $p'_1(\bar{\nu})$.

Let us show that $p_0^{*\prime}$ approximates the p^* we look for. So let $\langle \bar{\nu} \rangle \in T^{p_0^{*\prime}}$ and assume

$$\begin{aligned} (6.4.1a) \quad q'_1 &\leq^* p'_1(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_1, \\ (6.4.1b) \quad q'_0 &\leq^* ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_0, \\ (6.4.1c) \quad q'_1 \cap q'_0 &\in D. \end{aligned}$$

Let ξ be such that $\kappa^0(\bar{\nu}) = \tau_\xi$. Set s as

$$\begin{aligned} s(1) &= \{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}}(\bar{\nu}) \rangle \mid \bar{\alpha} \in \text{supp } p_0^* \}, \\ s(2) &= \{ \langle \pi_{\bar{\alpha}^{*\prime}, \bar{\alpha}^0}(\bar{\nu}) \rangle \}, \end{aligned}$$

where $\xi_0 = \xi + 1$, $\rho_0 = \rho + 1$. By our construction there is ρ such that

$$(u^{\xi_0, \rho_0})_{\langle s^{\xi_0, \rho} \rangle} = (u^{\xi_0, \rho_0})_{\langle s \rangle}.$$

Let us set

$$r = ((u_{\langle s^{\xi_0, \rho} \rangle}^{\xi_0, \rho_0})_1 \cup \{ T_1^{\xi_0, \rho_0} \}) \cap ((u_{\langle s^{\xi_0, \rho} \rangle}^{\xi_0, \rho_0})_0 \cup \{ T_0^{\xi_0, \rho_0} \}).$$

By construction we have

$$(p'_1(\bar{\nu}) \cup ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_1) \cap ((p_0^{*\prime})_{\langle \bar{\nu} \rangle})_0 \leq^* r.$$

So what we have is

$$D \ni q'_1 \cap q'_0 \leq^* r.$$

This is a positive answer to the question in the induction, hence

$$r \in D,$$

which gives us, by openness of D , that

$$(6.4.2) \quad (p'_1(\bar{\nu}) \cup ((p_0^*)_{\langle \bar{\nu} \rangle})_1) \frown ((p_0^*)_{\langle \bar{\nu} \rangle})_0 \in D.$$

Having proved this approximation property of $p_0^{*'}$, let us consider the set

$$B = \{ \langle \bar{\nu} \rangle \in T^{p_0^{*'}} \mid \exists q \leq^* (p'_1(\bar{\nu}) \cup ((p_0^*)_{\langle \bar{\nu} \rangle})_1) \frown ((p_0^*)_{\langle \bar{\nu} \rangle})_0 \ q \in D \}.$$

Let $\bar{\alpha}^{*'} = \text{mc}(p_0^{*'})$. There are 2 cases to be considered:

$$(1) \ \exists \zeta < 1(\bar{E}) \ B \in E_{\bar{\alpha}^{*'}}(\zeta).$$

Let us set

$$\begin{aligned} p_1^\zeta &= j(p'_1)(\bar{E}_{\bar{\alpha}^{*'}} \upharpoonright \zeta), \\ \bar{\beta}^{\zeta'} &= \text{mc}(p_1^\zeta), \\ A^\zeta &= \{ \langle \bar{\nu} \rangle \mid ((p_1^\zeta)_{\langle \bar{\nu} \rangle})_1 = p'_1(\pi_{\bar{\beta}^{\zeta'}, \bar{\alpha}^{*'}}(\bar{\nu})) \}. \end{aligned}$$

Clearly

$$A^\zeta \in E_{\bar{\beta}^{\zeta'}}(\zeta).$$

Let $\bar{\beta}^\zeta \succ_E \bar{\beta}^{\zeta'}$, $\bar{\alpha}^{*'}$. Set

$$T^\zeta = \pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}^{-1}(T^{p_0^{*'}}).$$

As $T^{p_0^{*'}}$ is not legal as a tree we give the necessary modification making T^ζ legal

$$\begin{aligned} \forall \bar{\nu} \in \pi_{\bar{\beta}^\zeta, \bar{\beta}^{\zeta'}}^{-1}(A^\zeta) \ T^\zeta(\bar{\nu}) &= \pi_{\bar{\nu}, \bar{\alpha}'_1(\pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu}))}^{-1} T^{p_0^{*'}}(\pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu})), \\ \forall \bar{\nu} \notin \pi_{\bar{\beta}^\zeta, \bar{\beta}^{\zeta'}}^{-1}(A^\zeta) \ T^\zeta(\bar{\nu}) &= \pi_{\bar{\nu}, \bar{\alpha}'_1(\pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu}))}^{-1} T^{p_0}(\pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu})), \\ p^\zeta &= p_1^\zeta \cup p_0^{*'} \cup \{ \langle \beta^\zeta, t \rangle \} \cup \{ T^\zeta \} \ \text{where} \ \max t^0 = p_0^0. \end{aligned}$$

The nice property of p^ζ is that when $\langle \bar{\nu} \rangle \in T^\zeta \upharpoonright (\pi_{\bar{\beta}^\zeta, \bar{\beta}^{\zeta'}})^{-1}(A^\zeta)$ we get

$$(p^\zeta)_{\langle \bar{\nu} \rangle} \leq^* (p'_1(\pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu})) \cup ((p_0^*)_{\langle \pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu}) \rangle})_1) \frown ((p_0^*)_{\langle \pi_{\bar{\beta}^\zeta, \bar{\alpha}^{*'}}(\bar{\nu}) \rangle})_0.$$

We set

$$\begin{aligned} p^* &= p^\zeta, \\ A &= \text{Lev}_0(T^\zeta) \cap (\pi_{\bar{\beta}^\zeta, \bar{\beta}^{\zeta'}})^{-1} A^\zeta, \end{aligned}$$

and show that the claim is satisfied: Assume that

$$\begin{aligned} \langle \bar{\nu} \rangle &\in A, \\ q'_1 &\leq^* ((p^*)_{\langle \bar{\nu} \rangle})_1, \\ q'_0 &\leq^* ((p^*)_{\langle \bar{\nu} \rangle})_0, \\ q'_1 \frown q'_0 &\in D. \end{aligned}$$

Note that

$$\begin{aligned} ((p^*)_{\langle \bar{v} \rangle})_1 &\leq^* p'_1(\pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}) \cup ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_1, \\ ((p^*)_{\langle \bar{v} \rangle})_0 &\leq^* ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_0. \end{aligned}$$

Hence, we know that

$$\begin{aligned} q'_1 &\leq^* p'_1(\pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v})) \cup ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_1, \\ q'_0 &\leq^* ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_0, \\ q'_1 \wedge q'_0 &\in D. \end{aligned}$$

This is the assumption (6.4.1). So from (6.4.2) we know that

$$(p'_1(\pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v})) \cup ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_1) \wedge ((p_0^{*'})_{\langle \pi_{\bar{\beta}\zeta, \bar{\alpha}^{*'}}(\bar{v}) \rangle})_0 \in D$$

hence by openness of D

$$(p^*)_{\langle \bar{v} \rangle} \in D.$$

(2) $\forall \zeta < l(\bar{E}) \ B \notin E_{\bar{\alpha}^{*'}}(\zeta)$ which is the same as saying that

$$\{\langle \bar{v} \rangle \in T^{p_0^{*'}} \mid \forall q \leq^* (p'_1(\bar{v}) \cup ((p_0^{*'})_{\langle \bar{v} \rangle})_1) \wedge ((p_0^{*'})_{\langle \bar{v} \rangle})_0 \ q \notin D\} \in \bar{E}_{\bar{\alpha}^{*'}}.$$

In fact from the construction we can see that

$$\{\langle \bar{v} \rangle \in T^{p_0^{*'}} \mid p'_1(\bar{v}) = \emptyset\} \in \bar{E}_{\bar{\alpha}^{*'}}.$$

So we really have

$$A = \{\langle \bar{v} \rangle \in T^{p_0^{*'}} \mid \forall q \leq^* (p_0^{*'})_{\langle \bar{v} \rangle} \ q \notin D\} \in \bar{E}_{\bar{\alpha}^{*'}}.$$

and the completion is quite easy now, we set

$$\begin{aligned} T^{p^*} &= T^{p^{*'}} \upharpoonright A, \\ p^* &= p_0^{*'} \cup \{T^{p^*}\}. \end{aligned}$$

□

Claim 6.5. *Let D be dense open in $P_{\bar{E}}/P_{\bar{e}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{e}}$. Then there is $p^* \leq^* p$ such that one and only one of the following is true:*

- (1) *There are $n < \omega$, $S \subseteq T^{p^*} \upharpoonright [V_{\bar{\kappa}}]^n$ such that*
 - (1.1) $\forall k < n \exists \xi < l(\bar{E}) \ \text{Suc}_S(\langle \bar{v}_1, \dots, \bar{v}_k \rangle) \in E_{\text{mc}(p^*)}(\xi),$
 - (1.2) $\forall \langle \bar{v}_1, \dots, \bar{v}_n \rangle \in S \ (p^*)_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle} \in D,$
- (2) $\forall n < \omega \ \forall \langle \bar{v}_1, \dots, \bar{v}_n \rangle \in T^{p^*} \forall q \leq^* (p^*)_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle} \ q \notin D.$

Proof. Let $p^0 = p$.

Generate $p^{n+1} \leq^* p^n$ by invoking 6.4 for $n + 1$ levels.

Take $\forall n < \omega \ p^* \leq p^n$. □

Claim 6.6. *Let D be dense open in $P_{\bar{E}}/P_{\bar{e}}$, $p = p_0 \in P_{\bar{E}}/P_{\bar{e}}$. Then there are $n < \omega$, $p^* \leq^* p$, $S \subseteq T^{p^*} \upharpoonright [V_{\bar{\kappa}}]^n$ such that*

- (1) $\forall k < n \exists \xi < l(\bar{E}) \ \text{Suc}_S(\langle \bar{v}_1, \dots, \bar{v}_k \rangle) \in E_{\text{mc}(p^*)}(\xi),$
- (2) $\forall \langle \bar{v}_1, \dots, \bar{v}_n \rangle \in S \ (p^*)_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle} \in D.$

Proof. Towards a contradiction, let us assume that the conclusion is false. That means that for all $p^* \leq^* p$, for all $n < \omega$, for all $S \subseteq T^{p^*} \upharpoonright [V_\kappa]^n$ such that

$$\forall k < n \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle \in S \exists \xi < l(\bar{E}) \text{ Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\text{mc}(p^*)}(\xi)$$

we have

$$\exists \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \notin D.$$

We construct a \leq^* -decreasing sequence as follows: We set $p^0 = p$. We construct p^{n+1} from p^n using 6.4 for $n+1$ levels. Due to our assumption we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^n} \forall q \leq^* (p^n)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} q \notin D.$$

We choose $p^{*'}$ such that $\forall n < \omega p^{*'} \leq^* p^n$ we get

$$\forall n < \omega \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^{*'}} \forall q \leq^* (p^{*'})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} q \notin D.$$

Construct tree T from $T^{p^{*'}}$ using 4.10. Let us call p^* the condition $p^{*'}$ with T substituted for $T^{p^{*'}}$. Now if we have

$$q \leq p^*$$

then there is $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in T^{p^{*'}}$ such that

$$q \leq^* (p^{*'})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}.$$

Hence

$$q \notin D.$$

However, D is dense. Contradiction. \square

Claim 6.7. *Let D be dense open in $P_{\bar{E}}$, $p = p_1 \hat{\wedge} p_0 \in P_{\bar{E}}$. Then there is $p^* \leq^* p$ such that*

$$\begin{aligned} \exists S^1 \exists n_1 \forall \langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle \in S^1 \dots \exists S^0 \exists n_0 \forall \langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle \in S^0 \\ (p_1^*)_{\langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle} \hat{\wedge} \dots \hat{\wedge} (p_0^*)_{\langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle} \in D \end{aligned}$$

where

- (1) $S^i \subseteq T^{p_i^*} \upharpoonright [V_\kappa]^{n_i}$,
- (2) $\forall l < n_i \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle \in S^i \exists \xi \text{ Suc}_{S^i}(\langle \bar{\nu}_1, \dots, \bar{\nu}_l \rangle) \in E_{\text{mc}(p_i^*)}(\xi)$.

Proof. Let \bar{e} be such that $p_1 \in P_{\bar{e}}$. We prove that there are $n < \omega$, $p_0^* \leq^* p_0$, $q_1 \leq p_1$, $S \subseteq T^{p^*} \upharpoonright [V_\kappa]^n$ such that

- (1) $\forall k < n \exists \xi < l(\bar{E}) \text{ Suc}_S(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\text{mc}(p^*)}(\xi)$,
- (2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S q_1 \hat{\wedge} (p^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D$.

Set

$$E = \{r \in P_{\bar{E}} \mid \exists q_1 q_1 \hat{\wedge} r \in D, q_1 \leq p_1\}.$$

This E is dense open in $P_{\bar{E}}/P_{\bar{e}}$: Let $r \in P_{\bar{E}}/P_{\bar{e}}$. Then $p_1 \hat{\wedge} r \in P_{\bar{E}}$. By density of D , there is $q_1 \hat{\wedge} s \in D$ such that $q_1 \leq p_1$, $s \leq r$. By the definition of E , $s \in E$. Hence E is dense. Openness of E is immediate from openness of D .

By 6.6 there are $p_0^* \leq^* p_0$, S' , $n < \omega$ such that

- (1) $\forall k < n \exists \xi < l(\bar{E}) \text{ Suc}_{S'}(\langle \bar{\nu}_1, \dots, \bar{\nu}_k \rangle) \in E_{\text{mc}(p^*)}(\xi)$,
- (2) $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S' (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in E$.

This means that $\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S'$ there is $q_1(\bar{\nu}_1, \dots, \bar{\nu}_n) \leq p_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S' \quad q_1(\bar{\nu}_1, \dots, \bar{\nu}_n) \wedge (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

As $|P_{\bar{\epsilon}}| < \kappa$, $q_1(\bar{\nu}_1, \dots, \bar{\nu}_n)$ is in fact almost always constant. Hence, by shrinking S' to S and letting q_1 be this constant value, we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \quad q_1 \wedge (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

With this, we finished the first part of the proof. We use this claim for all conditions in $P_{\bar{\epsilon}}$.

Let $P_{\bar{\epsilon}} = \{p_1^\zeta \mid \zeta < \lambda\}$ where $\lambda < \kappa$.

We construct by induction a \leq^* -decreasing sequence $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Set

$$p_0^0 = p_0.$$

Assume we have constructed $\langle p_0^\zeta \mid \zeta < \zeta_0 \rangle$.

ζ_0 is limit: Choose $p_0^{\zeta_0} \leq^* p_0^\zeta$ for all $\zeta < \zeta_0$.

$\zeta_0 = \zeta + 1$: Use the first part of the proof on $p_1^\zeta \wedge p_0^\zeta$ to construct $p_0^{\zeta_0}$.

When the induction terminates we have $\langle p_0^\zeta \mid \zeta < \lambda \rangle$. Choose

$$\forall \zeta < \lambda \quad p_0^* \leq^* p_0^\zeta.$$

Let

$$D_{\bar{\epsilon}} = \{q_1 \in P_{\bar{\epsilon}} \mid \exists n \exists S \forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \quad q_1 \wedge (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D\}.$$

$D_{\bar{\epsilon}}$ is dense open: Let $q_1 \in P_{\bar{\epsilon}}$. Then there is ζ such that $q_1 = p_1^\zeta$. By the induction we have that there are $n, S, r_1 \leq q_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \quad r_1 \wedge (p_0^{\zeta_0})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

By openness of D we get

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in S \quad r_1 \wedge (p_0^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle} \in D.$$

Hence

$$r_1 \in D_{\bar{\epsilon}}.$$

As $D_{\bar{\epsilon}}$ is dense open we can use 6.6. Hence there are $p_1^* \leq p_1, S^1, n_1$ such that

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_{n_1} \rangle \in S^1 \quad (p_1^*)_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_1} \rangle} \in D_{\bar{\epsilon}}.$$

This means that

$$\forall \langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle \in S^1 \quad \exists n \exists S \forall \langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n} \rangle \in S \\ (p_1^*)_{\langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle} \wedge (p_0^*)_{\langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n} \rangle} \in D,$$

which is what we need to prove. \square

Finally, we add the last touch.

Proof of 6.1. The proof is done by induction on k . The case $k = 1$ is 6.7. We assume, then, that the theorem is proved for k and prove it for $k + 1$.

Then $p = p_{k+1} \wedge p_k \wedge \dots \wedge p_0$. Let $\bar{\epsilon}$ be such that $p_{k+1} \in P_{\bar{\epsilon}}$. We just repeat the proof of 6.7 with $P_{\bar{\epsilon}}$ and use the induction hypotheses to conclude the proof. \square

7. PRIKRY'S CONDITION

Theorem 7.1. *Let $p \in P_{\bar{E}}$ and σ a formula in the forcing language. Then there is $p^* \leq^* p$ such that $p^* \parallel \sigma$.*

Proof. The set $\{q \in P_{\bar{E}} \mid q \parallel \sigma\}$ is dense open. Assuming $p = p_k \hat{\ } \cdots \hat{\ } p_0$ and using 6.1 we get that there is $q \leq^* p$ such that

$$\exists S^k \exists n_k \forall \langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle \in S^k \dots \exists S^0 \exists n_0 \forall \langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle \in S^0 \\ (p_k^*)_{\langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle} \hat{\ } \cdots \hat{\ } (p_0^*)_{\langle \bar{\nu}_{0,1}, \dots, \bar{\nu}_{0,n_0} \rangle} \parallel \sigma.$$

Recall that we really should write

$$S^{k-1}(\bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k}), \\ S^{k-2}(\bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k}, \bar{\nu}_{k-1,1}, \dots, \bar{\nu}_{k-1,n_k}), \\ \vdots$$

In order to avoid (too much) clutter, we use the following convention in the proof. When we write

$$\vec{\nu} \in \prod_{1 \leq l \leq k} S^l$$

we mean that

$$\langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle \in S^k, \\ \vdots \\ \langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle \in S^1,$$

and $r(\vec{\nu})$ is

$$(q_k)_{\langle \bar{\nu}_{k,1}, \dots, \bar{\nu}_{k,n_k} \rangle} \hat{\ } \cdots \hat{\ } (q_1)_{\langle \bar{\nu}_{1,1}, \dots, \bar{\nu}_{1,n_1} \rangle}.$$

We start by naming q_0 as $q_0^{n_0}$ and T^{q_0} as T^{0,n_0} . For $\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle \in S^0$ set

$$A^0 = \{ \langle \bar{\nu}_{n_0} \rangle \in \text{Suc}_{T^{0,n_0}}(\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle) \mid r(\vec{\nu}) \hat{\ } (q_0^{n_0})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0} \rangle} \Vdash \sigma \}, \\ A^1 = \{ \langle \bar{\nu}_{n_0} \rangle \in \text{Suc}_{T^{0,n_0}}(\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle) \mid r(\vec{\nu}) \hat{\ } (q_0^{n_0})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0} \rangle} \Vdash \neg \sigma \}.$$

Note that

$$\text{Suc}_{S^0}(\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle) \subseteq A^0 \cup A^1, \\ A^0 \cap A^1 = \emptyset$$

Hence, there is $\xi < l(\bar{E})$ such that one and only one of the following is true:

- (1) $A^0 \in E_{\text{mc}(q_0^{n_0})}(\xi)$,
- (2) $A^1 \in E_{\text{mc}(q_0^{n_0})}(\xi)$.

In either case, using 4.8, we can shrink $T_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle}^{0,n_0}$ and get a condition q'_0 such that $r(\vec{\nu}) \hat{\ } q'_0_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle} \parallel \sigma$.

So we shrink now $T_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle}^{0,n_0}$ for all $\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle \in S^0$ and we call this tree T^{0,n_0-1} . The name of the condition $q_0^{n_0}$ with T^{0,n_0-1} substituted for T^{0,n_0} is $q_0^{n_0-1}$. $q_0^{n_0-1}$ satisfies

$$\forall \langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle \in S^0 \ r(\vec{\nu}) \hat{\ } (q_0^{n_0-1})_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n_0-1} \rangle} \parallel \sigma.$$

We are now in the same position as we were when setting $q_0^{n_0}$. So by repeating the above arguments we get

$$p_0 \geq^* q_0 = q_0^{n_0} \geq^* q_0^{n_0-1} \geq^* \cdots \geq^* q_0^1 \geq^* q_0^0$$

such that for each $l = n_0, n_0 - 1, \dots, 1, 0$

$$\forall \langle \bar{v}_1, \dots, \bar{v}_l \rangle \in S^0 \ r(\bar{v}) \wedge (q_0^l)_{\langle \bar{v}_1, \dots, \bar{v}_l \rangle} \parallel \sigma.$$

Specifically we get

$$r(\bar{v}) \wedge (q_0^0) \parallel \sigma.$$

Of course q_0^0 depends on \bar{v} . Note that we got from $q_0^{n_0}$ to q_0^0 only by shrinking the trees. So, we repeat this process for all \bar{v} calling the resulting condition $q_0^0(\bar{v})$. So we have

$$\forall \bar{v} \in \prod_{1 \leq l \leq k} S^l \ r(\bar{v}) \wedge q_0^0(\bar{v}) \parallel \sigma.$$

By setting

$$T^{p_0^*} = \bigcap_{\bar{v}} T^{q_0^0(\bar{v})}$$

and letting p_0^* be q_0 with $T^{p_0^*}$ substituted for T^{q_0} we get

$$\forall \bar{v} \in \prod_{1 \leq l \leq k} S^l \ r(\bar{v}) \wedge p_0^* \parallel \sigma.$$

We are in the same position as in the beginning of the proof. So we can generate in the same way p_1^* from p_1 and so on until we have

$$p_k^* \wedge \cdots \wedge p_0^* \parallel \sigma.$$

□

8. PROPERNESS

The notions $\langle N, P \rangle$ -generic and properness, as defined by Shelah [9], are as follows:

Definition 8.1. Let $N \prec H_\chi$ such that

- (1) $|N| = \omega$,
- (2) $P \in N$.

Then $p \in P$ is called $\langle N, P \rangle$ -generic if

$$p \Vdash \ulcorner \forall D \in \widehat{N} \ D \text{ is dense open in } \widehat{P} \implies D \cap \widetilde{G} \cap \widehat{N} \neq \emptyset \urcorner.$$

Definition 8.2. A forcing notion P is called proper if for all $N \prec H_\chi$ such that

- (1) $|N| = \omega$,
- (2) $P \in N$,

and for all $q \in P \cap N$ there is $p \leq q$ which is $\langle N, P \rangle$ -generic.

We adapt these definitions for our needs (namely, larger submodels), keeping the original names:

Definition 8.3. Let $N \prec H_\chi$ such that

- (1) $|N| = \kappa$,
- (2) $N \supseteq V_\kappa$,

- (3) $N \supseteq N^{<\kappa}$,
- (4) $P \in N$.

Then $p \in P$ is called $\langle N, P \rangle$ -generic if

$$p \Vdash \ulcorner \forall D \in \widehat{N} \ D \text{ is dense open in } \widehat{P} \implies D \cap \widetilde{G} \cap \widehat{N} \neq \emptyset \urcorner.$$

Definition 8.4. A forcing notion P is called proper if for all $N \prec H_\chi$ such that

- (1) $|N| = \kappa$,
- (2) $N \supseteq V_\kappa$,
- (3) $N \supseteq N^{<\kappa}$,
- (4) $P \in N$,

and for all $q \in P \cap N$ there is $p \leq q$ which is $\langle N, P \rangle$ -generic.

It was brought to our attention by the referee that Woodin initiated the use of proper forcing arguments to show cardinal preservation in Radin-style forcing.

Claim 8.5. Let $p \in P_{\bar{E}}$, $N \prec H_\chi$ such that

- (1) $|N| = \kappa$,
- (2) $N \supseteq V_\kappa$,
- (3) $N \supseteq N^{<\kappa}$,
- (4) $P_{\bar{E}} \in N$,
- (5) $p \in P_{\bar{E}} \cap N$.

Then there is $p^* \leq^* p$ such that p^* is $\langle N, P_{\bar{E}} \rangle$ -generic

Proof. Let $p = p_{k(p)} \frown \dots \frown p_1 \frown p_0$.

Let $\langle D_\xi \mid \xi < \kappa \rangle$ be an enumeration of all dense open subsets of $P_{\bar{E}}$ which are in N . Note that for $\xi_0 < \kappa$ we have that $\langle D_\xi \mid \xi < \xi_0 \rangle \in N$.

We start now an induction on ξ in which we build

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \kappa \rangle.$$

The construction is done ensuring that $\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle \in N$ for all $\xi_0 < \kappa$. We start by setting

$$\begin{aligned} u^0 &= p_0 \setminus \{T^{p_0}\}, \\ \bar{\alpha}^0 &= \text{mc}(p_0), \\ T^0 &= T^{p_0} \upharpoonright \pi_{\bar{\alpha}^0, 0}^{-1} \{\bar{\nu} \mid \kappa^0(\bar{\nu}) \text{ is inaccessible}\}, \end{aligned}$$

and taking an increasing enumeration in N

$$\{\kappa^0(\bar{\nu}) \mid \langle \bar{\nu} \rangle \in T^0\} = \langle \tau_\xi \mid \xi < \kappa \rangle.$$

Assume then that we have

$$\langle \bar{\alpha}^\xi, u^\xi \mid \xi < \xi_0 \rangle.$$

The construction splits now according to whether ξ_0 is limit or successor. In both cases the work is done inside N .

ξ_0 is limit: Choose $\bar{\alpha}^{\xi_0} >_{\bar{E}} \bar{\alpha}^\xi$ for all $\xi < \xi_0$ and set

$$u^{\xi_0} = \bigcup_{\xi < \xi_0} u^\xi \cup \{\langle \bar{\alpha}^{\xi_0}, t \rangle\} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

$\xi_0 = \xi + 1$: For each $\bar{\nu}_1, \dots, \bar{\nu}_n$ such that $\kappa^0(\bar{\nu}_1) < \dots < \kappa^0(\bar{\nu}_n) = \tau_\xi$ we set

$$\begin{aligned} S(\bar{\nu}_1, \dots, \bar{\nu}_n) = & \left(\prod_{\bar{\alpha} \in \text{supp } u^\xi} \{\bar{\mu}_1 \mid \kappa^0(\bar{\mu}_1) = \kappa^0(\bar{\nu}_1)\} \right) \times \\ & \left(\prod_{\bar{\alpha} \in \text{supp } u^\xi} \{\bar{\mu}_2 \mid \kappa^0(\bar{\mu}_2) = \kappa^0(\bar{\nu}_2)\} \right) \times \\ & \vdots \\ & \left(\prod_{\bar{\alpha} \in \text{supp } u^\xi} \{\bar{\mu}_n \mid \kappa^0(\bar{\mu}_n) = \kappa^0(\bar{\nu}_n)\} \right) \times \\ & \{\bar{\nu}_1, \dots, \bar{\nu}_n\}. \end{aligned}$$

Let

$$S = \bigcup_{\kappa^0(\bar{\nu}_1) < \dots < \kappa^0(\bar{\nu}_n) = \tau_\xi} S(\bar{\nu}_1, \dots, \bar{\nu}_n)$$

and set enumeration of S

$$S = \langle s^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle.$$

We do induction on ρ which builds

$$\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle,$$

from which we build $\langle \bar{\alpha}^{\xi_0}, u^{\xi_0} \rangle$. Set

$$\begin{aligned} \bar{\alpha}^{\xi_0, 0} &= \bar{\alpha}^\xi, \\ u_0^{\xi_0, 0} &= u_0^\xi. \end{aligned}$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T_0^{\xi_0, \rho} \mid \rho < \rho_0 \rangle$.

ρ_0 is limit: Set

$$\begin{aligned} \forall \rho < \rho_0 \quad \bar{\alpha}^{\xi_0, \rho_0} &>_{\bar{E}} \bar{\alpha}^{\xi_0, \rho}, \\ u^{\xi_0, \rho_0} &= \bigcup_{\rho < \rho_0} u^{\xi_0, \rho} \cup \{ \langle \bar{\alpha}^{\xi_0, \rho_0}, t \rangle \} \text{ where } \kappa^0(t) = \tau_{\xi_0}. \end{aligned}$$

We set T^{ξ_0, ρ_0} to anything we like as we do not use it later.

$\rho_0 = \rho + 1$: Let $s^{\xi_0, \rho}(n(s) + 1) = \langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle$.

$$\begin{aligned} u'' &= (u_0^{\xi_0, \rho})_{\langle s^{\xi_0, \rho} \rangle}, \\ T_0'' &= \pi_{\bar{\alpha}^\xi, \bar{\alpha}^0}^{-1}(T_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}^0), \\ T_1'' &= \pi_{\text{mc}(u_1'', \bar{\nu}_n)}^{-1}(T_{\langle \bar{\nu}_1, \dots, \bar{\nu}_{n-1} \rangle}^0(\bar{\nu}_n)), \\ &\vdots \\ T_{n-1}'' &= \pi_{\text{mc}(u_{n-1}'', \bar{\nu}_2)}^{-1}(T_{\langle \bar{\nu}_1 \rangle}^0(\bar{\nu}_2)), \\ T_n'' &= \pi_{\text{mc}(u_n'', \bar{\nu}_1)}^{-1}(T^0(\bar{\nu}_1)). \end{aligned}$$

Take enumeration

$$\{D_\sigma \mid \sigma < \tau_\xi\} \times \\ \{q \mid q \leq p_{k(p)} \hat{\wedge} \cdots \hat{\wedge} p_1 \hat{\wedge} u''_{n(s)} \cup \{T''_n\} \hat{\wedge} \cdots \hat{\wedge} u''_1 \cup \{T''_1\}\} = \\ \langle \langle E^{\xi_0, \rho_0, \zeta}, q^{\xi_0, \rho_0, \zeta} \mid \zeta < \tau_{\xi_0} \rangle \rangle.$$

We start induction on ζ . Set

$$\bar{\alpha}^{\xi_0, \rho_0, 0} = \bar{\alpha}^{\xi_0, \rho}, \\ u_0^{\xi_0, \rho_0, 0} = u^{\xi_0, \rho}.$$

Assume we have constructed $\langle \bar{\alpha}^{\xi_0, \rho_0, \zeta}, u_0^{\xi_0, \rho_0, \zeta}, T_0^{\xi_0, \rho_0, \zeta} \mid \zeta < \zeta_0 \rangle$.
 ζ_0 is limit:

$$\forall \zeta < \zeta_0 \ \bar{\alpha}^{\xi_0, \rho_0, \zeta_0} \underset{\bar{E}}{>} \bar{\alpha}^{\xi_0, \rho_0, \zeta}, \\ u_0^{\xi_0, \rho_0, \zeta_0} = \bigcup_{\zeta < \zeta_0} u_0^{\xi_0, \rho_0, \zeta} \cup \{ \langle \bar{\alpha}^{\xi_0, \rho_0, \zeta_0}, t \rangle \} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

We set $T^{\xi_0, \rho_0, \zeta}$ to whatever we want as no use of it is made later.
 $\zeta_0 = \zeta + 1$: We set

$$u'' = (u_0^{\xi_0, \rho_0, \zeta})_{s^{\xi_0, \rho}}, \\ T_0'' = \pi_{\bar{\alpha}^{\xi_0, \rho_0, \zeta}, \bar{\alpha}^0}^{-1} T_{(\bar{v}_1, \dots, \bar{v}_n)}^0.$$

If there is

$$u'_0 \leq^* u''_0 \cup \{T_0''\}$$

such that

$$q^{\xi_0, \rho_0, \zeta} \cup u'_0 \in E^{\xi_0, \rho_0, \zeta}$$

then set

$$\bar{\alpha}^{\xi_0, \rho_0, \zeta_0} = \text{mc}(u'_0), \\ u_0^{\xi_0, \rho_0, \zeta_0} = u_0^{\xi_0, \rho_0, \zeta} \cup (u'_0 \setminus (u''_0 \cup \{T^{u'_0}\})), \\ T^{\xi_0, \rho_0, \zeta_0} = T^{u'_0},$$

otherwise set

$$\bar{\alpha}^{\xi_0, \rho_0, \zeta_0} = \bar{\alpha}^{\xi_0, \rho_0, \zeta}, \\ u_0^{\xi_0, \rho_0, \zeta_0} = u_0^{\xi_0, \rho_0, \zeta}, \\ T_0^{\xi_0, \rho_0, \zeta_0} = T_0''.$$

When the induction on ζ terminates we have $\langle \bar{\alpha}^{\xi_0, \rho_0, \zeta}, u_0^{\xi_0, \rho_0, \zeta}, T^{\xi_0, \rho_0, \zeta} \mid \zeta < \tau_{\xi_0} \rangle$

We continue with the induction on ρ . We set

$$\forall \zeta < \tau_{\xi_0} \ \bar{\alpha}^{\xi_0, \rho_0} \underset{\bar{E}}{>} \bar{\alpha}^{\xi_0, \rho_0, \zeta}, \\ u_0^{\xi_0, \rho_0} = \bigcup_{\zeta < \tau_{\xi_0}} u_0^{\xi_0, \rho_0, \zeta} \cup \{ \langle \bar{\alpha}^{\xi_0, \rho_0}, t \rangle \} \text{ where } \kappa^0(t) = \tau_{\xi_0}.$$

When the induction on ρ terminates we have $\langle \bar{\alpha}^{\xi_0, \rho}, u_0^{\xi_0, \rho}, T^{\xi_0, \rho} \mid \rho < \tau_{\xi_0} \rangle$. We continue with the induction on ξ . We set

$$\begin{aligned} \forall \rho < \tau_{\xi_0} \quad \bar{\alpha}^{\xi_0} >_{\mathbb{E}} \bar{\alpha}^{\xi_0, \rho}, \\ u_0^{\xi_0} = \bigcup_{\rho < \tau_{\xi_0}} u_0^{\xi_0, \rho} \cup \{ \langle \bar{\alpha}^{\xi_0}, t \rangle \} \quad \text{where } \kappa^0(t) = \tau_{\xi_0}. \end{aligned}$$

When the induction on ξ terminates we have $\langle \bar{\alpha}^\xi, u_0^\xi \mid \xi < \kappa \rangle$. We note that this sequence is not in N . Let

$$\begin{aligned} \forall \xi < \kappa \quad \bar{\alpha}^* >_{\mathbb{E}} \bar{\alpha}^\xi, \\ p_0^* = \bigcup_{\xi < \kappa} u_0^\xi \cup \{ \langle \bar{\alpha}^*, t \rangle \} \quad \text{where } \kappa^0(t) = \max p_0^*. \end{aligned}$$

We construct a series of trees, R^n , and $T^{p_0^*}$ is $\bigcap_{n < \omega} R^n$.

$$\text{Lev}_0(R^0) = \pi_{\bar{\alpha}^*, \bar{\alpha}^0}^{-1} \text{Lev}_0(T^0).$$

Let us consider $\langle \bar{\nu}_1 \rangle \in \text{Lev}_0(R^0)$. There is ξ such that $\kappa^0(\bar{\nu}_1) = \tau_\xi$. We set

$$\begin{aligned} s(0) &= \{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*, \bar{\alpha}}(\bar{\nu}_1) \rangle \mid \bar{\alpha} \in \text{supp } p_0^* \}, \\ s(1) &= \{ \langle \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_1) \rangle \}. \end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0, \rho_0})_{\langle s \rangle} = (u_0^{\xi_0, \rho_0})_{\langle s^{\xi_0, \rho} \rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R_{\langle \bar{\nu}_1 \rangle}^1 = \pi_{\bar{\alpha}^*, \bar{\alpha}^{\xi_0, \rho_0}}^{-1} (T^{\xi_0, \rho_0}) \cap \pi_{\bar{\alpha}^*, \bar{\alpha}^0}^{-1} (T_{\langle \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_1) \rangle}^0).$$

Assume that we have constructed R^n . We set the first n levels of R^{n+1} to be the same as the first n levels of R^n and we complete the tree as follows. Let us consider $\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle \in R^n$. There is ξ such that $\kappa^0(\bar{\nu}_n) = \tau_\xi$. We set s as follows

$$\begin{aligned} \forall 1 \leq k \leq n \quad s(k) &= \{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*, \bar{\alpha}}(\bar{\nu}_k) \rangle \mid \bar{\alpha} \in \text{supp } p_0^* \}, \\ s(n+1) &= \{ \langle \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_1), \dots, \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_n) \rangle \}. \end{aligned}$$

Let $\xi_0 = \xi + 1$. By our construction there is ρ such that

$$(u_0^{\xi_0, \rho_0})_{\langle s \rangle} = (u_0^{\xi_0, \rho_0})_{\langle s^{\xi_0, \rho} \rangle},$$

where $\rho_0 = \rho + 1$. We set

$$R_{\langle \bar{\nu}_1, \dots, \bar{\nu}_n \rangle}^{n+1} = \pi_{\bar{\alpha}^*, \bar{\alpha}^{\xi_0, \rho_0}}^{-1} (T^{\xi_0, \rho_0}) \cap \pi_{\bar{\alpha}^*, \bar{\alpha}^0}^{-1} (T_{\langle \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_1), \dots, \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{\nu}_n) \rangle}^0).$$

After ω stages we set

$$T^{p_0^*} = \bigcap_{n < \omega} R^n.$$

We finish the construction by setting

$$p^* = p_{k(p)} \widehat{\ } \cdots \widehat{\ } p_1 \widehat{\ } p_0^*.$$

We show that p^* is as required.

Let G be $P_{\mathbb{E}}$ -generic such that $p^* \in G$. Let $D \in N$ be dense open in $P_{\mathbb{E}}$. We want to show $D \cap G \cap N \neq \emptyset$.

Choose $q \hat{\wedge} r_0 \in D \cap G$ such that

$$\begin{aligned} r_0 &\leq^* p_0'', \\ q &\leq p_{k(p)} \hat{\wedge} \cdots \hat{\wedge} p_1 \hat{\wedge} p_n'' \hat{\wedge} \cdots \hat{\wedge} p_1'', \end{aligned}$$

where

$$\begin{aligned} \langle \bar{v}_1, \dots, \bar{v}_n \rangle &\in \text{dom } T^{p_0^*}, \\ \kappa^0(\bar{v}_n) &= \tau_\xi, \\ D &\in \langle D_\zeta \mid \zeta < \tau_\xi \rangle, \\ p'' &= (p_0^*)_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle}. \end{aligned}$$

We set s to be

$$\begin{aligned} \forall 1 \leq k \leq n \quad s(k) &= \{ \langle \bar{\alpha}, \pi_{\bar{\alpha}^*, \bar{\alpha}}(\bar{v}_k) \rangle \mid \bar{\alpha} \in \text{supp } p_0^* \}, \\ s(n+1) &= \{ \langle \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{v}_1), \dots, \pi_{\bar{\alpha}^*, \bar{\alpha}^0}(\bar{v}_n) \rangle \}. \end{aligned}$$

We get that

$$(p_0^*)_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle} = (p_0^*)_{\langle s \rangle}.$$

We let $\xi_0 = \xi + 1$. Recall the enumeration of S in the construction. There is ρ such that

$$(p_0^* \setminus \{ \langle \text{mc}(p_0^*), (p_0^*)^{\text{mc}} \rangle \})_{\langle s \rangle} \cup \{ \langle \text{mc}(p_0^*), (p_0^*)^{\text{mc}} \rangle \} = (p_0^*)_{\langle s^{\xi_0, \rho} \rangle}.$$

We let $\rho_0 = \rho + 1$. Considering the construction of $T^{p_0^*}$, we see that

$$T_{\langle \bar{v}_1, \dots, \bar{v}_n \rangle}^{p_0^*} \leq T^{\xi_0, \rho_0}$$

hence

$$(p_0^*)_{\langle s \rangle} \leq^* (u_0^{\xi_0, \rho_0})_{\langle s^{\xi_0, \rho_0} \rangle}.$$

We note that

$$\forall 1 \leq k \leq n \quad p_k'' = ((p_0^*)_{\langle s \rangle})_k.$$

Recalling that q was chosen so that

$$q \leq p_{k(p)} \hat{\wedge} \cdots \hat{\wedge} p_1 \hat{\wedge} p_n'' \hat{\wedge} \cdots \hat{\wedge} p_1''$$

we conclude that there is ζ such that $q = q^{\xi_0, \rho_0, \zeta}$ and $D = E^{\xi_0, \rho_0, \zeta}$. That is

$$E^{\xi_0, \rho_0, \zeta} \ni q^{\xi_0, \rho_0, \zeta} \hat{\wedge} r_0 \leq^* q^{\xi_0, \rho_0, \zeta} \hat{\wedge} ((u^{\xi_0, \rho_0, \zeta})_{\langle s^{\xi_0, \rho} \rangle})_0.$$

Note that this is an answer to the question we asked in the construction. Hence, due to elementarity of N , there was such a condition in N . Hence

$$q^{\xi_0, \rho_0, \zeta} \hat{\wedge} ((u^{\xi_0, \rho_0, \zeta})_{\langle s^{\xi_0, \rho} \rangle})_0 \in D \cap N.$$

The last point to note is that

$$q^{\xi_0, \rho_0, \zeta} \hat{\wedge} ((u^{\xi_0, \rho_0, \zeta})_{\langle s^{\xi_0, \rho} \rangle})_0 \geq^* .q^{\xi_0, \rho_0, \zeta} \hat{\wedge} p_0'' \geq^* q \hat{\wedge} r_0 \in G$$

Hence

$$q^{\xi_0, \rho_0, \zeta} \hat{\wedge} ((u^{\xi_0, \rho_0, \zeta})_{\langle s^{\xi_0, \rho} \rangle})_0 \in G.$$

□

Corollary 8.6. $P_{\bar{E}}$ is proper.

9. CARDINALS IN $V^{P_{\bar{E}}}$

Lemma 9.1. κ^+ remains a cardinal in $V^{P_{\bar{E}}}$.

Proof. The proof really has no connection to the specific structure of $P_{\bar{E}}$. It is an exercise in properness.

Let

$$p \Vdash \ulcorner \tilde{f}: \widehat{\kappa} \rightarrow \widehat{\kappa^+} \urcorner.$$

Choose χ large enough so that H_χ contains everything we are interested in. Take $N \prec H_\chi$ such that

- (1) $p, P_{\bar{E}}, \tilde{f} \in N$,
- (2) $|N| = \kappa$,
- (3) $N \supseteq V_\kappa$,
- (4) $N \supseteq N^{<\kappa}$.

By 8.6 there is $q \leq p$ which is $\langle N, P_{\bar{E}} \rangle$ -generic. Let us set

$$\lambda = N \cap \kappa^+,$$

where λ is an ordinal $< \kappa^+$.

Let G be $P_{\bar{E}}$ -generic with $q \in G$. The $\langle N, P_{\bar{E}} \rangle$ -genericity ensures us that for all $\xi < \kappa$ $\tilde{f}(\xi)^{N[G]} \in N$ and $\tilde{f}(\xi)^{N[G]} = \tilde{f}(\xi)^{H_\chi[G]}$. Hence $\text{ran } \tilde{f}^{V[G]} \subseteq \lambda$. That is

$$q \Vdash \ulcorner \tilde{f} \text{ is bounded in } \kappa^+ \urcorner.$$

□

Lemma 9.2. No cardinals $> \kappa$ are collapsed by $P_{\bar{E}}$.

Proof. κ^+ is not collapsed by 9.1. No cardinals $\geq \kappa^{++}$ are collapsed as $P_{\bar{E}}$ satisfies κ^{++} -c.c. □

Lemma 9.3. Let $\xi < \kappa$ and ζ the ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \xi < \kappa^0(\bar{E}_G(\zeta + 1))$. Then $\mathcal{P}(\xi) \cap V[G] = \mathcal{P}(\xi) \cap V[G \upharpoonright \zeta]$.

Proof. Take $p = p_n \hat{\ } \cdots \hat{\ } p_{k+1} \hat{\ } p_k \hat{\ } \cdots \hat{\ } p_0 \in G$ such that $\bar{E}(p_{k+1}) = \bar{E}_G(\zeta)$, $\bar{E}(p_k) = \bar{E}_G(\zeta + 1)$. We know that $V[G] = V[G/p]$. So we work in $P_{\bar{E}}/p$. Set $p^l = p_n \hat{\ } \cdots \hat{\ } p_{k+1}$, $p^h = \{\langle \bar{E}_G(\zeta), \emptyset \rangle\} \hat{\ } p_k \hat{\ } \cdots \hat{\ } p_0$. Then $P_{\bar{E}}/p = P_{\bar{E}} \upharpoonright p^l \times P_{\bar{E}} \upharpoonright p^h$. Note that $\langle P_{\bar{E}}/p^h, \leq^* \rangle$ is $\kappa^0(\bar{E}_G(\zeta + 1))$ -closed. In particular it is ξ^+ -closed.

Let $A \in V[G]$, $A \subseteq \xi$. Choose \tilde{A} , a canonical $P_{\bar{E}}/p$ -name for A . Let $q \in P_{\bar{E}}/p^h$. By induction we construct $\langle q_\tau \mid \tau < \xi \rangle$ satisfying

- (1) $\tau_0 < \tau_1 \implies q_{\tau_1} \leq^* q_{\tau_0}$,
- (2) $q_\tau \Vdash \ulcorner \hat{\tau} \in \tilde{A} \urcorner$.

Choose $q_\xi \leq^* q_\tau$ for all $\tau < \xi$.

By density argument we can construct, \tilde{B} , a $P \upharpoonright p^l$ -name such that $A = \tilde{A}[G] = \tilde{B}[G \upharpoonright p^l]$. □

Corollary 9.4. No cardinals $\leq \kappa$ are collapsed by $P_{\bar{E}}$.

Proof. Let $G \subseteq P_{\bar{E}}$ be generic. Assume $\lambda < \kappa$ is a collapsed cardinal. Let $\mu = |\lambda|^{V[G]}$. We have $\mu < \lambda$, and there is $A \in \mathcal{P}(\mu)^{V[G]}$ which codifies the order type λ . Let ζ be the unique ordinal such that $\kappa^0(\bar{E}_G(\zeta)) \leq \mu < \kappa^0(\bar{E}_G(\zeta + 1))$. By 9.3 $A \in V[G \upharpoonright \zeta]$. Hence λ is collapsed already in $V[G \upharpoonright \zeta]$. However, by 9.2, $P_{\bar{E}_G(\zeta)}$ collapses no cardinals above $\kappa^0(\bar{E}_G(\zeta))$. Contradiction.

So, no cardinal $< \kappa$ is collapsed. As κ is a limit of cardinals which are not collapsed, it is not collapsed. \square

We have just shown

Theorem 9.5. *No cardinals are collapsed in $V^{P_{\bar{E}}}$.*

10. PROPERTIES OF κ IN $V^{P_{\bar{E}}}$

Theorem 10.1. *If $1(\bar{E}) = \kappa^+$ then $V^{P_{\bar{E}}} \models \ulcorner \kappa \text{ is regular} \urcorner$.*

Proof. Let $\lambda < \kappa$, \tilde{f} be such that

$$\Vdash_{P_{\bar{E}}} \ulcorner \tilde{f}: \hat{\lambda} \rightarrow \hat{\kappa} \urcorner.$$

Let

$$D_0 = \{p \mid \exists i \ p \Vdash \ulcorner \tilde{f}(0) = \hat{i} \urcorner\}.$$

As D_0 is a dense open set we can invoke 6.1 to get p^0 , n_0 , $S^0 \subset T^{p^0}$ such that

$$\begin{aligned} \forall k < n_0 \ \forall \langle \nu_1, \dots, \nu_k \rangle \in S^0 \ \exists \xi < 1(\bar{E}) \ \text{Suc}_S(\langle \nu_1, \dots, \nu_k \rangle) \in E_{\text{mc}(p^0)}(\xi), \\ \forall \langle \nu_1, \dots, \nu_{n_0} \rangle \in S^0 \ (p^0)_{\langle \nu_1, \dots, \nu_{n_0} \rangle} \in D_0. \end{aligned}$$

Let us set

$$A'_0 = \{(p^0)_{\langle \nu_1, \dots, \nu_{n_0} \rangle} \mid \langle \nu_1, \dots, \nu_{n_0} \rangle \in S^0\}.$$

A_0 is an anti-chain. By shrinking T^{p^0} as was done in the proof of 7.1 we can make A_0 into a maximal anti-chain below p^0 . As $\lambda < \kappa$ and $\langle P_{\bar{E}}^*, \leq^* \rangle$ is κ -closed we can construct a \leq^* -decreasing sequence

$$p^0 \geq^* p^1 \geq^* \dots \geq^* p^\tau \geq^* \dots \quad \tau < \lambda.$$

and n_τ , $S^{\tau} \subseteq T^{p^\tau}$ such that

$$\begin{aligned} \forall k < n_\tau \ \forall \langle \nu_1, \dots, \nu_k \rangle \in S^\tau \ \exists \xi < 1(\bar{E}) \ \text{Suc}_{S^\tau}(\langle \nu_1, \dots, \nu_k \rangle) \in E_{\text{mc}(p^\tau)}(\xi), \\ \forall \langle \nu_1, \dots, \nu_{n_\tau} \rangle \in S^\tau \ \exists i \ (p^\tau)_{\langle \nu_1, \dots, \nu_{n_\tau} \rangle} \Vdash \ulcorner \tilde{f}(\hat{\tau}) = \hat{i} \urcorner, \end{aligned}$$

and

$$A'_\tau = \{(p^\tau)_{\langle \nu_1, \dots, \nu_{n_\tau} \rangle} \mid \langle \nu_1, \dots, \nu_{n_\tau} \rangle \in S^\tau\}$$

is a maximal anti-chain below p^τ .

Let $p' \leq^* p^\tau$ for all $\tau < \lambda$. We set $S^\tau = \pi_{\text{mc}(p'), \text{mc}(p^\tau)}^{-1}(S^{\tau})$ and p^τ to be p' with $\pi_{\text{mc}(p'), \text{mc}(p^\tau)}^{-1}(T^{p^\tau})$ substituted for $T^{p'}$ and maybe shrunken a bit so that

$$A_\tau = \{(p^\tau)_{\langle \nu_1, \dots, \nu_{n_\tau} \rangle} \mid \langle \nu_1, \dots, \nu_{n_\tau} \rangle \in S^\tau\}$$

is a maximal anti-chain below p^τ .

Let $p \leq^* p^\tau$ for all $\tau < \lambda$ and let \tilde{g} be the following $P_{\bar{E}}$ -name

$$\tilde{g} = \bigcup_{\tau < \lambda} \{ \langle \langle \hat{\tau}, \hat{i} \rangle, (p^\tau)_{\langle \nu_1, \dots, \nu_{n_\tau} \rangle} \rangle \mid A_\tau \ni (p^\tau)_{\langle \nu_1, \dots, \nu_{n_\tau} \rangle} \Vdash \ulcorner \tilde{f}(\tau) = \hat{i} \urcorner \}.$$

Then

$$p \Vdash \ulcorner \tilde{f} = \tilde{g} \urcorner.$$

Let P^* be the following forcing notion:

$$P^* = \{q \leq_R p \mid q \in P_{\bar{E}}\}.$$

By 5.2 $\langle P^*, \leq_R \rangle$ is sub-forcing of $\langle P_{\bar{E}}/p, \leq \rangle$. Hence if G is $P_{\bar{E}}$ -generic then $G^* = G \cap P^*$ is P^* -generic. \tilde{g} is in fact a P^* -name and as can be seen from its definition $\tilde{g}[G] = \tilde{g}[G^*] \in V[G^*]$. So in order to complete the proof it is enough to show that $\Vdash_{P^*} \ulcorner \tilde{g} \text{ is bounded} \urcorner$.

By 5.3 there is $r \in R_{\text{mc}(p)}$ such that $P^* \simeq R_{\text{mc}(p)}/r$. Now we use the following fact about Radin forcing: When the measure sequence is of length κ^+ , κ is regular in the generic extension. Necessarily, $\Vdash_{P^*} \ulcorner \tilde{g} \text{ is bounded} \urcorner$. \square

Definition 10.2. We say that $\tau < l(\bar{E})$ is a repeat point of \bar{E} if $P_{\bar{E}} = P_{\bar{E} \upharpoonright \tau}$.

We note, again, our convention that when $\tau_1 < \tau_2$ we have $P_{\bar{E} \upharpoonright \tau_2} \subseteq P_{\bar{E} \upharpoonright \tau_1}$. So the equality in the above definition is meant to be with this convention. We point out that if τ is a repeat point then $P_{\bar{E} \upharpoonright \tau} \in M$.

Theorem 10.3. *If \bar{E} has a repeat point then $V^{P_{\bar{E}}} \models \ulcorner \kappa \text{ is measurable} \urcorner$.*

Proof. Let τ be a repeat point of \bar{E} . The crucial observation regarding τ being a repeat point is as follows. Let $p = p_{n..0} \in P_{\bar{E}}$. Set a function $f(\bar{\nu}) = p_{\langle \bar{\nu} \rangle n+1..1}$ for all $\langle \bar{\nu} \rangle \in T^{p_0}$. Then $j(f)(\text{mc}(p) \upharpoonright \tau) \in P_{\bar{E} \upharpoonright \tau}$. By our convention it means $j(f)(\text{mc}(p) \upharpoonright \tau) \in P_{\bar{E}}$. Moreover, $j(f)(\text{mc}(p) \upharpoonright \tau) = p!$

We start by showing that for each $p \in P_{\bar{E}}$, $\Vdash_{P_{\bar{E}}} \ulcorner \tilde{A} \subseteq \widehat{\kappa} \urcorner$ there is $p^* \leq^* p$ such that $j(p^*)_{\langle \text{mc}(p^*) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \widehat{\kappa} \in j(\tilde{A}) \urcorner$. Note that $j(p)_{\langle \text{mc}(p) \upharpoonright \tau \rangle} = p \wedge (j(p)_{\langle \text{mc}(p) \upharpoonright \tau \rangle})_0$.

Choose χ large enough so that H_χ contains everything we are interested in. Take $N \prec H_\chi$ such that $p, \tilde{A}, P_{\bar{E}} \in N$, $|N| = \kappa$, $N \supseteq V_\kappa$, $N \supseteq N^{<\kappa}$. By 8.6 there is $p' \leq^* p$ which is $\langle N, P_{\bar{E}} \rangle$ -generic.

Let us fix $\langle \bar{\nu} \rangle \in T^{p'}$. By 7.1 there is $q \leq^* p'_{\langle \bar{\nu} \rangle}$ such that $q \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \tilde{A} \urcorner$. Then, by the N -genericity of p' , there is $q' \in N$, $q' \leq^* p_{\langle \pi_{\text{mc}(p')}, \text{mc}(p) \rangle(\bar{\nu})}$, $q' \Vdash q$, $q' \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \tilde{A} \urcorner$. Hence there is $q^* \leq^* q', p'_{\langle \bar{\nu} \rangle}$ such that $\text{supp } q_0^* = \text{supp } p'_0$, $q^* \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \tilde{A} \urcorner$.

So, for each $\langle \bar{\nu} \rangle \in T^{p'}$ let $q^*(\bar{\nu}) \leq^* p'_{\langle \bar{\nu} \rangle}$ such that $\text{supp } q_0^*(\bar{\nu}) = \text{supp } p'_0$, $q^*(\bar{\nu}) \Vdash_{P_{\bar{E}}} \ulcorner \kappa^0(\bar{\nu}) \in \tilde{A} \urcorner$. Of course this means $j(q^*)(\text{mc}(p') \upharpoonright \tau) \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}) \urcorner$. Let us set $r_{n..0} = j(q^*)(\text{mc}(p') \upharpoonright \tau) \in j(P_{\bar{E}})$. By our convention $r_{n..1} \in P_{\bar{E}}$. Let us set $p^* = r_{n..1}$. We have $j(p^*)_{\langle \text{mc}(p^*) \upharpoonright \tau \rangle} \leq^* j(q^*)(\text{mc}(p') \upharpoonright \tau)$, hence $j(p^*)_{\langle \text{mc}(p^*) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}) \urcorner$ as needed.

Let \tilde{U} be a $P_{\bar{E}}$ -name defined by

$$p \Vdash_{P_{\bar{E}}} \ulcorner \tilde{A} \in \tilde{U} \urcorner \text{ if } \{q \in P_{\bar{E}} \mid j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}) \urcorner\} \text{ is dense below } p.$$

We note that as $\{q \in P_{\bar{E}} \mid j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}) \urcorner\}$ is dense we have

$$p \Vdash_{P_{\bar{E}}} \ulcorner \tilde{A} \notin \tilde{U} \urcorner \text{ if } \{q \in P_{\bar{E}} \mid j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\tilde{A}) \urcorner\} \text{ is dense below } p.$$

We show \tilde{U} is a name for a measure on κ :

- (1) $\Vdash_{P_{\bar{E}}} \ulcorner \emptyset \notin \tilde{U} \urcorner$: Immediate due to $\Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\emptyset) \urcorner$.
- (2) $p \Vdash_{P_{\bar{E}}} \ulcorner \tilde{A} \notin \tilde{U} \urcorner \implies p \Vdash_{P_{\bar{E}}} \ulcorner \kappa \setminus \tilde{A} \in \tilde{U} \urcorner$: So, there is X , a maximal anti-chain below p , such that $q \in X \implies j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \notin j(\tilde{A}) \urcorner$. Then $q \in X$ implies $j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\kappa \setminus \tilde{A}) \urcorner$. This means $p \Vdash_{P_{\bar{E}}} \ulcorner \kappa \setminus \tilde{A} \in \tilde{U} \urcorner$.
- (3) $p \Vdash \ulcorner \tilde{A} \in \tilde{U}, \tilde{A} \subseteq \tilde{B} \urcorner \implies p \Vdash \ulcorner \tilde{B} \in \tilde{U} \urcorner$: Obvious.

(4) $\lambda < \kappa$, $p \Vdash_{P_{\bar{E}}} \ulcorner \forall \xi < \lambda \tilde{A}_\xi \in \tilde{U} \urcorner \implies p \Vdash_{P_{\bar{E}}} \ulcorner \cap_{\xi < \lambda} \tilde{A}_\xi \in \tilde{U} \urcorner$: So, for all $\xi < \lambda$ there is X_ξ , a maximal anti-chain below p , such that $q \in X_\xi \implies j(q)_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}_\xi) \urcorner$. It is enough to show $\{r \in P_{\bar{E}} \mid j(r)_{\langle \text{mc}(r) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \forall \xi < \lambda \kappa \in j(\tilde{A}_\xi) \urcorner\}$ is dense below p .

Let $q = q_{n..0} \leq p$. As $\{r \in P_{\bar{E}} \mid j(r)_{\langle \text{mc}(r) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}_\xi) \urcorner\}$ is dense below p for all $\xi < \lambda$, we can construct $\langle r^\xi \mid \xi \leq \lambda \rangle$, $\langle Y_\xi \mid \xi < \lambda \rangle$ such that:

- $\xi = 0$: $r^0 = q_0$.
- ξ is limit ordinal: $\forall \xi' < \xi \ r^\xi \leq^* r^{\xi'}$.
- $\xi = \xi' + 1$: $r^\xi \leq^* r^{\xi'}$, Y_ξ is a maximal anti-chain below $q_{n..1}$ and for all $s \in Y_\xi \ s \cap j(r^{\xi'})_{\langle \text{mc}(q) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \kappa \in j(\tilde{A}_{\xi'}) \urcorner$.

Then we have $q_{n..1} \cap j(r^\lambda)_{\langle \text{mc}(r^\lambda) \upharpoonright \tau \rangle} \Vdash_{j(P_{\bar{E}})} \ulcorner \forall \xi < \lambda \ \kappa \in j(\tilde{A}_\xi) \urcorner$. □

We note that actually the measure \tilde{U} constructed in the above proof is normal. Moreover, we can construct an extender, lifting $E(0)$, by deciding formulas of the form $\alpha \in j(\tilde{A})$ for $\alpha \in \text{dom } E(0)$.

11. WHAT HAVE WE PROVED?

We can sum everything as follows:

We can control independently two properties of κ in a generic extension. The first is the size of 2^κ which is controlled by $|\bar{E}|$. The second is how ‘big’ we want κ to be which is controlled by $l(\bar{E})$.

12. GENERIC BY ITERATION

Recall that if R is Radin forcing generated from $j:V \rightarrow M$ then there is τ and $G \in V$ such that G is $j_{0,\tau}(R)$ -generic over M_τ .

Our original aim was to find some form of this claim for our forcing. We have a partial result in this direction. Namely, when $l(\bar{E}) = 1$ we have a generic filter in V over an elementary submodel in M_ω .

In this section we assume that $l(\bar{E}) = 1$.

Let us take an iteration of $j = j_{0,1}$

$$\langle \langle M_n \mid n < \omega \rangle, \langle j_{n,m} \mid n \leq m < \omega \rangle \rangle.$$

Choose χ large enough so that everything interesting is in H_χ (i.e. $P_{\bar{E}} \in H_\chi$) and set

$$\begin{aligned} \kappa_n &= j_{0,n}(\kappa), \\ \bar{E}^n &= j_{0,n}(\bar{E}), \\ P^n &= j_{0,n}(P_{\bar{E}}) (= P_{\bar{E}^n}), \\ \chi_n &= j_{0,n}(\chi). \end{aligned}$$

Definition 12.1 (when $l(\bar{E}) = 1$). We call $\langle N, p \rangle$ a k -pair if

- (1) $M_k \models \ulcorner N \prec H_{\chi_k}^{M_k} \urcorner$,
- (2) $M_k \models \ulcorner |N| = \kappa_k \urcorner$,
- (3) $M_k \models \ulcorner N \supseteq V_{\kappa_k}^{M_k} \urcorner$,
- (4) $M_k \models \ulcorner N \supseteq N^{< \kappa_k} \urcorner$,

- (5) $p \in P^{k+1} \cap j_{k,k+1}(N)$,
- (6) If $D \in N$, D is dense open in P^k then there are $n, S \leq T^p$ such that

$$\forall \langle \nu_1, \dots, \nu_n \rangle \in S \ (p)_{\langle \nu_1, \dots, \nu_n \rangle} \in j_{k,k+1}(D).$$

Claim 12.2. Let $N \in M_k$, $k < \omega$ and $q \in j_{k,k+1}(N) \cap P^{k+1}$ such that M_k satisfies:

- (1) $N \prec H_{\chi^k}^{M_k}$,
- (2) $|N| = \kappa_k$,
- (3) $N \supseteq V_{\kappa_k}^{M_k}$,
- (4) $N \supseteq N^{< \kappa_k}$,
- (5) $P^k \in N$.

Then there is $p \in j_{k,k+1}(N) \cap P^{k+1}$ such that

- (1) $p \leq^* q$,
- (2) $\langle N, p \rangle$ is a k -pair.

Proof. Let $\langle D_\xi^k \mid \xi < \kappa_k \rangle$ be an enumeration, in M_k , of all the dense open subsets of P^k which are in N . Set $N_{k+1} = j_{k,k+1}(N)$. As we have

$$\begin{aligned} \langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle &\subseteq N_{k+1}, \\ \langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle &\in M_{k+1}, \\ M_{k+1} &\models \ulcorner N_{k+1} \supseteq N_{k+1}^{< \kappa_{k+1}} \urcorner, \end{aligned}$$

we get that

$$\langle j_{k,k+1}(D_\xi^k) \mid \xi < \kappa_k \rangle \in N_{k+1}.$$

Starting with q we construct in N_{k+1} a \leq^* -decreasing sequence $\langle p_\xi \mid \xi < \kappa_k \rangle$ using 6.1. Note that we have no problem at the limit stages as $N_{k+1} \models \ulcorner \langle P^{k+1}, \leq^* \rangle$ is κ_{k+1} -closed \urcorner . Choose now $p \in P^{k+1} \cap N_{k+1}$ such that $\forall \xi < \kappa_k \ p \leq^* p_\xi$. We get that $\langle N, p \rangle$ is a k -pair. \square

Definition 12.3. We call $\langle \vec{N}, \vec{p} \rangle$ a P^ω -generic approximation sequence if

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that for all $k_0 \leq k < \omega$

- (1) $M_k \models \ulcorner N_k \prec H_{\chi^k}^{M_k} \urcorner$,
- (2) $M_k \models \ulcorner |N_k| = \kappa_k \urcorner$,
- (3) $M_k \models \ulcorner N_k \supseteq V_{\kappa_k}^{M_k} \urcorner$,
- (4) $M_k \models \ulcorner N_k \supseteq N^{< \kappa_k} \urcorner$,
- (5) $P^k \in N_k$,
- (6) $\langle N_k, p^{k+1} \rangle$ is a k -pair,
- (7) $j_{k_1, k_2}(N_{k_1}) = N_{k_2}$,
- (8) $p^{k+2} \leq^* (j_{k+1, k+2}(p^{k+1}))_{\langle \text{mc}(p^{k+1}) \rangle}$.

Definition 12.4. Let $\langle \vec{N}, \vec{r} \rangle$ be a P^ω -generic approximating sequence. Then

$$G(\langle \vec{N}, \vec{r} \rangle) = \{p \in P^\omega \mid \exists k \ j_{k, \omega}(r^k) \leq^* p\}.$$

Claim 12.5. Let $k_0 < \omega$, $q \in P^{k_0} \cap N_{k_0}$ and assume for all $k_0 \leq k < \omega$

- (1) $P^k \in N_k$,
- (2) $j_{k, k+1}(N_k) = N_{k+1}$,
- (3) $M_k \models \ulcorner N_k \prec H_{\chi^k}^{M_k} \urcorner$,

- (4) $M_k \models \ulcorner |N_k| = \kappa_k \urcorner$,
- (5) $M_k \models \ulcorner N_k \supseteq V_{\kappa_k}^{M_k} \urcorner$,
- (6) $M_k \models \ulcorner N_k \supseteq N_k^{<\kappa_k} \urcorner$.

Then there is a P^ω -generic approximating sequence

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that $p^{k_0+1} \leq^* j_{k_0, k_0+1}(q)$.

Proof. We construct the p^{k+1} by induction. We set $p^{k_0} = q$.

Assume that we have constructed $\langle p^{k'} \mid k_0 \leq k' \leq k \rangle$. We set $q^{k+1} = j_{k, k+1}(p^k)_{(\text{mc}(p^k))}$. Invoke 12.2 to get $p^{k+1} \leq^* q^{k+1}$ such that $\langle N_k, p^{k+1} \rangle$ is a k -pair.

When the induction terminates we have $\langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$ as required. \square

Claim 12.6. *Let $k_0 < \omega$ and assume*

- (1) $M_{k_0} \models \ulcorner N_{k_0} \prec H_{\chi_{k_0}}^{M_{k_0}} \urcorner$,
- (2) $M_{k_0} \models \ulcorner |N_{k_0}| = \kappa_{k_0} \urcorner$,
- (3) $M_{k_0} \models \ulcorner N_{k_0} \supseteq V_{\kappa_{k_0}}^{M_{k_0}} \urcorner$,
- (4) $M_{k_0} \models \ulcorner N_{k_0} \supseteq N_{k_0}^{<\kappa_{k_0}} \urcorner$,
- (5) $P^{k_0} \in N_{k_0}$,
- (6) $q \in P^{k_0} \cap N_{k_0}$.

Then there is a P^ω -generic approximating sequence

$$\langle \vec{N}, \vec{p} \rangle = \langle \langle N_k, p^{k+1} \rangle \mid k_0 \leq k < \omega \rangle$$

such that $p^{k_0+1} \leq^* j_{k_0, k_0+1}(q)$.

Proof. We set $N_k = j_{k_0, k}(N_{k_0})$ for all $k_0 < k < \omega$ and then we invoke 12.5. \square

Theorem 12.7. *Assume*

- (1) $M_\omega \models \ulcorner N \prec H_{\chi_\omega}^{M_\omega} \urcorner$,
- (2) $M_\omega \models \ulcorner |N| = \kappa_\omega \urcorner$,
- (3) $M_\omega \models \ulcorner N \supseteq V_{\kappa_\omega}^{M_\omega} \urcorner$,
- (4) $M_\omega \models \ulcorner N \supseteq N^{<\kappa_\omega} \urcorner$,
- (5) $P^\omega \in N$,
- (6) $q \in P^\omega \cap N$.

Then there is, in V , a filter $G \subseteq P^\omega$ such that

- (1) $q \in G$,
- (2) $\forall D \in N \ D \text{ is dense open in } P^\omega \implies G \cap D \cap N \neq \emptyset$.

Proof. We find, $\langle \vec{N}, \vec{p} \rangle$, a P^ω -generic approximating sequence. $G(\langle \vec{N}, \vec{p} \rangle)$ is the required filter.

Find k_0 and N_{k_0}, q^{k_0} such that

$$\begin{aligned} P^{k_0} &\in N_{k_0}, \\ j_{k_0, \omega}(N_{k_0}) &= N, \\ j_{k_0, \omega}(q^{k_0}) &= q. \end{aligned}$$

Invoke 12.6 to get from N^{k_0}, q^{k_0} a generic approximating sequence $\langle \vec{N}, \vec{p} \rangle$.

Let $D \in N$ be dense open in P^ω .

Find $k \geq k_0$ and D^k such that $j_{k,\omega}(D^k) = D$. As usual we set $D^{k+l} = j_{k,k+l}(D^k)$. By construction there is n such that

$$\forall \langle \nu_1, \dots, \nu_n \rangle \in T^{p^{k+1}} \quad (p^{k+1})_{\langle \nu_1, \dots, \nu_n \rangle} \in D^{k+1} \cap N_{k+1},$$

which means that

$$j_{k+1,k+1+n}(p^{k+1})_{\langle \text{mc}(p^{k+1}), \dots, j_{k+1,k+n}(\text{mc}(p^{k+1})) \rangle} \in D^{k+1+n} \cap N_{k+1+n}.$$

Hence

$$(12.7.1) \quad j_{k+1,\omega}(p^{k+1})_{\langle \text{mc}(p^{k+1}), \dots, j_{k+1,k+n}(\text{mc}(p^{k+1})) \rangle} \in D \cap N.$$

As $\langle \vec{N}, \vec{p} \rangle$ is a P^ω -generic approximation sequence it satisfies

$$p^{k+1+n} \leq^* j_{k+1,k+1+n}(p^{k+1})_{\langle \text{mc}(p^{k+1}), \dots, j_{k+1,k+1+n}(\text{mc}(p^{k+1})) \rangle}.$$

Hence

$$j_{k+1+n,\omega}(p^{k+1+n}) \leq^* j_{k+1,\omega}(p^{k+1})_{\langle \text{mc}(p^{k+1}), \dots, j_{k+1,k+n}(\text{mc}(p^{k+1})) \rangle},$$

giving us that

$$(12.7.2) \quad j_{k+1,\omega}(p^{k+1})_{\langle \text{mc}(p^{k+1}), \dots, j_{k+1,k+n}(\text{mc}(p^{k+1})) \rangle} \in G(\langle \vec{N}, \vec{p} \rangle),$$

so $G(\langle \vec{N}, \vec{p} \rangle) \cap D \cap N \neq \emptyset$ by 12.7.1 and 12.7.2. \square

13. CONCLUDING REMARKS

- (1) A definition of repeat point that depends only on the extender sequence and is equivalent to the one we gave (which mentions $P_{\bar{E}}$) will probably be useful.
- (2) It is not completely clear what $l(\bar{E})$ should be in order to make sure that \bar{E} has a repeat point.
- (3) A finer analysis in the case of measurability and stronger properties is needed. For example, extending the elementary embedding to the generic extension, and not just constructing a normal ultrafilter.
- (4) We do not know how to get a generic by iteration when $l(\bar{E}) > 1$.
- (5) Making this forcing more ‘precise’ by adding ‘gentle’ collapses so we get a prescribed behaviour on *all* cardinals below κ in the generic extension is in preparation.

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