POSSIBLE VALUES FOR $2^{\aleph_n}$ and $2^{\aleph_\omega}$

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Abstract. From GCH and $P^m(\kappa)$-measurable $(1 < m < \omega)$, we construct a model satisfying $2^{\aleph_n} = \aleph_{a(n)}$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$ for monotonic $a : \omega \to \omega$ satisfying $a(n) > n$.

1. Introduction

Determining the possibilities for the function $\kappa \mapsto 2^\kappa$ is still an open question. History of the work done on this problem can be found in [G-Ma], [C], [Sh2].

In this work, which is a generalization of [G-Ma], we prove the following: Given $1 < m < \omega$ and a monotonic function $a : \omega \to \omega$ and assuming that κ is a $P^m(\kappa)$-hypermeasurable cardinal, we can build a generic extension in which $2^{\aleph_n} = \aleph_{a(n)}$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$. For $m > 2$ this assumption is needed by [G-Mi]. For $m = 2$ using [G1] one can reduce the assumption to $o(\kappa) = \kappa^{++}$ which is the best possible.

We tried to make the paper as self contained as possible, assuming that forcing technology and large cardinals techniques are known.

The structure of this work is as follows. In section 2 we give definitions and notations which are either well known or are from [G-Ma]. In section 3 we'll extend V in order to have generics we need. In section 4 we'll define the forcing notion which actually does the job.

This paper is a somewhat generalized version of the 2nd author M.Sc. thesis done at Tel-Aviv university under the direction of M. Gitik. The 2nd author would like to thank again M. Gitik for his help with this work.

2. Extender Preliminaries

Let κ be measurable cardinal and assume $\langle \mathbf{A}, < \rangle$ is κ^+-directed partial order.

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Definition 2.1: A sequence $\langle\langle U_\alpha \mid \alpha\in \mathbf{A}\rangle, \langle \pi_{\alpha,\beta} \mid \alpha,\beta\in \mathbf{A} \;\; \alpha\geq\beta\rangle\rangle$ of κ-complete ultrafilters over sets of cardinality κ is called Rudin-Kiesler directed commutative if

(1) $\pi^{-1}_{\alpha,\beta}{}''(X)\in U_\alpha \Leftrightarrow X\in U_\beta$

(2) $\forall\alpha\in \boldsymbol{A}\;\; \pi_{\alpha,\alpha}=\mathrm{id}$

(3) $\forall\alpha,\beta,\gamma\in \boldsymbol{A}\;\; \alpha>\beta>\gamma$ there's $X\in U_\alpha$ such that

$$\forall\nu\in X\;\; \pi_{\alpha,\gamma}(\nu)=\pi_{\beta,\gamma}(\pi_{\alpha,\beta}(\nu))$$

(4) $\forall\alpha,\beta,\gamma\in \boldsymbol{A}\;\; \beta\neq\gamma,\; \alpha>\beta,\gamma$ there's $X\in U_\alpha$ such that

$$\forall\nu\in X\;\; \pi_{\alpha,\gamma}(\nu)\neq\pi_{\alpha,\beta}(\nu)\;\;\Diamond$$

Definition 2.2: An ultrafilter U will be called P-point if for any $f:\kappa\to\kappa$ there's $X\in U$ such that $\forall\nu<\kappa\; |X\cap f^{-1}{}''(\nu)|<\kappa$.

Definition 2.3: We call $\langle\langle U_\alpha \mid \alpha\in \mathbf{A}\rangle, \langle \pi_{\alpha,\beta} \mid \alpha,\beta\in \mathbf{A} \;\; \alpha\geq\beta\rangle\rangle$ a nice system of length $|\boldsymbol{A}|$ if

(1) $\boldsymbol{A}$ has a minimal element, 0.

(2) $\langle\langle U_\alpha \mid \alpha\in \mathbf{A}\rangle, \langle \pi_{\alpha,\beta} \mid \alpha,\beta\in \mathbf{A} \;\; \alpha\geq\beta\rangle\rangle$ is Rudin-Kiesler directed commutative.

(3) U_0 is a normal measure over κ.

(4) $\forall\alpha\in \boldsymbol{A}\;\; U_\alpha$ is A P-point ultra-filter over κ.

(5) $\forall\alpha\geq\beta\;\; \forall\nu<\kappa\;\; \pi_{\alpha,0}(\nu)=\pi_{\beta,0}(\pi_{\alpha,\beta}(\nu))$

(6) $\forall\alpha,\beta\in \boldsymbol{A}\;\; \forall\nu<\kappa\;\; \pi_{\alpha,0}(\nu)=\pi_{\beta,0}(\nu)$.$\Diamond$

Definition 2.4: We'll write ν^0 instead of $\pi_{\alpha,0}(\nu)$. Note that it's independent of α.

Definition 2.5: $\langle\nu_0,\ldots,\nu_k\rangle\in[\kappa]^{<\omega}$ will be called ^{0} -increasing if $\langle\nu^0_0,\ldots,\nu^0_k\rangle\in[\kappa]^{<\omega}$

Definition 2.6: Let $s\in[\kappa]^{<\omega}$ be ^{0} -increasing. We'll say ν is permitted to s if $\nu^0>\max s^0$.

Definition 2.7: Suppose $\forall\xi<\kappa\;\; A_\xi\subseteq\kappa$ then $\underset{\xi<\kappa}{\Delta^0} A_\xi=\{\zeta<\kappa \mid \forall\xi<\zeta^0\;\; \zeta\in A_\xi\}$.

Note 2.8: Δ^0 is a kind of diagonal intersection and if $\forall\xi<\kappa\;\; A_\xi\in U_\alpha$ then $\underset{\xi<\kappa}{\Delta^0} A_\xi\in U_\alpha$.

Notation 2.9:

$$\mathrm{Col}(\alpha,\beta)=\{f \mid \mathrm{dom} f\subset\alpha, \mathrm{ran} f\subset\beta, |f|<\alpha\}$$

$$\mathrm{Col}(\alpha,<\beta)=\{f \mid \mathrm{dom} f\subset\beta\times\alpha, \mathrm{ran} f\subset\beta, f(\xi,\zeta)<\xi, |f|<\alpha\}$$
$$\mathrm{C}(\alpha,\beta)=\{f \mid \mathrm{dom} f\subset\beta\times\alpha, \mathrm{ran} f\subset 2, |f|<\alpha\}$$

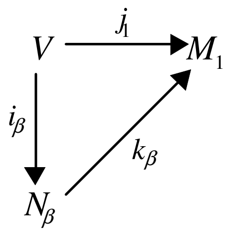
Definition 2.10: Let $\rho: P_1 \to P_2$ be an embedding of forcing notions and $G_1 \subseteq P_1$ filter. Then

$$<\rho'' G_1>=\{p_2\in P_2 \mid \exists p_1\in G_1\, \rho(p_1)\le p_2\}$$

Fix $1<m<\omega$ and assume we have an extender $\mathbf{E}=\langle E_a \mid a\in[\kappa^{+m}]^{<\omega}\rangle$ and its' natural embedding $j_1: V\to M_1\cong \mathrm{Ult}(V,\boldsymbol{E})$. Moreover assume $\kappa^{+m}=(\kappa^{+m})_{M_1}$ and $M_1 \models$"$2^\kappa\le\kappa^{+m}$". We iterate j_1 to get $j_{1,2}: M_1\to M_2$ and we set $j_2=j_{1,2}\circ j_1$, $\kappa_n=j_n(\kappa)$.

Claim 2.11: There's a nice system $\boldsymbol{U}=\langle\langle U_\alpha \mid \alpha\in \boldsymbol{A}\rangle,\langle\pi_{\alpha,\beta}\mid \alpha,\beta\in\boldsymbol{A}\ \ \alpha\ge\beta\rangle\rangle$ on κ where $\boldsymbol{A}\subseteq\kappa^{+m}$, $|\boldsymbol{A}|=\kappa^{+m}$ with a minimal element κ and $M_1=\mathrm{Ult}(V,\boldsymbol{U})$.

Proof: As $M_1\models$"$2^\kappa\le\kappa^{+m}$" we have that $X=\{\alpha<\kappa \mid 2^\alpha\le\alpha^{+m}\ \ \alpha \text{ inaccesible}\}$ is $E_{<\kappa>}$-large set. Build by induction a function $T:\kappa \xrightarrow[onto]{1-1}[\kappa]^{<\kappa}$ such that $\forall\alpha\in X$ $T:\alpha^{+m}\xrightarrow[onto]{1-1}[\alpha^{+m}]^{\le\alpha}$ and $T(\alpha)=\langle\alpha\rangle$. We get that $j_1(T): j_1(\kappa)\xrightarrow[onto]{1-1}[j_1(\kappa)]^{<j_1(\kappa)}$ and $j_1(T)(\kappa)=\langle\kappa\rangle$. For our purpose the restriction of $j_1(T)$ to κ^{+m} is the important part. We have $j_1(T)\restriction\kappa^{+m}:\kappa^{+m}\xrightarrow[onto]{1-1}[\kappa^{+m}]^{\le\kappa}$. We define a partial order on $\kappa^{+m}-\kappa$ by $\alpha\le\beta \iff j_1(T)(\alpha)\subseteq j_1(T)(\beta)$. Clearly this partial order is κ^+-directed. We'll take $\boldsymbol{A}=\{\alpha<\kappa^{+m} \mid \min j_1(T)(\alpha)=\kappa\}$ with the same partial ordering. On $\boldsymbol{A}$ the partial order $\le$ has κ as a minimal element. For each $\alpha\in\boldsymbol{A}$ define $X\in U_\alpha \iff \alpha\in j_1(X)$. The definition of $\pi_{\beta,\alpha}$ for $\beta>\alpha$ will be done in 2 steps. First we define $\pi_{\beta,\kappa}$ and then $\pi_{\beta,\alpha}$ for $\beta>\alpha>\kappa$. Let $\beta\in\boldsymbol{A}$. Set $\pi_{\beta,\kappa}(\nu)=\min T(\nu)$. Note that this definition isn't dependent on β. We get $j_1(\pi_{\beta,\kappa})(\beta)=\min j_1(T)(\beta)=\kappa$ as needed. Let $\kappa<\alpha<\beta$ and set $b=j_1(T)(\beta)$. Hence



$i_\beta : V \to N_\beta \cong \mathrm{Ult}(V, U_\beta)$

$k_\beta([f]_{U_\beta}) = j_1(f)(\beta)$

$k_\beta(i_\beta(T)([\mathrm{id}]_{U_\beta})) = b$. As $[\mathrm{id}]_{U_\kappa} = \kappa$ we get that $k_\beta(\kappa) \le k_\beta([\pi_{\beta,\kappa}]) = \kappa$ hence $\mathrm{crit}(k_\beta) > \kappa$. As $|b| \le \kappa$ we have that $k_\beta''(i_\beta(T)([\mathrm{id}]_{U_\beta})) = b$. We set $i_\beta(T)([\mathrm{id}]_{U_\beta}) = b'$ and we know that $|b'| \le \kappa$. Setting $a = j_1(T)(\alpha)$ we know that $a \subseteq b$. Hence there's $a' \subseteq b'$ such that $k_\beta''(a') = a$. As $|a'| \le \kappa$ we have that $a' \in N_\beta$ giving us $k_\beta(a') = a$. By setting $[f]_{U_\beta} = a'$ we get that $j_1(f)(\beta) = a = j_1(T)(\alpha)$. Hence $j_1(T^{-1} \circ f)(\beta) = \alpha$.

We note that

$$j_1(\pi_{\alpha,\kappa} \circ T^{-1} \circ f)(\beta) = \kappa = j_1(\pi_{\beta,\kappa})(\beta)$$

hence $X = \{\nu < \kappa \mid \pi_{\alpha,\kappa} \circ T^{-1} \circ f(\nu) = \pi_{\beta,\kappa}(\nu)\} \in U_\beta$. Thus we define

$$\pi_{\beta,\alpha}(\nu) = \begin{cases} (T^{-1} \circ f)(\nu) & \nu \in X \\ \nu & \nu \notin X \end{cases}$$

This gives us $\forall \nu < \kappa \;\; \pi_{\beta,\kappa}(\nu) = (\pi_{\alpha,\kappa} \circ \pi_{\beta,\alpha})(\nu)$ for $\beta > \alpha$. The last thing to show is that $M_1 \cong \mathrm{Ult}(V, \boldsymbol{U})$. Let $x \in M_1$. Thus $x = j_1(f)(a)$. Without loss of generality $\min(a) = \kappa$, so we can pick $\alpha \in \boldsymbol{A}$ such that $j_1(T)(\alpha) = a$. Hence $x = j_1(f \circ T)(\alpha)$.◊

Lemma 2.12: $j_1''\boldsymbol{A}$ is dense in $j_1(\boldsymbol{A})$.

Proof: Let $\delta \in j_1(\boldsymbol{A})$. Then $\delta = j_1(f)(\alpha)$ for $f : \kappa \to \boldsymbol{A}$. As $\boldsymbol{A}$ is κ^+-directed there's $\gamma \in \boldsymbol{A}$ such that $\forall \xi < \kappa \;\; \gamma \ge f(\xi)$. So $\delta = j_1(f)(\alpha) \le j_1(\gamma)$.◊

Proposition 2.13: $j_2''(\boldsymbol{A})$ is dense in $j_2(\boldsymbol{A})$

Proof: From elementarity we get that $j_{1,2}''(j_1(\boldsymbol{A}))$ is dense in $j_{1,2}(j_1(\boldsymbol{A})) = j_2(\boldsymbol{A})$ so from previous lemma $j_{1,2}''(j_1''(\boldsymbol{A})) = j_2''(\boldsymbol{A})$ is dense in $j_2(\boldsymbol{A})$.◊

Claim 2.14: $M_2 = \{j_2(f)(\alpha, j_1(\alpha)) \mid f \in V, \alpha \in \boldsymbol{A}\}$

Proof: Let $x \in M_2$. Then there are $h \in M_1, \delta \in j_1(\boldsymbol{A})$ such that $x = j_{1,2}(h)(\delta)$. Due to denseness of $j_1''\boldsymbol{A}$ in $j_1(\boldsymbol{A})$ we can assume that $x = j_{1,2}(h)(j_1(\gamma))$ for $\gamma \in \boldsymbol{A}$. Now there are $g \in V, \beta \in \boldsymbol{A}$ such that $h = j_1(g)(\beta)$. So $x = j_{1,2}(j_1(g)(\beta))(j_1(\gamma)) = j_2(g)(j_{1,2}(\beta))(j_1(\gamma)) = j_2(g)(\beta)(j_1(\gamma))$. The last equality is because

$\beta < \kappa^{+m} < j_1(\kappa) = \kappa_1 = \mathrm{crit}(j_{1,2})$. Now take $\alpha \geq \beta, \gamma$ and define $f(\xi, \zeta) = g(\pi_{\alpha,\beta}(\xi), \pi_{\alpha,\gamma}(\zeta))$. We'll get $x = j_2(f)(\alpha, j_1(\alpha))$.◊

Definition 2.15: $X \in U_\alpha^2 \Leftrightarrow \{\nu_0 \mid \{\nu_1 \mid \langle \nu_0, \nu_1 \rangle \in X\} \in U_\alpha\} \in U_\alpha$

Note the above is equivalent to $X \in U_\alpha^2 \Leftrightarrow \langle \alpha, j_1(\alpha) \rangle \in j_2(X)$.

Proposition 2.16: For any $\alpha \in \boldsymbol{A}$ $\{\langle \nu_0, \nu_1 \rangle \in [\kappa]^2 \mid \nu_0 < \nu_1^0\} \in U_\alpha^2$

Proof: This reflects the fact that $M_2 \models$"$\alpha < \kappa_1$".◊

Proposition 2.17: Let $\alpha \in \boldsymbol{A}$, $X \in U_\alpha$ and $f: X \to \kappa$ such that $\forall \nu \in X$ $f(\nu) < \nu^0$. Then there's $\xi < \kappa$ and $X \supseteq Y \in U_\alpha$ such that $\forall \nu \in Y$ $f(\nu) = \xi$.

Proof: We get that $j_1(f)(\alpha) < \kappa$. Let $\xi = j_1(f)(\alpha)$. As $\xi < \kappa$ we have $\xi = j_1(c_\xi)(\alpha)$ (where c_ξ is a constant function with value ξ). So $j_1(f)(\alpha) = j_1(c_\xi)(\alpha)$.◊

The following claim is a typical usage of the previous one. Several variations of it are used later.

Claim 2.18: Let $\alpha \in \boldsymbol{A}$, $X \in U_\alpha$ and $F: X \to \mathrm{Col}(\mu, \kappa) \times \mathrm{C}(\mu, \kappa)$ such that $\forall \nu \in X$ $F(\nu) \in \mathrm{Col}(\mu, \nu^0) \times \mathrm{C}(\mu, \nu^0)$. Then there's $f \in \mathrm{Col}(\mu, \kappa) \times \mathrm{C}(\mu, \kappa)$ and $X \supseteq Y \in U_\alpha$ such that $\forall \nu \in Y$ $F(\nu) = f$.

Proof: Take enumeration $\left\langle f_\xi \middle| \xi < \kappa \right\rangle$ of $\mathrm{Col}(\mu, \kappa) \times \mathrm{C}(\mu, \kappa)$ satisfying $\xi < \nu^0 \Leftrightarrow f_\xi \in \mathrm{Col}(\mu, \nu^0) \times \mathrm{C}(\mu, \nu^0)$. Defining now $f(\zeta) = \xi \Leftrightarrow F(\zeta) = f_\xi$ yields a function on which the previous proposition works, giving $X \supseteq Y \in U_\alpha$ and ξ such that $\forall \zeta \in Y$ $f(\zeta) = \xi$ and so $\forall \zeta \in Y$ $F(\zeta) = f_\xi$.◊

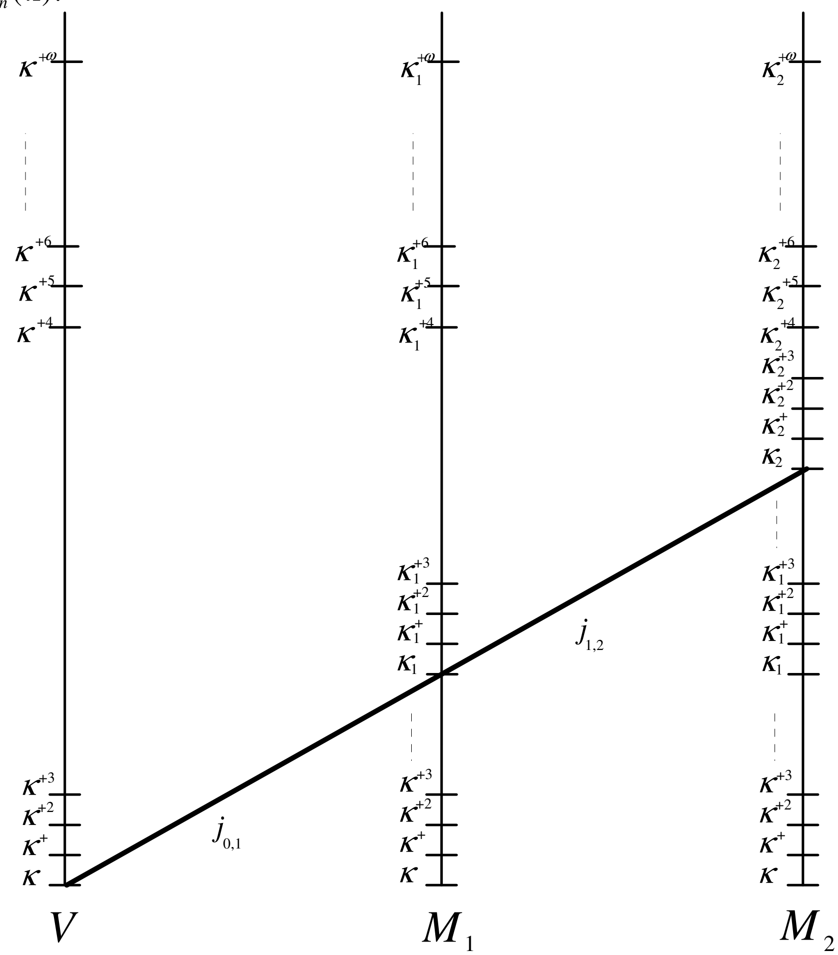
3. Preparation Forcing

We start from a universe V satisfying GCH which has an extender $\boldsymbol{E} = \{E_a \mid a \in [\kappa^{+m}]^{<\omega}\}$ $(1 < m < \omega)$ which catches V up to $V_{\kappa+m}$. That is we have $j_{0,1}: V \to \mathrm{Ult}(V, \boldsymbol{E}) \cong M_1 \supseteq V_{\kappa+m}$ (i.e. κ is a κ^{+m}-strong cardinal).

Occasionally we'll write j_k when we mean $j_{0,k}$ in the sequel.

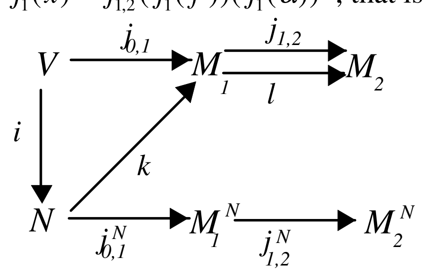
Our assumptions are enough for invoking 2.11 so we drive the nice system $\boldsymbol{U}$ from the extender $\boldsymbol{E}$.

We'll iterate $j_{0,1}$ and have $V \xrightarrow{j_{0,1}} M_1 \xrightarrow{j_{1,2}} M_2$ and set $j_{0,2} = j_{1,2} \circ j_{0,1}$, $\kappa_n = j_n(\kappa)$.



Cardinal Structure for $m = 3$

Using $\boldsymbol{U}$ we define $l: M_1 \to M_2$ as $l(j_1(f)(\alpha)) = j_2(f)(j_1(\alpha))$. Note that $l = j_1 | M_1$. (That's because for any $x \in M_1$ $x = j_1(f)(\alpha)$ hence by elementarity $M_1 \models$ "$j_1(x) = j_{1,2}(j_1(f))(j_1(\alpha))$", that is $j_1(x) = j_2(f)(j_1(\alpha)) = l(x)$.)



$i: V \to N \cong \mathrm{Ult}(V, U_\kappa)$

$k([f]_{U_\kappa}) = j_1(f)(\kappa)$

M_1^N is the model generated in N using the extender $i(E)$. The other models and elementary embeddings with the N superscript are created analogously. We set $\kappa_1^N = j_{0,1}^N(i(\kappa))$.

Set the following:

$$P_0 = (\prod_{m<n<\omega} \mathrm{Col}(\kappa^{+n}, < \kappa_1))_{M_2} = (\prod_{m<n<\omega} \mathrm{Col}(\kappa^{+n}, < \kappa_1))_{M_1}$$
$$P_1 = (\prod_{m<n<\omega} \mathrm{C}(\kappa^{+n}, \kappa_1^{+\omega}))_{M_2} = (\prod_{m<n<\omega} \mathrm{C}(\kappa^{+n}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$$
$$P_2 = (\mathrm{C}(\kappa^{+m}, \kappa_1^{+\omega}))_{M_2} = (\mathrm{C}(\kappa^{+m}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$$
$$P_3 = (\mathrm{C}(\kappa^{+(m-1)}, \kappa_1^{+\omega}))_{M_2} = (\mathrm{C}(\kappa^{+(m-1)}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$P_m = (\mathrm{C}(\kappa^{++}, \kappa_1^{+\omega}))_{M_2} = (\mathrm{C}(\kappa^{++}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$$
$$P_{m+1} = (\mathrm{C}(\kappa^{+}, \kappa_1^{+\omega}))_{M_2} = (\mathrm{C}(\kappa^{+}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$$
$$Q = (\prod_{1\le n<\omega} \mathrm{Col}(\kappa_1, \kappa_1^{+}))_{M_2} = (\prod_{1\le n<\omega} \mathrm{Col}(\kappa_1, \kappa_1^{+}))_{M_1}$$

(The equalities above hold since $M_1 \models$ "$M_2^{\kappa_1} \subseteq M_2$".)

$$R_0 = (\prod_{m<n<\omega} \mathrm{Col}(\kappa_1^{+n}, < \kappa_2))_{M_3} = (\prod_{m<n<\omega} \mathrm{Col}(\kappa_1^{+n}, < \kappa_2))_{M_2} = l(P_0)$$
$$R_1 = (\prod_{m<n<\omega} \mathrm{C}(\kappa_1^{+n}, \kappa_2^{+\omega}))_{M_3} = (\prod_{m<n<\omega} \mathrm{C}(\kappa_1^{+n}, (\kappa_2^{+\omega})_{M_3}))_{M_2} = l(P_1)$$
$$R_2 = (\mathrm{C}(\kappa_1^{+m}, \kappa_2^{+\omega}))_{M_3} = (\mathrm{C}(\kappa_1^{+m}, (\kappa_2^{+\omega})_{M_3}))_{M_2} = l(P_2)$$
$$R_3 = (\mathrm{C}(\kappa_1^{+(m-1)}, \kappa_2^{+\omega}))_{M_3} = (\mathrm{C}(\kappa_1^{+(m-1)}, (\kappa_2^{+\omega})_{M_3}))_{M_2} = l(P_3)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$R_m = (\mathrm{C}(\kappa_1^{++}, \kappa_2^{+\omega}))_{M_3} = (\mathrm{C}(\kappa_1^{++}, (\kappa_2^{+\omega})_{M_3}))_{M_2} = l(P_m)$$
$$R_{m+1} = (\mathrm{C}(\kappa_1^{+}, \kappa_2^{+\omega}))_{M_3} = (\mathrm{C}(\kappa_1^{+}, (\kappa_2^{+\omega})_{M_3}))_{M_2} = (\mathrm{C}(\kappa_1^{+}, (\kappa_2^{+\omega})_{M_3}))_{M_1} = l(P_{m+1})$$

We want to extend V so we would have a $P_0 \times \ldots \times P_{m+1} \times Q \times R_0 \times \ldots \times R_{m+1}$-generic filter over M_2. We would force with the following in order to get this filter:

$$\overline{P}_1 = \prod_{1\le n\le m} \mathrm{C}(\kappa^{+}, \kappa^{+n}) \times \prod_{m<n<\omega} \mathrm{C}(\kappa^{+}, \kappa^{+m})$$
$$\overline{P}_2 = \mathrm{C}(\kappa^{+m}, (\kappa_1^{+\omega})_{M_2}) \cong \mathrm{C}(\kappa^{+m}, \kappa^{+m})$$
$$\overline{P}_3 = \mathrm{C}(\kappa^{+(m-1)}, (\kappa_1^{+\omega})_{M_2}) \cong \mathrm{C}(\kappa^{+(m-1)}, \kappa^{+m})$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\overline{P}_m = \mathrm{C}(\kappa^{++}, (\kappa_1^{+\omega})_{M_2}) \cong \mathrm{C}(\kappa^{++}, \kappa^{+m})$$

$$\overline{P}_{m+1} = \mathrm{C}(\kappa^+, (\kappa_1^{+\omega})_{M_2}) \cong \mathrm{C}(\kappa^+, \kappa^{+m})$$

$$\overline{Q}_n = \mathrm{C}(\kappa^+, \kappa^+)$$

$\overline{P} \times \overline{Q} = \overline{P}_2 \times \cdots \times \overline{P}_{m+1} \times \overline{P}_1 \times \prod_{1 \le n < \omega} \overline{Q}_n$ (We hope the rather strange indexing we chose will be clear from the proof). Let $\bar{I} \times \bar{J} = \bar{I}_2 \times \ldots \times \bar{I}_{m+1} \times \bar{I}_1 \times \bar{I}_{m+2} \times \prod_{1 \le n < \omega} \bar{J}_n$ be $\overline{P} \times \overline{Q}$-generic over V.

Lemma 3.1: A λ-closed separative forcing notion of size λ is isomorphic to $\mathrm{C}(\lambda, \lambda)$.

Proof: Essentially lemma 25.11 in [J] .◊

In [C] it was proved that $(\mathrm{C}(i(\kappa), i(\kappa)^{++}))_N \cong \mathrm{C}(\kappa^+, \kappa^{++})$. The following is a slight generalization of it. The proof technique is the one used in [C].

Lemma 3.2: $1 \le n < \omega$ $|\mu| = \kappa^+$, $N \models$"μ is cardinal", $(\mathrm{C}(\mu, i(\kappa)^{+n}))_N \cong \mathrm{C}(\kappa^+, \kappa^{+n})$.

Proof: The proof will be done by induction on n. For $n = 1$ we use the previous lemma. Let $n > 1$ and assume the lemma is proved for values below n. As $i(\kappa^{+n}) = \kappa^{+n}$ the set $C' = \{\alpha < \kappa^{+n} \mid i(\alpha) = \alpha\}$ is unbounded. Let C be C' with its' limit points and take increasing enumeration of it $C = \langle \alpha_\xi \mid \xi < \kappa^{+n} \rangle$. We have that $i(\kappa^{+n}) = \bigcup_{\xi < \kappa^{+n}} [\alpha_\xi, \alpha_{\xi+1})$. Consider $(\mathrm{C}(\mu, [\alpha_\xi, \alpha_{\xi+1})))_N$. By induction there's $n' < n$, σ_ξ, $P_\xi = \mathrm{C}(\kappa^+, \kappa^{+n'})$ such that $\sigma_\xi : (\mathrm{C}(\mu, [\alpha_\xi, \alpha_{\xi+1})))_N \cong P_\xi$. Let $Q = \{q \in \prod_{\xi < \kappa^{+n}} P_\xi \mid |\mathrm{supp}\, q| \le \kappa\}$. It is clear that $Q \cong \mathrm{C}(\kappa^+, \kappa^{+n})$. So by proving now that $\sigma : (\mathrm{C}(\mu, i(\kappa)^{+n}))_N \to Q$ defined by $\sigma(p) = \langle \sigma_\xi(p | [\alpha_\xi, \alpha_{\xi+1}) \times \mu) \mid \xi < \kappa^{+n} \rangle$ is isomorphism we finish the lemma. The non-obvious things are that the range of σ is in Q and that it is onto Q. So let $p \in (\mathrm{C}(\mu, i(\kappa)^{+n}))_N$. In order to show that $\sigma(p) \in Q$ we need to show that $|\mathrm{supp}\, \sigma(p)| \le \kappa$. As $N \models$"$|p| < \mu$" there's $\xi < i(\kappa^{+n})$ such that $p \subseteq p | \alpha_\xi \times \mu$ hence $\mathrm{supp}\, \sigma(p) \subseteq \mathrm{supp}\, \sigma(p | \alpha_\xi \times \mu)$, so it's enough to prove $|\mathrm{supp}\, \sigma(p | \alpha_\xi \times \mu)| \le \kappa$ which we'll prove by induction on ξ. For $\xi = 0$ it's obvious. For the successor case we know

$$\mathrm{supp}\, \sigma(\mathrm{p} | \alpha_{\xi+1} \times \mu) = \mathrm{supp}\, \sigma(\mathrm{p} | \alpha_\xi \times \mu) \cup \mathrm{supp}\, \sigma_\xi(\mathrm{p} | [\alpha_\xi, \alpha_{\xi+1}) \times \mu).$$

By induction |supp $\sigma(p|\alpha_\xi \times \mu)| \le \kappa$ and from definition |supp $\sigma_\xi(p|[\alpha_\xi, \alpha_{\xi+1}) \times \mu)| \le \kappa$ hence |supp $\sigma(p|\alpha_{\xi+1} \times \mu)| \le \kappa$. For ξ limit we'll split the proof. Let ξ be limit ordinal with $\lambda = \text{cf } \xi \le \kappa$. Take $\alpha_\xi = \bigcup_{\nu<\lambda} \alpha_{\xi_\nu}$, so supp $\sigma(p|\alpha_\xi \times \mu) = \bigcup_{\nu<\lambda} \text{supp } \sigma(p|\alpha_{\xi_\nu} \times \mu)$, by induction and $\lambda \le \kappa$ we get |supp $\sigma(p|\alpha_\xi \times \mu)| \le \kappa$. Now let ξ be limit with $\lambda = \text{cf } \xi > \kappa$. Take $\alpha_\xi = \bigcup_{\nu<\lambda} \alpha_{\xi_\nu}$ such that $i(\alpha_{\xi_\nu}) = \alpha_{\xi_\nu}$. As $\lambda \ne \kappa$ we have $i(\alpha_\xi) = \bigcup_{\nu<\lambda} i(\alpha_{\xi_\nu}) = \bigcup_{\nu<\lambda} \alpha_{\xi_\nu} = \alpha_\xi$. As $\kappa < \text{cf } \alpha_\xi$ also $i(\kappa) < i(\text{cf } \alpha_\xi) = \text{cf}_N i(\alpha_\xi) = \text{cf}_N(\alpha_\xi)$. As $N \models$"$|p|\alpha_\xi \times \mu| < \mu$" and $N \models$"cf $\alpha_\xi > i(\kappa)$" there's $\xi' < \xi$ such that $p|\alpha_\xi \times \mu \subseteq p|\alpha_{\xi'} \times \mu$ giving us supp $\sigma(p|\alpha_\xi \times \mu) \subseteq \text{supp } \sigma(p|\alpha_{\xi'} \times \mu)$ and by induction we finish.◊

Lemma 3.3: There's a $\rho_1 : (\prod_{m<n<\omega} \text{C}(\kappa^{+n}, (i(\kappa)^{+\omega})_{M_1^N}))_N \cong \overline{P}_1$.

Proof: This is a corollary of 3.2,

$$(\prod_{1\le k<\omega} \text{C}(\kappa^{+n}, (i(\kappa)^{+k})_{M_1^N}))_N \cong (\text{C}(\kappa^{+n}, (i(\kappa)^{+\omega})_{M_1^N}))_N,$$

$$\forall k > m \ (\text{C}(\kappa^{+n}, (i(\kappa)^{+k})_{M_1^N}))_N \cong (\text{C}(\kappa^{+n}, i(\kappa)^{+m}))_N,$$

$$\prod_{1\le k\le m} \text{C}(\kappa^+, \kappa^{+k}) \times \prod_{m<k<\omega} \text{C}(\kappa^+, \kappa^{+m}) \cong \text{C}(\kappa^+, \kappa^{+m}).◊$$

Lemma 3.4: There's a $\rho_2 : (\prod_{1\le n<\omega} \text{Col}(i(\kappa), i(\kappa)^+))_N \cong \overline{Q}$.

Proof: From 3.1 we get $(\text{Col}(i(\kappa), i(\kappa)^+))_N \cong \overline{Q}_n$.◊

Claim 3.5: $V[\bar{I} \times \bar{J}]$ has the same cardinal structure as V and contains a $R_0 \times \ldots \times R_{m+1} \times Q \times P_0 \times \ldots \times P_{m+1}$-generic filter $I = I_0 \times \ldots \times I_{m+1} \times J \times G_0 \times \ldots G_{m+1}$ over M_2. Specifically, if $M_2 \models$"D is dense open in $P_0 \times \ldots \times P_{m+1} \times Q \times R_0 \times \ldots \times R_{m+1}$" then there's element in $D \cap (I_0 \times \ldots \times I_{m+1} \times J \times G_0 \times \ldots \times G_{m+1})$ of the form $\langle j_1(f_0)(\kappa), j_1(f_1)(\kappa), j_1(f_2)(\alpha), \ldots, j_1(f_{m+1})(\alpha), j_1(h)(\kappa),$
$j_2(g_0)(j_1(\alpha)), \ldots, j_2(g_{m+1})(j_1(\alpha))\rangle$ and $\langle j_1(g_0)(\alpha), \ldots, j_1(g_{m+1})(\alpha) > \in I_0 \times \ldots \times I_{m+1}$.

Proof: $\overline{P} \times \overline{Q}$ collapses no cardinals using the usual arguments for multiple Cohen forcings.

We will show that we have the generic set by constructing it step by step in $V[\bar{I} \times \bar{J}]$.

Step 1: There's $I_0' \in V$ which is $T = (\prod_{m<n<\omega} \text{Col}(\kappa^{+n}, < i(\kappa)))_N$-generic over N.

Proof: Working in N: T satisfies $i(\kappa)$-c.c. and $|T|=i(\kappa)$. So T has at most $i(\kappa)$ maximal anti-chains.

Working in V: T is κ^+-closed and the number of anti-chains we had found in N is $|i(\kappa)|=\kappa^+$. Hence we can build a filter $I_0' \in V$ which is T-generic over N.

Step 2: There's I_1' which is $T=(\prod_{m<n<\omega} \mathrm{C}(\kappa^{+n},(i(\kappa)^{+\omega})_{M_1^N}))_N$-generic over $N[I_0']$.

Proof: Use the ρ_1 from 3.3 to set $I_1'=\langle \rho_1^{-1''} \bar{I}_1\rangle$. We'll show I_1' is T-generic over $N[I_0']$. Take $D \in N[I_0']$ dense open in T. As $I_0' \in V$ we have $D \in V$. So $\rho_1''D \in V$ is dense open in $\bar{P}_1$ and using genericity of $\bar{I}_1$ over V we have $\rho_1''D \cap \bar{I}_1 \neq \varnothing$, hence $D \cap I_1' \neq \varnothing$.

Step 3: Set the following:

$I_0=\langle k''I_0'\rangle$

$I_1=\langle k''I_1'\rangle$

$I_2=\bar{I}_2|P_2$

$I_3=\bar{I}_3|P_3$

$\vdots$

$I_m=\bar{I}_m|P_m$

$I_{m+1}=\bar{I}_{m+1}$ (here we have $P_{m+1}=\bar{P}_{m+1}$)

$J'=\langle \rho_2^{-1''} \bar{J}\rangle$

$J=\langle k''J'\rangle$

$G_0=\langle l''I_0\rangle$

$G_1=\langle l''I_1\rangle$

$G_2=\langle l''I_2\rangle$

$\vdots$

$G_m=\langle l''I_m\rangle$

$G_{m+1}'=\langle i''I_{m+1}\rangle$

$G_{m+1} = \langle l'' I_{m+1} \rangle = \langle k'' G'_{m+1} \rangle$ (because $l = j_1 = k \circ i$)

Step 4: $I_0 \times I_1$ is $P_0 \times P_1$-generic over M_1.

Proof: Let $D \in M_1$ be dense open in $P_0 \times P_1$. Take f such that $D = j_1(f)(\kappa, \alpha_1, \ldots, \alpha_n)$. Then

$$X = \{\langle \xi, \xi_1, \ldots, \xi_n \rangle \mid f(\xi, \xi_1, \ldots, \xi_n) \text{ dense open in } \prod_{m<n<\omega} \mathrm{Col}(\xi^{+n}, <\kappa) \times \prod_{m<n<\omega} \mathrm{C}(\xi^{+n}, (\kappa^{+\omega})_{M_1})\}$$

is $E_{<\kappa, \alpha_1, \ldots, \alpha_n>}$-big. Define $f^*(\xi) = \bigwedge_{\langle \xi, \xi_1, \ldots \xi_n \rangle \in X} f(\xi, \xi_1, \ldots \xi_n)$ and note that $j_1(f^*)(\kappa) \subseteq j_1(f)(\kappa, \alpha_1, \ldots, \alpha_n)$. For each ξ the forcing $\prod_{m<n<\omega} \mathrm{Col}(\xi^{+n}, <\kappa) \times \prod_{m<n<\omega} \mathrm{C}(\xi^{+n}, (\kappa^{+\omega})_{M_1})$ is ξ^{+m+1}-closed and $\xi < \xi_1 < \ldots < \xi_n < \xi^{+m}$ so $\{\xi \mid f^*(\xi)$ dense open in $\prod_{m<n<\omega} \mathrm{Col}(\xi^{+n}, <\kappa) \times \prod_{m<n<\omega} \mathrm{C}(\xi^{+n}, (\kappa^{+\omega})_{M_1})\} \in E_{<\kappa>}$ meaning $i(f^*)(\kappa) \in N$ is dense open in $(\prod_{m<n<\omega} \mathrm{Col}(\kappa^{+n}, <i(\kappa)) \times \prod_{m<n<w} \mathrm{C}(\kappa^{+n}, (i(\kappa)^{+\omega})_{M_1^N}))_N$ so from genericity of $I'_0 \times I'_1$ over N we get that $i(f^*)(\kappa) \cap (I'_0 \times I'_1) \neq \emptyset$ yielding $j_1(f^*)(\kappa) \cap (I_0 \times I_1) \neq \emptyset$.

Step 5: I_2 is P_2-generic over $M_1[I_0 \times I_1]$.

Proof: Let $P_2 \supseteq A \in M_1[I_0 \times I_1]$ such that $M_1[I_0 \times I_1] \models$"$A$ is maximal anti-chain". As $M_1 \models$"$P_0 \times P_1$ is κ^{+m+1}-closed" we have that $A \in M_1$. Take enumeration $A = \{a_\xi \mid \xi < \kappa^{+m}\}$ and set $B = \bigcup \{\mathrm{dom}\, a_\xi \mid \xi < \kappa^{+m}\}$. Take $\varphi : B \times 2 \leftrightarrow V_{\kappa+m}$ $\varphi \in M_1$. We'll show that A is also a maximal anti chain in $C(\kappa^{+m}, \kappa^{+m})$. (That is in V !). Suppose there's $p \in C(\kappa^{+m}, \kappa^{+m})$ such that $\forall a \in A\ p \perp a$. Then from the definition of B we have that $\forall a \in A\ p \restriction B \perp a$. As $|p| < \kappa^{+m}$ we have that $|\varphi'' p \restriction B| < \kappa^{+m}$ and $\varphi'' p \restriction B \subseteq V_{\kappa+m}$. We know that $M_1 \supseteq V_{\kappa+m}^{V_{\kappa+(m-1)}}$ and so $p \restriction B \in (\mathrm{C}(\kappa^{+m}, (\kappa_1^{+\omega})_{M_2}))_{M_1}$. So there's $a \in A$ such that $p \restriction B \parallel a$. Contradiction. So $A \cap \bar{I}_2 \neq \emptyset$ and obviously $A \cap I_2 \neq \emptyset$.

Step 6: $\forall 3 \leq k \leq m+1$ I_k is P_k-generic over $M_1[I_0 \times I_1 \times \ldots \times I_{k-1}]$.

Proof: This is exactly as in step 6.

Note: Thus far we had shown that $I_0 \times \cdots \times I_{m+1} \in V[\bar{I}]$ is $P_0 \times \cdots \times P_{m+1}$-generic over M_1.

Step 7: $G_0 \times \ldots \times G_m$ is $R_0 \times \ldots \times R_m$-generic over M_2.

Note: We don't handle here the generic over R_{m+1} due to Q. Later, we'll tackle $R_{m+1} \times Q$ together.

Proof: Let $D \in M_2$ be dense open in $R_0 \times \ldots \times R_m$. Take f such that $D = j_2(f)(\alpha, j_1(\alpha))$. Let

$$X = \{<\nu_0, \nu_1> \mid f(\nu_0, \nu_1) \text{ dense open in } \prod_{m<n<\omega} \mathrm{Col}(\nu_1^{0+n}, <\kappa) \times \prod_{m<n<\omega} \mathrm{C}(\nu_1^{0+n}, (\kappa^{+\omega})_{M_1}) \times \mathrm{C}(\nu_1^{0+m}, (\kappa^{+\omega})_{M_1}) \times \mathrm{C}(\nu_1^{0+m-1}, (\kappa^{+\omega})_{M_1}) \times \ldots \times \mathrm{C}(\nu_1^{0++}, (\kappa^{+\omega})_{M_1})\}$$

then $X \in U^2_\alpha$. Recalling that $\{\langle \nu_0, \nu_1 \rangle \mid \nu_0 < \nu_1^0\} \in U^2_\alpha$ we set $f^*(\nu_1) = \bigcap_{\substack{\langle \nu_0, \nu_1 \rangle \in X \\ \nu_0 < \nu_1^0}} f(\nu_0, \nu_1)$ and we note that $j_2(f^*)(j_1(\alpha)) \subseteq j_2(f)(\alpha, j_1(\alpha))$. Thanks to ν_1^{0+}-closure we have

$$\{\langle \nu_1 \rangle \mid f^*(\nu_1) \text{ dense open in } \prod_{m<n<\omega} \mathrm{Col}(\nu_1^{0+n}, \kappa) \times \prod_{m<n<\omega} \mathrm{C}(\nu_1^{0+n}, (\kappa^{+\omega})_{M_1}) \times \mathrm{C}(\nu_1^{0+m}, (\kappa^{+\omega})_{M_1}) \times \mathrm{C}(\nu_1^{0+m-1}, (\kappa^{+\omega})_{M_1}) \times \ldots \times \mathrm{C}(\nu_1^{0++}, (\kappa^{+\omega})_{M_1}) \in U_\alpha$$

hence $j_1(f^*)(\alpha) \in M_1$ is dense open in $P_0 \times \ldots \times P_m$ so from genericity of $I_0 \times \ldots \times I_m$ over M_1 we get that $j_1(f^*)(\alpha) \cap (I_0 \times \ldots \times I_m) \neq \emptyset$ yielding $j_2(f^*)(j_1(\alpha)) \cap (G_0 \times \ldots \times G_m) \neq \emptyset$.

Step 8: G'_{m+1} is $T = (\mathrm{C}(i(\kappa)^+, (\kappa_1^{N+\omega})_{M_2^N}))_N$-generic over N. Moreover, if $D \in N$ is dense open in T then there's $p \in I_{m+1}$ such that $l(p) \in G_{m+1}$ and $i(p) \in D \cap G'_{m+1}$.

Proof: Let $D \in N$ be dense open in T. Take $D = i(f)(\kappa)$. Hence $X = \{\xi \mid f(\xi) \text{ dense open in } \mathrm{C}(\kappa^+, (\kappa_1^{+\omega})_{M_2})\} \in U_\kappa$. Set $f^* = \bigcap_{\xi \in X} f(\xi)$. Note that $i(f^*) \subseteq D$ and due to κ^+-closeness f^* is dense open in $\mathrm{C}(\kappa^+, (\kappa_1^{+\omega})_{M_2})$. From genericity of $\bar{I}_{m+1}$ over V we have $p \in f^* \cap \bar{I}_{m+1}$ hence $i(p) \in i(f^*) \cap G'_{m+1}$. Trivially

$p \in I_{m+1}$. $j_1(p) = k(i(p)) \in k''G'_{m+1} \subseteq G_{m+1}$. As $p \in M_1$ and $j_1 | M_1 = l$ we get that $l(p) \in G_{m+1}$.

Step 9: J' is $T = (\prod_{1 \le n < \omega} \mathrm{Col}(i(\kappa), i(\kappa)^+))_N$ -generic over $V[\bar{I}_{m+1}]$ (and thus over $N[G'_{m+1}]$).

Proof: Let $D \in V[\bar{I}_{m+1}]$ be dense open in T. Then $\rho_2'' D \in V[\bar{I}_{m+1}]$ is dense open in $\overline{Q}$ and as $\bar{J}$ is $\overline{Q}$-generic over $V[\bar{I}_{m+1}]$ we get that $\rho_2'' D \cap \bar{J} \neq \emptyset$ hence $D \cap J' \neq \emptyset$.

Step 10: $G_{m+1} \times J$ is $R_{m+1} \times Q$-generic over M_1.

Proof: Let $M_1 \models$ "D dense open in $R_{m+1} \times Q$" and $D = j_1(f)(\kappa, \alpha_1, \ldots, \alpha_n)$. So

$X = \{\langle \xi, \xi_1, \ldots, \xi_n \rangle \mid$
$f(\xi, \xi_1, \ldots, \xi_n)$ dense open in $\mathrm{C}(\kappa^+, (\kappa_1^{+\omega})_{M_2}) \times \prod_{1 \le n < \omega} \mathrm{Col}(\kappa, \kappa^+)\} \in E_{\langle \kappa, \alpha_1, \ldots \alpha_n \rangle}$.

Set $f^*(\xi) = \bigcap_{\langle \xi, \xi_1, \ldots, \xi_n \rangle \in X} f(\xi, \xi_1, \ldots, \xi_n)$ and due to κ-closeness we have $\{\xi | f^*(\xi)$ dense open in $\mathrm{C}(\kappa^+, (\kappa_1^{+\omega})_{M_2}) \times \prod_{1 \le n < \omega} \mathrm{Col}(\kappa, \kappa^+)\} \in E_{\langle \kappa \rangle}$, so $j_1(f^*)(\kappa) \subseteq j_1(f)(\kappa, \alpha_1, \ldots, \alpha_n)$ and $i(f^*)(\kappa)$ is dense open in $(\mathrm{C}(i(\kappa)^+, (\kappa_1^{N+\omega})_{M_2^N}) \times \prod_{1 \le n < \omega} \mathrm{Col}(i(\kappa), i(\kappa)^+))_N$. Hence there's $p \in I_{m+1}$ such that $l(p) \in G_{m+1}$ and $\langle i(p), h' \rangle \in i(f^*)(\kappa) \cap (G'_{m+1} \times J')$ which yields $\langle j_1(p), k(h') \rangle \in j_1(f)(\kappa, \alpha_1, \ldots, \alpha_n) \cap (G_{m+1} \times J)$. Note that as $p \in M_1$ we have $p = j_1(g)(\alpha)$ hence $j_1(p) = l(p) = j_2(g)(j_1(\alpha))$.

Step 11: $G_{m+1} \times J$ is $R_{m+1} \times Q$-generic over $M_2[G_0 \times \ldots \times G_m]$.

Proof: Let $M_2[G_0 \times \ldots \times G_m] \models$ "$A \subseteq R_{m+1} \times Q$ is a maximal anti-chain". As $M_2 \models$ "$R_{m+1} \times Q$ is κ_1^{++}-c.c." and $M_2 \models$ "$R_0 \times \ldots \times R_m$ is κ_1^{++}-closed" we have that $A \in M_2$. Trivially $A \in M_1$ and from genericity over M_1 we get $A \cap (G_{m+1} \times J) \neq \emptyset$.

Step 12: $I_0 \times \ldots \times I_{m+1}$ is $P_0 \times \ldots \times P_{m+1}$-generic over M_2.

Proof: Let $M_2 \models$ "D is dense open in $P_0 \times \ldots \times P_{m+1}$". Trivially $M_1 \models$ "D is dense open in $P_0 \times \ldots \times P_{m+1}$" hence $D \cap (I_0 \times \ldots \times I_{m+1}) \neq \emptyset$.

Step 13: $I_0 \times \ldots \times I_{m+1}$ is $P_0 \times \ldots \times P_{m+1}$-generic over $M_2[G_0 \times \ldots \times G_{m+1} \times J]$.

Proof: Let $M_2[G_0 \times \ldots \times G_{m+1} \times J] \models$ "$A \subseteq P_0 \times \ldots \times P_{m+1}$ is a maximal anti-chain". As $M_2 \models$ "$R_0 \times \ldots \times R_{m+1} \times Q$ is κ_1-closed" we have that $A \in M_2$ and from the previous step we get $A \cap (I_0 \times \ldots \times I_{m+1}) \neq \emptyset$.

Step 14: $I_0 \times \ldots \times I_{m+1} \times J \times G_0 \times \ldots \times G_{m+1}$ is $P_0 \times \ldots \times P_{m+1} \times Q \times R_0 \times \ldots \times R_{m+1}$-generic over M_2.

Proof: That's just a rewording of the previous step.

By this we proved the claim.◊

Set $\overline{P}^{(n)} \times \overline{Q}^{(n)} = j_n(\overline{P} \times \overline{Q})$ and $\bar{I}^{(n)} \times \bar{J}^{(n)} = \langle j_n''\bar{I} \rangle \times \langle j_n''\bar{J} \rangle$.

The forcing notion $\overline{P} \times \overline{Q}$ is κ^+-closed and j_1 is an elementary embedding derived from the extender $\boldsymbol{E} = \{E_a \mid a \in [\kappa^{+m}]^{<\omega}\}$ hence we can extend j_1 to j_1^* with domain $V[\bar{I} \times \bar{J}]$ and j_1^* will be also derived from an extender $\overline{\boldsymbol{E}} = \{\overline{E}_a \mid a \in [\kappa^{+m}]^{<\omega}\}$. Thus we lift $V \xrightarrow{j_{0,1}} M_1 \xrightarrow{j_{1,2}} M_2$ to $V[\bar{I} \times \bar{J}] \xrightarrow{j_{0,1}^*} M_1[\bar{I}^{(1)} \times \bar{J}^{(1)}] \xrightarrow{j_{1,2}^*} M_2[\bar{I}^{(2)} \times \bar{J}^{(2)}]$. Set $M_2^* = M_2[\bar{I}^{(2)} \times \bar{J}^{(2)}]$. The forcing notions $P_0, \ldots, P_{m+1}$, Q, $R_0, \ldots, R_{m+1}$ were defined in M_2. Substituting M_2^* for M_2 in those definitions will leave us with the same sets because $M_2 \models$"$\overline{P}^{(2)} \times \overline{Q}^{(2)}$ is κ_2^+-closed".

Claim 3.6: $I_0 \times \ldots \times I_{m+1} \times J \times G_0 \times \ldots \times G_{m+1}$ is $P_0 \times \ldots \times P_{m+1} \times Q \times R_0 \times \ldots \times R_{m+1}$-generic over M_2^*.

Proof: Let $M_2^* \models$"$A \subseteq P_0 \times \ldots \times P_{m+1} \times Q \times R_0 \times \ldots \times R_{m+1}$ is a maximal anti-chain". As $M_2 \models$"$\overline{P}^{(2)} \times \overline{Q}^{(2)}$ is κ_2^+-closed" we get that $A \in M_2$ and so $A \cap (I_0 \times \ldots \times I_{m+1} \times J \times G_0 \times \ldots \times G_{m+1}) \neq \varnothing$ by genericity over M_2.◊

Let use define

$$V^1 = V[\bar{I}][\prod_{1 \le n < \omega} \bar{J}_n] = V^*$$

$$V^2 = V[\bar{I}][\prod_{2 \le n < \omega} \bar{J}_n]$$

$$V^k = V[\bar{I}][\prod_{k \le n < \omega} \bar{J}_n] \text{ for } 2 \le k < \omega.$$

Note that we got from the construction that if $D \in V_{n+1}$ is dense in $(\mathrm{Col}(i(\kappa), i(\kappa)^+))_N$ then there's $i(f)(\kappa) \in D \cap J_n'$ such that $j(f)(\kappa) \in J_n$.

Corollary 3.7:

It is consistent that there's V^* with the following power-set function

$$2^{\lambda}=\begin{cases}\lambda^{+} & \lambda\le\kappa\\ \kappa^{+m} & \kappa<\lambda<\kappa^{+m}\\ \lambda^{+} & \kappa^{+m}\le\lambda\end{cases}$$

which contains an elementary embedding $j_1^*: V^* \to M_1^*$, $\mathrm{crit}(j_1^*)=\kappa$ derived from an extender $\overline{\boldsymbol{E}}=\{\overline{E}_a \mid a\in[\kappa^{+m}]^{<\omega}\}$ and $(\kappa^{+m})_{M_1^*}=\kappa^{+m}$. Moreover iterating j_1^* we have $V^* \xrightarrow{j_{0,1}^*} M_1^* \xrightarrow{j_{1,2}^*} M_2^*$ and there's a filter $I_0\times\ldots\times I_{m+1}\times J\times G_0\times\ldots\times G_{m+1}\in V^*$ which is $P_0\times\ldots\times P_{m+1}\times Q\times R_0\times\ldots\times R_{m+1}$-generic over M_2^*.

4. The Forcing

The forcing notion we're presenting here is essentially the Gitik-Magidor forcing [G-Ma] with added Cohen forcings. As the exact definition contains lots of details we'll describe it here from scratch in a somewhat non technical way.

Our starting point is Prikry forcing. We will extend it in two directions independently and then we'll merge both extensions into one humongous forcing.

We're starting from a measurable cardinal κ and conditions of the form $\langle t,T\rangle$ where T is a tree of possible continuations of t with splittings in some ultrafilter. As is well known the trees in the conditions don't affect the conditions' compatibility. Taking the sequences in the generic object give us a cofinal sequence $\langle\tau_n \mid n<\omega\rangle$ into κ. We'd like to modify this forcing so that κ will become $\aleph_\omega$ of the generic extension. If the τ_n will become $\aleph_n$ and κ won't be collapsed then it'll be $\aleph_\omega$. So we'll add Levy collapses to the conditions. (i.e. a typical condition will look like $\langle\langle\tau_0,\tau_1\rangle,\langle f_0,f_1\rangle,T\rangle$ where $f_0\in\mathrm{Col}(\tau_0,<\tau_1)$, $f_1\in\mathrm{Col}(\tau_1,<\kappa)$). The trees in the original Prikry forcing allowed us to prove that any statement in the forcing language can be decided by a condition with arbitrarily chosen sequence length. (Henceforth we will call this Prikry condition). The Prikry condition is essential in order to control the behavior of the power-set function below κ. In order to have this condition we need some analog of the trees for the functions. For this we'll add F to the condition which is a function with domain T such that $F(\langle\nu_1,\ldots,\nu_n\rangle)\in\mathrm{Col}(\nu_n,<\kappa)$. This idea goes back to

Magidor's paper [Ma1]. So our typical condition will look like $\langle\langle\tau_0,\tau_1\rangle,\langle f_0,f_1\rangle,T,F\rangle$ with $\langle\langle\tau_0,\tau_1,\tau_2\rangle,\langle g_0,g_1,g_2\rangle,S,G\rangle\leq$ $\langle\langle\tau_0,\tau_1\rangle,\langle f_0,f_1\rangle,T,F\rangle$ if we also add the requirements that $g_2\leq F(\langle\tau_2\rangle)$ and that $G(\langle\nu_1,\ldots,\nu_n\rangle)\leq F(\langle\tau_2,\nu_1,\ldots,\nu_n\rangle)$ for each $\langle\nu_1,\ldots\nu_2\rangle$ in S. This is not enough. We want that this F won't affect the conditions' compatibility. (Remember, the F is analogous to T). In order to get this we will restrict the allowed F. The idea of restricting such a function to values in a filter is due to H. Woodin [C-Wo]. We'll define $F_{\langle\nu_1,\ldots,\nu_n\rangle}(\nu)=\mathrm{F}(\langle\nu_1,\ldots,\nu_n,\nu\rangle)$. Let $i_1{:}V\to N_1$ witness the measurability of κ. We iterate the embedding to get $i_{1,2}{:}N_1\to N_2$ and we define $i_2=i_{1,2}\circ i_1$. If we could have a filter, I, which is $(\mathrm{Col}(\kappa,<i_1(\kappa)))_{N_1}$-generic over N_1 then we would have required $i_1(F_{\langle\nu_1,\ldots,\nu_n\rangle})(\kappa)\in I$. Alas, we have no such filter. However, we do have a filter, I, which is $(\mathrm{Col}(\kappa^+,<i_1(\kappa)))_{N_1}$-generic over N_1. In order to be able to add the requirement $i_1(F_{\langle\nu_1,\ldots,\nu_n\rangle})(\kappa)\in I$ we should change the definition of F and hence also the definition of f. So in the typical condition $\langle\langle\tau_0,\tau_1\rangle,\langle f_0,f_1\rangle,T,F\rangle$ we have $f_0\in\mathrm{Col}(\tau_0^+,<\tau_1)$, $f_1\in\mathrm{Col}(\tau_1^+,<\kappa)$ with the obvious change in F's definition. The cardinal structure in the generic extension will now be: $\aleph_0=\tau_0$, $\aleph_1=\tau_0^+$, $\aleph_2=\tau_1$, $\aleph_3=\tau_1^+$... etc. and still if κ won't collapse it'll be $\aleph_\omega$. We note here that in fact, for every $1\leq m<\omega$ we have a filter which is $(\mathrm{Col}(\kappa^{+m},<i_1(\kappa)))_{N_1}$-generic over N_1 and after the appropriate changes to the f's and F's we could use this filter. Our next step is to monkey with $2^{\aleph_n}$ of the generic extension. Our first aim will be to have $\forall n<\omega\ 2^{\aleph_n}=\aleph_{n+3}$. Clearly what's needed to be done is to add to our forcing condition Cohen functions and a function whose domain is the tree. That is, a typical condition will look something like

$$\langle\langle\tau_0,\tau_1\rangle,\langle f_0,f_1\rangle,\langle g_{00},g_{01},g_{10},g_{11}\rangle,T,F,G^0,G^1\rangle.$$

As our cardinal structure will be $\langle\tau_0,\tau_0^+,\tau_1,\tau_1^+,\mathrm{K}\rangle$ our g's should be $g_{00}\in\mathrm{C}(\tau_0,\tau_1^+)$, $g_{01}\in\mathrm{C}(\tau_0^+,\kappa)$, $g_{10}\in\mathrm{C}(\tau_1,\kappa^+)$, $g_{11}\in\mathrm{C}(\tau_1^+,\kappa^{++})$. These last 2 functions look a bit weird and generate some technical problems, mainly, it's not clear how to extend such functions when the Prikry sequence is enlarged.. In order to overcome these problems we will change them to g^*_{10}, g^*_{11}. These 2 functions have as their domain the 1st level of T and for each $\langle \nu \rangle \in T$ we require $g^*_{10}(\nu) \in \mathrm{C}(\tau_1, \nu^+)$, $g^*_{11}(\nu) \in \mathrm{C}(\tau_1^+, \nu^{++})$. (The observant reader will see that $i_1(g^*_{10})(\kappa)$, $i_1(g^*_{11})(\kappa)$ are the g_{10}, g_{11}). Defining now for $0 \le l \le 1$ $G^l_{\langle \nu_1, \ldots \nu_n \rangle}(\mu_0, \mu_1) = G^l(\langle \nu_1, \ldots \nu_n, \mu_0, \mu_1 \rangle)$ the compatibility requirement will be $i_2(G^l)(\kappa, i_1(\kappa)) \in J^l$ where $J^1 \times J^2$ is $(C(\kappa, \kappa_1^+) \times C(\kappa^+, \kappa_1^{++}))_{N_2}$-generic over N_2. Once again this generic doesn't exist. The solution to this problem is to make a preparation forcing which will 'bring in' the needed generics. After this preparation forcing we loose GCH above κ but we still have the elementary embeddings. As we couldn't get the generic for $(\mathrm{C}(\kappa, \kappa_1^+))_{N_2}$ without loosing also the elementary embeddings (that is the measurability of κ) we won't have this generic. This leaves us with the inability to control 2^{τ_n}. It was pointed out by Woodin that τ_n^+ can be collapsed to τ_n using $(\mathrm{Col}(i(\kappa), i(\kappa)^+)_{N_2}$ for which we have a generic filter. So we will control the power set size of $\langle \tau_0^+, \tau_1^+, \ldots \rangle$ and then we'll collapse all the τ_n^+ to τ_n. So the cardinal structure will be $\langle \tau_0, \tau_1, \ldots \rangle$. Unfortunately we don't have enough cardinals left after τ_1 in order to describe the power set of τ_0^+. We solve this problem by redefining the f's as $\langle f_0, f_1 \rangle \in \mathrm{Col}(\tau_0^{+3}, <\tau_1) \times \mathrm{Col}(\tau_1^{+3}, <\kappa)$. Now the cardinal structure will be $\langle \tau_0, \tau_0^{++}, \tau_0^{+3}, \tau_1, \tau_1^{++}, \tau_1^{+3}, \ldots \rangle$ and our typical forcing condition looks like $\langle \langle \tau_0, \tau_1 \rangle, \langle f_0, f_1 \rangle, \langle g_{01}, g_{02}, g_{03}, g^*_{11}, g^*_{12}, g^*_{13} \rangle, T, F, G^1, G^2, G^3 \rangle$ where $g_{01} \in \mathrm{C}(\tau_0^+, \tau_1)$, $g_{02} \in \mathrm{C}(\tau_0^{++}, \tau_1^{++})$, $g_{03} \in \mathrm{C}(\tau_0^{+3}, \tau_1^{+3})$, $g^*_{11}(\nu) \in \mathrm{C}(\tau_1^+, \nu)$, $g^*_{12}(\nu) \in \mathrm{C}(\tau_1^{++}, \nu^{++})$, $g^*_{13}(\nu) \in \mathrm{C}(\tau_1^{+3}, \nu^{+3})$. We now give the final touch which will collapse the τ_n^+. The forcing condition will look like $\langle \langle \tau_0, \tau_1 \rangle, \langle f_0, f_1 \rangle, \langle g_{01}, g_{02}, g_{03}, g^*_{11}, g^*_{12}, g^*_{13} \rangle, \langle h_0, h_1^* \rangle T, F, G^1, G^2, G^3, H \rangle$ where $h_0 \in \mathrm{Col}(\tau_1, \tau_1^+)$, $h_1^*(\nu) \in \mathrm{Col}(\nu, \nu^+)$ and $i_2(H_{\langle \nu_1, \ldots \nu_n \rangle})(\kappa, i_1(\kappa)) \in J$ where J is $(\mathrm{Col}(i_1(\kappa), i_1(\kappa)^+))_{N_2}$ -generic over N_2. We will now put the g's into the f's and change a bit the way a condition looks in order to simplify the notation. So a condition will look like $\langle \langle \tau_0, \tau_1 \rangle, \langle f_0, f^* \rangle, \langle h_0, h^* \rangle, T, F, H \rangle$ where $f_0 \in \mathrm{Col}(\tau_0^{+3}, <\tau_1) \times \mathrm{C}(\tau_0^+, \tau_1) \times \mathrm{C}(\tau_0^{++}, \tau_1^{++}) \times \mathrm{C}(\tau_0^{+3}, \tau_1^{+3})$,

$f^*(\nu) \in \mathrm{Col}(\tau_1^{+3}, <\nu) \times \mathrm{C}(\tau_1^{+}, \nu) \times \mathrm{C}(\tau_1^{++}, \nu^{++}) \times \mathrm{C}(\tau_1^{+3}, \nu^{+3})$, $h_0 \in \mathrm{Col}(\tau_1, \tau_1^{+})$, $h^*(\nu) \in \mathrm{Col}(\nu, \nu^{+})$ and $i_2(F_t)(\kappa, i_1(\kappa)) \in I$, $i_2(H_t)(\kappa, i_1(\kappa)) \in J$, where I is a $(\mathrm{Col}(\kappa^{+3}, <i_1(\kappa)) \times \mathrm{C}(\kappa^{+}, i_1(\kappa)) \times \mathrm{C}(\kappa^{++}, i_1(\kappa)^{++}) \times \mathrm{C}(\kappa^{+3}, i_1(\kappa)^{+3}))_{N_2}$-generic filter over N_2 and J is a $(\mathrm{Col}(i_1(\kappa), i_1(\kappa)^{+}))_{N_2}$-generic over N_2.

The above treatment will work as long as the wanted power set function have a bounded jump. That is there's a $k < \omega$ such that $2^{\aleph_n} \le \aleph_{n+k}$. In this case we choose Levy collapse which leave enough cardinals allowing us to describe the power function on the cardinals following τ_n using at most the cardinals following τ_{n+1}. If we want to lift this restriction we'll have also change the Levy collapses dynamically. As each 'chunk' might have a different length we can't use a fixed number of Cohen functions in a condition. We'll use $F^{(n)}$ to specify a specific function from the product forcing. Let's take as an example the case $2^{\aleph_0} = \aleph_1$ and $\forall n > 1\ 2^{\aleph_n} = \aleph_{n+n}$. We'll describe typical conditions to generate this case. The cardinal structure we suggest is

$$\begin{array}{cccccccccccccccccc} \tau_0 & \tau_0^{+} & \tau_0^{++} & \tau_1 & \tau_1^{+} & \tau_1^{++} & \tau_2 & \tau_2^{+} & \tau_2^{++} & \tau_2^{+3} & \tau_3 & \tau_3^{+} & \tau_3^{++} & \tau_3^{+3} & \tau_3^{+4} & \tau_3^{+5} & \tau_3^{+6} & \ldots \\ \aleph_0 & & \aleph_1 & \aleph_2 & & \aleph_3 & \aleph_4 & & \aleph_5 & \aleph_6 & \aleph_7 & & \aleph_8 & \aleph_9 & \aleph_{10} & \aleph_{11} & \aleph_{12} & \ldots \end{array}$$

and a condition will look like $\langle\langle \tau_0, \tau_1\rangle, \langle f_0, f^*\rangle, \langle h_0, h^*\rangle, T, F, H\rangle$ where

$$f_0 \in \mathrm{Col}(\tau_0^{++}, <\tau_1) \times \mathrm{C}(\tau_0^{+}, \tau_0^{++}) \times \mathrm{C}(\tau_0^{++}, \tau_1)$$

$$f^*(\nu) \in \mathrm{Col}(\tau_1^{++}, <\nu) \times \mathrm{C}(\tau_1^{+}, \nu) \times \mathrm{C}(\tau_1^{++}, \nu^{+3})$$

$$h_0 \in \mathrm{Col}(\tau_1, \tau_1^{+})$$

$$h^*(\nu) \in \mathrm{Col}(\nu, \nu^{+})$$

$$F(\langle \nu_0, \nu_1\rangle) \in \mathrm{Col}(\nu_0^{+3}, <\nu_1) \times \mathrm{C}(\nu_0^{+}, \nu_1^{++}) \times \mathrm{C}(\nu_0^{++}, \nu_1^{+4}) \times \mathrm{C}(\nu_0^{+3}, \nu_1^{+6})$$

$$F(\langle \nu_0, \nu_1, \nu_2\rangle) \in \mathrm{Col}(\nu_1^{+6}, <\nu_2) \times \ldots$$

$$i_2(F_{\langle\rangle})(\kappa, i_1(\kappa)) \in (\mathrm{Col}(\kappa^{+3}, <i_1(\kappa)) \times \mathrm{C}(\kappa^{+}, i(\kappa)^{++}) \times \mathrm{C}(\kappa^{++}, i(\kappa)^{+4}) \times \mathrm{C}(\kappa^{+3}, i(\kappa)^{+6}))_{N_2}$$

$$H(\langle \nu_0, \nu_1\rangle) \in \mathrm{Col}(\nu_1, \nu_1^{+}) \qquad \ldots$$

$$i_2(H_{\langle\rangle})(\kappa, j_1(\kappa)) \in (\mathrm{Col}(i_1(\kappa), i_1(\kappa)^{+}))_{N_2}$$

We retreat now to the original Prikry forcing. Our aim is to enlarge 2^κ without 'loosing control' below κ. Using Prikry forcing we can add an unbounded subset to κ. An obvious suggestion would be to iterate this forcing enough times. Unfortunately we are loosing control on what happens this way. (i.e. along with the Prikry sequences also the relations between them are added and these are new ω-sequences). In order to solve this we will add all the Prikry sequences in one step and we'll make sure that the relations between these new sequences are in the ground model. Our assumption is that we have a big enough set $\boldsymbol{A}$ equipped with a directed partial ordering having a minimal element 0 and for each $\alpha,\beta \in \boldsymbol{A}$ such that $\alpha > \beta$ we have a projection $\pi_{\alpha,\beta}$. For $s \subset \boldsymbol{A}$ a typical condition will look like $\{\langle \alpha, t^\alpha \rangle \mid \alpha \in s\}$. By taking only s with $|s| \leq \kappa$ we can demand that each such s have a maximal element. And a typical condition will look like $\{\langle \alpha, t^\alpha \rangle \mid \alpha \in s - \{\max s\}\} \cup \{\langle \max s, t^{\max s}, T \rangle\}$. The point in this forcing is that the Prikry sequences aren't enlarged independently. When $t^{\max s}$ is enlarged all the t^α are enlarged by the projection $\pi_{\max s,\alpha}$ of the enlargement. Imposing enough restrictions on the projections guarantee that these sequences will be different from each other which will blow 2^κ to $|\boldsymbol{A}|$. As the projections are already in the ground model we won't get new sets for the relations between the different sequences. The other parts of the definition of the partial order for this forcing are quite natural. (i.e. a stronger condition is one with larger support, sub-tree module the projection etc.). In order to build such an $\boldsymbol{A}$ of size κ^{+m} we had to assume that κ is a κ^{+m}-strong cardinal. Let $j_1 : V \to M_1$ witness the κ^{+m}-strongness of κ. We'll iterate it and have $j_{1,2} : M_1 \to M_2$, $j_2 = j_{1,2} \circ j_1$.

We will now combine together these 2 extension. For this we'll also require that we take only $s \subset \boldsymbol{A}$ with $0 \in s$. We will put the Levy and Cohen functions on the 0th coordinate. The function with the tree as domain will be carried on the maximal coordinate. So a typical condition will look like $\{\langle 0, \langle \tau_0, \tau_1 \rangle, \langle f_0, f^* \rangle, \langle h_0, h^* \rangle \rangle, \langle \max s, t^{\max s}, T, F, H \rangle\} \cup \{\langle \alpha, t^\alpha \rangle \mid \alpha \in s - \{0, \max s\}\}$.

As the tree T is being built on a different ultra-filter each time the compatibility requirement will use $\max s$ to restrict the functions into the proper generic. (i.e. $j_2(F_{\langle\rangle})(\max s, j_1(\max s)) \in (\mathrm{Col}(\kappa^{+3}, < j_1(\kappa)))_{M_2}$). Note that the generics we use here are over M_2 and the forcing we had done in the previous section give us all the generics we might need..

This forcing will blow 2^κ to $|\boldsymbol{A}|$, convert κ to $\aleph_\omega$ and set the value of $2^{\aleph_n}$ to the prescribed values in one step.

And now we'll look into the gory details.

The universe we're working is the one constructed in the previous section. That is we have $j_1\colon V \to M_1$, $\mathrm{crit}(j) = \kappa$, $(k^{+m})_{M_1} = \kappa^{+m}$. The power set function in V is

$$2^\lambda = \begin{cases} \lambda^+ & \lambda \le \kappa \\ \kappa^m & \kappa < \lambda < \kappa^{+m} \\ \lambda^+ & \kappa^{+m} \le \lambda \end{cases}$$

and j_1 is derived from the extender $\boldsymbol{E} = \{E_a | a \in [\kappa^{+m}]^{<\omega}\}$. We'll derive the nice system $\mathbf{U} = \langle\langle U_a | \alpha \in \mathbf{A}\rangle, \langle \pi_{\beta,\alpha} | \alpha,\beta \in \mathbf{A}\ \beta \ge \alpha\rangle\rangle$ from $\boldsymbol{E}$. Iterating j_1 we have $V \xrightarrow{j_{0,1}} M_1 \xrightarrow{j_{1,2}} M_2 \xrightarrow{j_{2,3}} M_3$ and setting $\kappa_0 = \kappa, \kappa_n = j_n(\kappa)$ we have a filter I which is

$$\begin{aligned}
&(\prod_{m<n<\omega} \mathrm{Col}(\kappa^{+n}, < \kappa_1) \times \prod_{1 \le n_1 \le n_2 < \omega} \mathrm{C}(\kappa^{+n_1}, \kappa^{+n_2}) \times \\
&\prod_{\substack{1 \le n_1 < \omega \\ 1 \le n_2 < \omega}} \mathrm{C}(\kappa^{+n_1}, \kappa_1^{+n_2}) \times \prod_{m<n<\omega} \mathrm{Col}(\kappa_1^{+n}, < \kappa_2) \times \prod_{1 \le n_1 \le n_2 < \omega} \mathrm{C}(\kappa_1^{+n_1}, \kappa_1^{+n_2}) \times \\
&\prod_{\substack{1 \le n_1 < \omega \\ 1 \le n_2 < \omega}} \mathrm{C}(\kappa_1^{+n_1}, (\kappa_2^{+n_2})_{M_3}) \times \prod_{1 \le n < \omega} \mathrm{Col}(\kappa_1, \kappa_1^{+}))_{M_2}
\end{aligned}$$

-generic over M_2.

We are given a monotonic function $a\colon \omega \to \omega$ and our aim is to build a generic extension in which $2^{\aleph_n} = \aleph_{a(n)}$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$. We derive functions $b, c_1, c_2\colon \omega \to \omega$ $d, e\colon \omega \times \omega \to \omega$ from a by induction:

$b(0) = m+1$ $\qquad\qquad$ $c_1(0) = 0$

$$b(n+1)=\begin{cases} m+1 & a(c(n))-c(n)\le m \\ a(c(n))-c(n) & a(c(n))-c(n)>m \end{cases} \qquad \begin{aligned} &c_2(0)=m \\ &c_1(n+1)=c_2(n)+1 \\ &c_2(n+1)=c_2(n)+b(n+1) \end{aligned}$$

$$d(n,k)=\begin{cases} b(d_1(n)+k-1)-d_1(n)+1 & b(d_1(n)+k-1)<c(n+1) \\ 0 & b(d_1(n)+k-1)=c(n+1) \\ b(d_1(n)+k-1)-d_1(n+1)+1 & b(d_1(n)+k-1)>c(n+1) \end{cases}$$

$$e(n,k)=\begin{cases} 0 & b(d_1(n)+k-1)<c(n+1) \\ 1 & b(d_1(n)+k-1)=c(n+1) \\ 1 & b(d_1(n)+k-1)>c(n+1) \end{cases}$$

While the domain of d,e isn't $\omega\times\omega$ we will use their values only where we defined them.

These functions describe the behavior of the 'chunks' we don't collapse in the normal Prikry sequence. We set I_n to be a $([\mathrm{Col}(\kappa^{+b(n)},<\kappa_1)\times\prod_{1\le i\le b(n)}\mathrm{C}(\kappa^{+i},\kappa_{e(n,i)}^{+d(n,i)})]\times\mathrm{Col}(\kappa_1,\kappa_1^{+}))_{M_2}$-generic filter over M_2, K_n

to be

$$([\mathrm{Col}(\kappa^{+b(n)},<\kappa_1)\times\prod_{1\le i\le b(n)}\mathrm{C}(\kappa^{+i},\kappa_{e(n,i)}^{+d(n,i)})]\times\mathrm{Col}(\kappa_1,\kappa_1^{+})\times$$
$$[\mathrm{Col}(\kappa_1^{+b(n)},<\kappa_2)\times\prod_{1\le i\le b(n)}\mathrm{C}(\kappa_1^{+i},(\kappa_{1+e(n,i)}^{+d(n,i)})_{M_3})])_{M_2}\text{-}$$

generic over M_2.

Recall that we have the sequence $V=V^1\supset V^2\supset\ldots\supset V^n\supset\ldots$ and let J_n' be $(\mathrm{Col}(i(\kappa),i(\kappa)^+))_N$-generic over V^{n+1}. Note that we choose the generic such that if $D\in V^{n+1}$ is dense in $(\mathrm{Col}(i(\kappa),i(\kappa)^+))_N$ then $i(f)(\kappa)\in D\cap J_n'$ and $j(f)(\kappa)$ is in the projection of I_n to $(\mathrm{Col}(\kappa_1,\kappa_1^{+}))_{M_2}$.

o

Example 4.1: $m=3$, $a(n)=n+n$

τ_0 τ_0^{+} τ_0^{+2} τ_0^{+3} $\tau_0^{+4=b(0)}$

$\aleph_{0=c_1(0)}$ $\aleph_1$ $\aleph_2$ $\aleph_{3=c_2(0)}$

τ_1 τ_1^{+} τ_1^{+2} τ_1^{+3} $\tau_1^{+4=b(1)}$

$\aleph_{4=c_1(1)}$ $\aleph_5$ $\aleph_6$ $\aleph_{7=c_2(1)}$

τ_2 τ_2^{+} τ_2^{+2} τ_2^{+3} τ_2^{+4} τ_2^{+5} τ_2^{+6} $\tau_2^{+7=b(2)}$
$\aleph_{8=c_1(2)}$ $\aleph_9$ $\aleph_{10}$ $\aleph_{11}$ $\aleph_{12}$ $\aleph_{13}$ $\aleph_{14=c_2(2)}$
τ_3 τ_3^{+} τ_3^{+2} τ_3^{+3} τ_3^{+4} τ_3^{+5} τ_3^{+6} τ_3^{+7} τ_3^{+8} τ_3^{+9} τ_3^{+10} τ_3^{+11} τ_3^{+12} τ_3^{+13} $\tau_3^{+14=b(3)}$
$\aleph_{15=c_1(3)}$ $\aleph_{16}$ $\aleph_{17}$ $\aleph_{18}$ $\aleph_{19}$ $\aleph_{20}$ $\aleph_{21}$ $\aleph_{22}$ $\aleph_{23}$ $\aleph_{24}$ $\aleph_{25}$ $\aleph_{26}$ $\aleph_{27}$ $\aleph_{28=c_2(3)}$

The d,e describe what Cohen functions we should add namely $1 \le k \le b(n)$
$C(\tau_n^{+k}, \tau_{n+e(n,k)}^{+d(n,k)})$.

Definition 4.2: Let $T \subseteq [\kappa]^{<\omega}$ ordered by end-extension. $t \in T$ then

$$\mathrm{Suc}_T(t) \stackrel{def}{=} \{ \nu < \kappa \mid t^\wedge \langle \nu \rangle \in T \}$$

Definition 4.3: Let $T \subseteq [\kappa]^{<\omega}$ ordered by end-extension. T will be called U_α-tree if:

$$t \in \mathrm{Lev}_n(T) \to |t| = n+1$$

$$t \in T \to t^0 \in [\kappa]^{<\omega}$$

$$t_1, t_2 \in T \quad t_1 \le t_2 \to \mathrm{Suc}_T(t_1) \supseteq \mathrm{Suc}_T(t_2)$$

$$\mathrm{Lev}_0(T) \in U_\alpha$$

$$t \in T \to \mathrm{Suc}_T(t) \in U_\alpha$$

Definition 4.4: Let T be a U_α-tree and $t \in T$. We'll define T_t a U_α-tree to be:

$$\mathrm{Lev}_0(T_t) = \mathrm{Suc}_T(t)$$
$$r \in T_t \to \mathrm{Suc}_{T_t}(r) = \mathrm{Suc}_T(t^\wedge r)$$

Definition 4.5: Let T be a U_α-tree and $A \in U_a$.. We'll define $T|A$ a U_α-tree to be:

$$\mathrm{Lev}_0(T|A) = \mathrm{Lev}_0(T) \cap A$$

$$t \in T|A \to \mathrm{Suc}_{T|A}(t) = \mathrm{Suc}_T(t) \cap A.$$

Definition 4.6: Let $\langle T_i \mid i < \lambda \rangle$ $\lambda < \kappa$ be U_α-trees . We'll define $\bigcap_{i<\lambda} T_i$ a U_α-tree to be:

$$\mathrm{Lev}_0(\bigcap_{i<\lambda} T_i) = \bigcap_{i<\lambda} \mathrm{Lev}_0(T_i)$$
$$t \in \bigcap_{i<\lambda} T_i \to \mathrm{Suc}_{\bigcap_{i<\lambda} T_i}(t) = \bigcap_{i<\lambda} \mathrm{Suc}_{T_i}(t)$$

Definition 4.7: Let T be a U_α-tree. We'll define $\pi_{\beta,\alpha}^{-1} T$ a U_β-tree to be:

$$\mathrm{Lev}_0(\pi_{\beta,\alpha}^{-1} T) = \pi_{\beta,\alpha}^{-1}(\mathrm{Lev}_0(T))$$
$$t \in \pi_{\beta,\alpha}^{-1} T \to \mathrm{Suc}_{\pi_{\beta\alpha}^{-1} T}(t) = \pi_{\beta,\alpha}^{-1}(\mathrm{Suc}_T(\pi_{\beta,\alpha}(t)))$$

Definition 4.8: Let T be a U_α-tree and F a function such that $\mathrm{dom}\, F = T$ $t \in T$, then:

$$\mathrm{dom}\, F_t = T_t$$

$\forall s \in T_t \ \ F_t(s) = F(t^\frown s)$

Definition 4.9: Let T be a U_α-tree and F a function such that $\operatorname{dom} F = T$, $t \in T$, $F_t(-,-)$ is defined by

$\operatorname{dom} \mathrm{F}_t(\text{-},\text{-}) = T_t \restriction [\kappa]^2$

$\forall \langle \nu_0, \nu_1 \rangle \in T_t \restriction [\kappa]^2 \ \ F_t(\nu_0, \nu_1) = F(t^\frown \langle \nu_0, \nu_1 \rangle)$

Definition 4.10: Our forcing notion P consists of elements p of the form

$\{\langle 0, \langle \tau_0^0, \ldots, \tau_n^0 \rangle, \langle f_0, \ldots, f_{n-1}, f_n^* \rangle, \langle h_0, \ldots, h_{n-1}, h_n^* \rangle \rangle, \langle \max s, p^{\max s}, T, F, H \rangle\} \cup$

$\{\langle \gamma, p^\gamma \rangle \mid \gamma \in s - 0, \max s\}$

where

(1) $s \subseteq \boldsymbol{A}$, $|s| \leq \kappa$, s has maximal element and $0 \in s$. We'll write $\mathrm{mc}(p)$ for $\max s$, p^{mc} for $p^{\max s}$ and $\mathrm{supp}(p)$ for s.

(2) $\forall \gamma \in s \ \ p^\gamma \in [\kappa]^{<\omega}$ is ^{0}-increasing.

(3) $\forall 0 \leq i < n \ \ f_i \in \mathrm{Col}(\tau_i^{0\ +b(i)}, < \tau_{i+1}^0) \times \prod_{1 \leq j \leq b(i)} \mathrm{C}(\tau_i^{0\ +j}, \tau_{i+e(i,j)}^{0\ +d(i,j)})$

(4) $\forall 0 \leq i < n \ \ h_i \in \mathrm{Col}(\tau_{i+1}^0, \tau_{i+1}^{0+})$

(5) T is $U_{\mathrm{mc}(p)}$-tree and $\forall \eta \in T \ \ p^{\mathrm{mc}\frown} \eta$ is ^{0}-increasing.

(6) $\forall \gamma \in s \ \max p^{mc}$ isn't permitted to p^γ

(7) $\forall \langle \nu \rangle \in T \ \ |\{\gamma \in s \mid \nu \text{ is permitted to } p^\gamma\}| \leq \nu^0$

(8) $(p^{mc})^0 = p^0 (= \langle \tau_0^0, \ldots \tau_n^0 \rangle)$ and $|p^0| \geq 1$.

(9) $\forall \langle \nu \rangle \in T \ \ f_n^*(\nu^0) \in \mathrm{Col}(\tau_n^{0\ +b(n)}, < \nu^0) \times \prod_{1 \leq j \leq b(n)} \mathrm{C}(\tau_n^{0\ +j}, \tau_{n+e(n,j)}^{0\ +d(n,j)})$ (Assume $\tau_{n+1}^0 = \nu^0$).

(10) $\forall \langle \nu \rangle \in T \ \ h_n^*(\nu^0) \in \mathrm{Col}(\nu^0, \nu^{0\ +})$

(11) $\forall \langle \nu_0, \ldots, \nu_{k+1} \rangle \in T$
$F(\nu_0, \ldots, \nu_k, \nu_{k+1}) \in \mathrm{Col}(\nu_k^{0\ +b(n+k+1)}, < \nu_{k+1}^0) \times \prod_{1 \leq j \leq b(n+k+1)} \mathrm{C}(\nu_k^{0\ +j}, \nu_{k+e(n+k+1,j)}^{0\ +d(n+k+1,j)})$

(12) $\forall \langle \nu_0, \ldots \nu_{k+1} \rangle \in T \ \ H(\nu_0^0, \ldots, \nu_{k+1}^0) \in \mathrm{Col}(\nu_{k+1}^0, \nu_{k+1}^{0\ +})$

(13) $\forall t \in T$ we require $\langle j_2(F_t)(\mathrm{mc}(p), j_1(\mathrm{mc}(p))), j_2(H_{t^0})(\kappa, j_1(\kappa)) \rangle \in I_{n+|t|+1}$.◊

Definition 4.11: $p, q \in P$. We say that $p \leq q$ (p is stronger than q) if

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$

(2) $\forall\gamma\in\text{supp}(q)$ p^γ is end-extension of q^γ

(3) $p^{\text{mc}(q)}-q^{\text{mc}}\in T^q$

(4) $\forall\gamma\in\text{supp}(q)$ $p^\gamma-q^\gamma=\pi_{\text{mc}(q),\gamma}{}''a$ where $a\subseteq p^{\text{mc}(q)}-q^{\text{mc}}$ is maximal permitted to q^γ

(5) $T^p\le T^q_{p^{\text{mc}(q)}-q^{\text{mc}}}$

(6) $\forall\gamma\in\text{supp}\,q$, $\forall\langle\nu\rangle\in T^p$ such that ν is permitted to p^γ

$$\pi_{\text{mc}(p),\gamma}(\nu)=\pi_{\text{mc}(q),\gamma}\circ\pi_{\text{mc}(p),\text{mc}(q)}(\nu)$$

(7) $\forall 0\le i<n^q$ $f_i^p\le f_i^q$

(8) If $n^p>n^q$ then $f^p_{n^q}\le f^{q*}_{n^q}(p^{\text{mc}(q)}(n^q+1))$

(9) $\forall n^q<i<n^p$ $f_i^p\le F^q((p^{\text{mc}(q)}-q^{\text{mc}})|i-n^q+1)$

(10) $\forall 0\le i<n^q$ $h_i^p\le h_i^q$

(11) If $n^p>n^q$ then $h^p_{n^q}\le h^{q*}_{n^q}(p^0(n^q+1))$

(12) $\forall n^q<i<n^p$ $h_i^p\le H^q((p^0-q^0)|i-n^q+1)$

(13) If $n^p=n^q$ then $\forall<\nu>\in T^p$ $f^{p*}_{n^p}(\nu)\le f^{q*}_{n^q}\circ\pi_{\text{mc}(p),\text{mc}(q)}(\nu)$, $h^{p*}_{n^p}(\nu^0)\le h^{q*}_{n^q}(\nu^0)$

(14) If $n^p>n^q$ then $\forall<\nu>\in T^p$ $f^{p*}_{n^p}(\nu)\le F^q_{p^{\text{mc}(q)}-q^{\text{mc}}}\circ\pi_{\text{mc}(p),\text{mc}(q)}(\nu)$,

$h^{p*}_{n^p}(\nu^0)\le H^q_{p^0-q^0}(\nu^0)$

(15) $\forall t\in T^p$ $F^p(t)\le F^q_{p^{\text{mc}(q)}-q^{\text{mc}}}\circ\pi_{\text{mc}(p),\text{mc}(q)}(t)$, $H^p(t^0)\le H^q_{p^0-q^0}(t^0)$ ◊

Claim 4.12: $\langle P,\le\rangle$ is a forcing notion.

Proof: It's easy to see that ≤ is reflexive. So we're left with showing transitivity of ≤.

Let $p\le q\le r$. We'll show $p\le r$.

(3) We need to show that $p^{\text{mc}(r)}-r^{\text{mc}}\in T^r$: We have

$p^{\text{mc}(r)}-r^{\text{mc}}=(p^{\text{mc}(r)}-q^{\text{mc}(r)})\cup(q^{\text{mc}(r)}-r^{\text{mc}})=$

$\pi''_{\text{mc}(q),\text{mc}(r)}(p^{\text{mc}(q)}-q^{\text{mc}})\cup(q^{\text{mc}(r)}-r^{\text{mc}})$

As $q^{\text{mc}(r)}-r^{\text{mc}}\in T^r$ and $p^{\text{mc}(q)}-q^{\text{mc}}\in T^q\le T^r_{q^{\text{mc}(r)}-r^{\text{mc}}}$ we have

$\pi''_{\text{mc}(q),\text{mc}(r)}(p^{\text{mc}(q)}-q^{\text{mc}})\cup(q^{\text{mc}(r)}-r^{\text{mc}})\in T^r$

(4) We need to show that $\forall\gamma\in\operatorname{supp}(r)\ \ p^\gamma - r^\gamma \in \pi_{\mathrm{mc}(r),\gamma}{}''a$ where $a \subseteq p^{\mathrm{mc}(r)} - r^{\mathrm{mc}}$ is maximal permitted to r^γ: Let $\gamma\in\operatorname{supp}(r)$. Then $q^\gamma - r^\gamma \subseteq \pi_{\mathrm{mc}(r),\gamma}{}''a$ where $a \subseteq q^{\mathrm{mc}(r)} - r^{\mathrm{mc}}$ maximal permitted to r^γ and $p^\gamma - q^\gamma \in \pi_{\mathrm{mc}(q),\gamma}{}''b$ where $b \subseteq p^{\mathrm{mc}(q)} - q^{\mathrm{mc}}$ maximal permitted to q^γ. So we have

$$p^\gamma - r^\gamma = (p^\gamma - q^\gamma)\cup(q^\gamma - r^\gamma) =$$
$$\pi_{\mathrm{mc}(q),\gamma}(a)\cup\pi_{\mathrm{mc}(r),\gamma}(b) = \pi_{\mathrm{mc}(r),\gamma}(\pi_{\mathrm{mc}(q),\mathrm{mc}(r)}(a)\cup b)$$

(5) We need to show that $T^p \le T^{r}_{p^{\mathrm{mc}(r)}-r^{\mathrm{mc}}}$. So:

$$T^p \le T^q_{p^{\mathrm{mc}(q)}-q^{\mathrm{mc}}} \le (T^r_{q^{\mathrm{mc}(r)}-r^{\mathrm{mc}}})_{p^{\mathrm{mc}(r)}-q^{\mathrm{mc}(r)}} = T^r_{p^{\mathrm{mc}(r)}-r^{\mathrm{mc}}}$$

(6) We need to show that $\forall\gamma\in\operatorname{supp} r,\ \forall <\nu>\in T^p$ if ν is permitted to r^γ then

$$\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(r),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(r)}(\nu))$$

So:

$$\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu)) =$$
$$\pi_{\mathrm{mc}(r),\gamma}(\pi_{\mathrm{mc}(q),\mathrm{mc}(r)}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu))) = \pi_{\mathrm{mc}(r),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(r)}(\nu))$$

◊

Definition 4.13: $p,q\in P$**.** We say that $p\le^* q$ (p is a direct extension of q) if

(1) $p\le q$.

(2) For every $\gamma\in\operatorname{supp}(q)$ $p^\gamma = q^\gamma$.◊

Lemma 4.14: Let $p\in P$ and $\alpha>\mathrm{mc}(p)$. Then there's $q\in P$ such that $q\le^* p$ and $\mathrm{mc}(q)=\alpha$.

Proof: Set $S=\pi^{-1}_{\alpha,\mathrm{mc}(p)}(T^p)$. For each $\langle\nu\rangle\in S$ define $B_\nu = \{\gamma\in\operatorname{supp}(p)\mid \nu^0 > \max(p^\gamma)^0\}$. These sets satisfy that $\nu_1^0\le\nu_2^0 \Rightarrow B_{\nu_1}\subseteq B_{\nu_2}$, $|B_\nu|\le\nu^0$, $\operatorname{supp} p = \bigcup_{<\nu>\in S} B_\nu$. Take enumeration $\operatorname{supp} p = \langle\gamma_\xi \mid \xi<\kappa\rangle$ which will satisfy $B_\nu\subseteq\{\gamma_\xi\mid\xi<\nu^0\}$. Set $A_\xi = \{\langle\nu\rangle\in S\mid \pi_{\alpha,\gamma_\xi}(\nu) = \pi_{\mathrm{mc}(p),\gamma_\xi}(\pi_{\alpha,\mathrm{mc}(p)}(\nu))\}$. For each $\xi, A_\xi\in U_\alpha$. Set $A = \Delta^0_{\xi<\kappa} A_\xi$ and so $A\in U_\alpha$. Shrink S to A. Now, suppose $\langle\nu\rangle\in S$ is permitted to p^γ. Then there's $\xi<\nu^0$ such that $\gamma=\gamma_\xi$. As $\nu\in A_\xi$ we have that $\pi_{\alpha,\gamma_\xi}(\nu) = \pi_{\mathrm{mc}(p),\gamma_\xi}(\pi_{\alpha,\mathrm{mc}(p)}(\nu))$. Find now a t such that $t^0=p^0$ and define

$$q = (p - \{\langle 0,p^0,f^p,f^{p^*},h^p,h^{p^*}\rangle,\langle\mathrm{mc}(p),p^{\mathrm{mc}},T^p,F^p,H^p\rangle\})\cup$$
$$\{\langle 0,p^0,f^p,f^{p^*}\circ\pi_{\alpha,\mathrm{mc}(p)},h^p,h^{p^*}\rangle,$$

$\langle \mathrm{mc}(p), p^{\mathrm{mc}}\rangle, \langle \alpha, t, S, F^p \circ \pi_{\alpha,\mathrm{mc}(p)}, H^p\rangle\}$

Then $q \le^* p$ as requested.◊

Convention: From now on whenever T will be a U_α-tree of a condition the meaning of $\pi^{-1}_{\beta,\alpha}(T)$ will be the shrunken tree as constructed above.

Lemma 4.15: Let $p \in P$ and $\beta \in A$. Then there's $q \in P$ such that $q \le^* p$ and $\beta \in \mathrm{supp}\, q$.

Proof: If $\beta \in \mathrm{supp}\, p$ then take $q = p$.

If $\beta \notin \mathrm{supp}\, p$ and $\beta < \mathrm{mc}(p)$ then take $q = p \cup \{\langle \beta, p^{\mathrm{mc}}\rangle\}$.

Otherwise take $\alpha > \beta, \mathrm{mc}(p)$ and using previous lemma find $q \le^* p$ with $\alpha = \mathrm{mc}(q)$ and now we can insert β into the support.◊

Lemma 4.16: $\langle P, \le\rangle$ has the κ^{++}-c.c.

Proof: Take $\langle p_\xi | \xi < \kappa^{++}\rangle \subseteq P$. Set $d_\xi = \mathrm{supp}\, p_\xi$. Using Δ-lemma we can extract a Δ-system of size κ^{++} from $\langle d_\xi | \xi < \kappa^{++}\rangle$. Without loss of generality assume we're starting with such sequence and its' kernel is d. The number of sequences we can put on d is $(\kappa^{<\omega})^\kappa = \kappa^+$. So now we can assume that $\forall \xi_1, \xi_2 < \kappa^{++}\ \forall \gamma \in d\ \ p^\gamma_{\xi_1} = p^\gamma_{\xi_2}$. As the number of possible f, h is κ we can also assume that $\forall \xi_1, \xi_2 < \kappa^{++}\ \ f^{p_{\xi_1}} = f^{p_{\xi_2}}$, $h^{p_{\xi_1}} = h^{p_{\xi_2}}$. Let $p^0_\xi = \langle \tau_0, \ldots, \tau_n\rangle$, $\tau_{n+1} = \kappa$. Then $\forall \xi < \kappa^{++}$ $\langle j_1(f^{p_\xi *})(\mathrm{mc}(p_\xi)), j_1(h^{p_\xi *})(\kappa)\rangle \in (\mathrm{Col}(\tau_n^{+b(n)}, <\kappa) \times \prod_{1\le i\le b(n)} \mathrm{C}(\tau_n^{+i}, \tau_{n+e(n,i)}^{+d(n,i)}) \times \mathrm{Col}(\kappa, \kappa^+))_{M_1}$. Set

$$Q = \mathrm{Col}(\tau_n^{+b(n)}, <\kappa) \times \prod_{1\le i\le b(n)} \mathrm{C}(\tau_n^{+i}, (\tau_{n+e(n,i)}^{+d(n,i)})_{M_1}) \times \mathrm{Col}(\kappa, \kappa^+)$$

Then due to $M_1 \supseteq M_1^\kappa$ we have

$$Q = (\mathrm{Col}(\tau_n^{+b(n)}, <\kappa) \times \prod_{1\le i\le b(n)} \mathrm{C}(\tau_n^{+i}, \tau_{n+e(n,i)}^{+d(n,i)}) \times \mathrm{Col}(\kappa, \kappa^+))_{M_1}$$

Q has the κ^{++}-c.c. so we must have ξ_1, ξ_2 such that $\langle j_1(f^{p_{\xi_1} *})(\mathrm{mc}(p_{\xi_1})), j_1(h^{p_{\xi_1} *})(\kappa)\rangle | \ \langle j_1(f^{p_{\xi_2} *})(\mathrm{mc}(p_{\xi_2})), j_1(h^{p_{\xi_2} *})(\kappa)\rangle$. The trees and the functions on them are compatible in all conditions so we have that $p_{\xi_1} \parallel p_{\xi_2}$.◊

Lemma 4.17: $\lambda < \kappa$, $n < \omega$ and $\forall \xi < \lambda\ F^\xi$ is a function with $\mathrm{dom}\, F^\xi = T^\xi$ which is a U_{α_ξ}-tree such that $\forall t \in T^\xi\ \ j_2(F^\xi_t)(\alpha_\xi, j_1(\alpha_\xi)) \in I_{n+|t|+1}$. Then there's F a function with dom $F = T$ which is a U_α-tree satisfying $j_2(F_t)(\alpha_\xi, j_1(\alpha_\xi)) \in I_{n+|t|+1}$ such that $\forall \xi < \lambda$ $F \leq F_\xi$.

Proof: First take β_0 such that $\forall \xi < \lambda$ $\beta_0 > \alpha_\xi$, then set $S^0 = \bigcap_{\xi<\lambda} \pi^{-1}_{\beta,\alpha_\xi}(T_\xi)$. The proof will be done by induction on the levels of S^0. Let S^i be a U_{β_i}-tree and $t \in S^i$, $|t| = i$. As $I_{n+|t|+1}$ leaves in a κ^+-closed forcing we can find $q \in I_{n+|t|+1}$ such that $\forall \xi < \lambda$ $q \leq j_2(F^\xi_{\pi_{\beta_i,\alpha_\xi}(t)})(\alpha_\xi, j_1(\alpha_\xi))$. Hence there're $f_t, \beta_t \geq \beta_i$ such that $q = j_2(f_t)(\beta_t, j_1(\beta_t))$. Then

$$A_t = \bigcap_{\xi<\lambda}\{\langle \nu_1, \nu_2\rangle \mid f_t(\nu_1,\nu_2) \leq F^\xi_{\pi_{\beta_t,\alpha_\xi}(t)}(\pi_{\beta_t,\alpha_\xi}(\nu_1), \pi_{\beta_t,\alpha_\xi}(\nu_2))\} \in U_{\beta_t}$$

. Pick now $\beta_{i+1} \geq \beta_i$ such that $\forall t \in S^i$ with $|t| = i$ we have $\beta_{i+1} \geq \beta_t$. Set $S' = \pi^{-1}_{\beta_{i+1},\beta_i}(S^i)$. Now we set $S^{i+1} | [\kappa]^i = S' | [\kappa]^i$ and $S^{i+1}_t = S'_t \cap \pi^{-1}_{\beta_{i+1},\beta_t}(A_t)$. Set $F^{i+1} | [\kappa]^i = F^i \circ \pi_{\beta_{i+1},\beta_i} | [\kappa]^i$ $\forall \langle \nu_1, \nu_2 \rangle \in S^{i+1}_t$ `. When the induction terminates we have $\langle \beta_i, S^i, F^i | i < \omega\rangle$. Pick α such that $\forall i < \omega$ $\alpha \geq \beta_i$ and set $T' = \bigcap_{i<\omega} \pi^{-1}_{\alpha,\beta_i}(S^i)$. Let $B_t = \bigcap_{\xi<\lambda} \{\langle \nu_1, \nu_2\rangle | F^{|t|+1}(\nu_1, \nu_2) \leq F^\xi_{\pi_{\alpha,\alpha_\xi}(t)}(\pi_{\alpha,\alpha_\xi}(\nu_1), \pi_{\alpha,\alpha_\xi}(\nu_2))\}$. Build by induction on levels $T_t = T'_t \cap B_t$ and set $F_t(\nu_1, \nu_2) = F^{|t|+1}_{\pi_{\alpha,\beta_{|t|+1}}(t)}(\pi_{\alpha,\beta_{|t|+1}}(\nu_1), \pi_{\alpha,\beta_{|t|+1}}(\nu_2))$.◊

Definition 4.18: Let $p \in P$. Then $P/p = \{q \in P \mid q \leq p\}$. If $\tau \in p^0$ then $(P/p)^{\geq \tau^{+k}}$ is P/p from which we dropped the f, h and parts of f^* working below τ^{+k} and $(P/p)^{<\tau^{+k}}$ are the dropped elements.

Note 4.19: $P/p = (P/p)^{<\tau} \times (P/p)^{\geq \tau}$.

Lemma 4.20: If $p \in P$ and $\tau \in p^0$ then $\langle (P/p)^{\geq \tau^{+k}}, \leq^* \rangle$ is τ^{+k}-closed and for $k \neq 1$ $(P/p)^{<\tau^{+k}}$ has τ^{+k}-c.c.

Proof: Obvious.◊

Definition 4.21: $P_n = \{p \in P \mid |p^0| \geq n\}$

Note 4.22: $P_n \in V^n$.

Definition 4.23: $p \in P$, $\gamma \geq \mathrm{mc}(p)$, $t \in [\kappa]^{<\omega}$ ^{0} -increasing

$$(p)_{\rangle\gamma,t\langle} \overset{def}{=} \{\langle \xi, p^\xi {}^\wedge s\rangle \mid \xi \in \mathrm{supp}\, p, s \subseteq \pi_{\gamma,\xi}(t) \text{ maximal permitted to } p^\xi\}$$

Claim 4.24: P satisfies Prikry condition.

Proof: The proof idea is to move on lots of possible extensions of u looking for those which decides σ and then combining all of them to one condition. That is if $\langle \nu_1 \rangle \in T^u$ then we search for a condition stronger than u with ν_1^0 as the second element in the normal Prikry sequence. The search is done on all the f's as we have relatively few of them. On the other hand we have too much of the f^* so we will move only on a maximal anti-chain there. Maximal anti-chains for h^* are too long so the h^* will be built as a monotonic decreasing sequence as they are closed enough and then using a denseness argument we'll find h^* in the generic. When we have such a condition which decides σ we accumulate its' Prikry sequences. All this will be done using 3 nested inductions. The first induction will be $\xi < \kappa$ and will move on most of the elements of $\mathrm{Lev}_0(T^u)$. The second induction will be on $\zeta < \zeta_\xi$. It will move on all Levy and Cohen functions which can appear when the Prikry sequence on the 0 coordinate is $\langle \nu_0^0, \mu_\xi^0 \rangle$ and on all the sequences of projections of ν to previous existing coordinates in the support. The last induction is on $\rho < \rho_\zeta$. The value of ρ_ζ isn't known before this step is started. We pick f_ρ^* and make sure that the sequence is anti-chain. As the forcing the f_ρ^* belongs to has the κ-c.c. we will reach a point when we won't be able to pick another f_ρ^*. The length of the maximal anti-chain constructed will be ρ_ζ. We make sure to throw away from the tree points which are below the length of the anti chains in order to avoid illegal conditions.

In what follows we use the following convention: If we have $\langle F^\zeta, T^\zeta, \beta_\zeta | \zeta < \zeta_0 \rangle$ then by writing $\forall \zeta < \zeta_0 \;\; F \le F^\zeta,\ \beta \ge \beta_\zeta$ and $T = \bigcap_{\zeta < \zeta_0} \pi^{-1}_{\beta, \beta_\zeta}(T^\zeta)$ we will mean that we got the F using lemma 4.17, that the picked β is larger than the filter $\operatorname{dom} F$ is using and that $T \le \operatorname{dom} F$.

Let $u \cup \{\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u*} \rangle, \langle h_0^{u*} \rangle \rangle, \langle \alpha, \langle \nu_0^\alpha \rangle, T^u, F^u, H^u \rangle\} \in P$ and σ a statement in the forcing language. It'll be clear from the proof that if we had chosen a condition with longer Prikry sequences (i.e. we would have $\langle \nu_0^0, \ldots, \nu_n^0 \rangle, \langle f_0, \ldots, f_{n-1}, f_n^* \rangle, \langle h_0, \ldots, h_{n-1}, h_n^* \rangle$) the only thing which would happen is that we would have to drag this sequence instead of just $\langle \nu_0^0\rangle,\langle f^{u^*}\rangle,\langle h^{u^*}\rangle$ along the proof. The proof will be done with 2 lemmas. Roughly speaking, in the first lemma we will find a direct extension $p\leq^* u$ such that if there's $\sigma \| q\leq p$ then the minimal enlargement of p which can accommodate the collapsing part of q also decides σ. In the second lemma we will find $p'\leq^* p$ such that if $q\leq p'$, $q\not\leq^* p'$ then q won't decide σ, which is a contradiction.

Lemma 4.24.1: There are p,β,S,F,H such that

1. We have

$$p\cup\{\langle 0,\langle \nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta,\langle\nu_0^\beta\rangle,S,F,H\rangle\}\leq^*$$
$$u\cup\{0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\alpha,\langle\nu_0^\alpha\rangle,T^u,F^u,H^u\rangle\}$$

2. If there are $q,\delta,\langle\nu_0^\delta,\dots,\nu_n^\delta\rangle,\langle f_0,\dots,f_{n-1},f_n'^*\rangle,\langle h_0,\dots,h_{n-1},h_n'^*\rangle,T',F',H'$ such that

$$\sigma\,\|\!-\, q\cup\{\langle 0,\langle\nu_0^0,\dots,\nu_n^0\rangle,\langle f_0,\dots,f_{n-1},f_n'^*\rangle,\langle h_0,\dots h_{n-1},h_n'^*\rangle,$$
$$\langle\delta,\langle\nu_0^\delta,\dots,\nu_n^\delta\rangle,T',F',H'\rangle\}\leq$$
$$p\cup\{0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta,\langle\nu_0^\beta\rangle,S,F,H\rangle\}$$

then $\exists f_n''^*\leq f_n'^*$, $\forall\delta'\geq\delta$, $\forall\langle\nu_0^\varepsilon,\dots\nu_n^\varepsilon\rangle$ we have

$$(p)_{\langle\beta,<\nu_0^\beta,\dots\nu_n^\beta>\rangle}\cup$$
$$\{\langle 0,\langle\nu_0^0,\dots,\nu_n^0\rangle,\langle f_0,\dots,f_{n-1},f_n''^*\circ\pi_{\delta',\delta}\rangle,\langle h_0,\dots h_{n-1}\rangle,H_{<\nu_0^0,\dots,\nu_n^0>}(-)\rangle\rangle,$$
$$\langle\beta,\langle\nu_0^\beta,\dots,\nu_n^\beta\rangle\rangle,$$
$$\langle\delta',\langle\nu_0^{\delta'},\dots,\nu_n^{\delta'}\rangle,\pi_{\delta',\beta}^{-1}S,F_{\langle\nu_0^\beta,\dots\nu_n^\beta\rangle}\circ\pi_{\delta',\beta},H_{\langle\nu_0^0,\dots\nu_n^0\rangle}\rangle\}\|\!-\sigma$$

Note: The values we choose to $\langle\nu_0^{\delta'},\dots\nu_n^{\delta'}\rangle$ can be anything. (As long as it is a condition)

Proof: We'll shrink T^u so that $\pi_{\alpha,0}T^u$ will contain only inaccessibles. Set ρ to be a well ordering of $T^u|[\kappa]$ such that $\mu_1\prec\mu_2\Rightarrow\mu_1^0\leq\mu_2^0$.

Work in V^2:

Pick $\tilde{h}$ such that $j_2(\tilde{h})(\kappa,\kappa_1)\leq j_2(H^u)(\kappa,\kappa_1)$.
Take $\mu_0^\alpha=\min^\rho T^u|[\kappa]$ and set $u_0=u$.

Let $\langle\langle f_0^{0,\zeta},h_0^{0,\zeta}\rangle|\zeta<\zeta_0\rangle$ be enumeration of conditions from

$$[\mathrm{Col}(\nu_0^{0+b(0)},<\mu_0^0)\times\prod_{1\le j\le b(0)}\mathrm{C}(\nu_0^{0+j},\mu_{0+e(0,j)}^{0+d(0,j)})]\times\mathrm{Col}(\mu_0^0,\mu_0^{0+})$$

which are stronger than $\langle f_0^{u^*}(\mu_0^\alpha),h_0^{u^*}(\mu_0^0)\rangle$.

Pick $f_1\in(\mathrm{Col}(\mu_0^{0+b(1)},<\kappa)\times\prod_{1\le j\le b(1)}\mathrm{C}(\mu_0^{0+j},\mu_{0+e(1,j)}^{+d(1,j)}))_{M_2}$ (Note that when $e(1,j)=1$ by $\mu_{0+e(1,j)}$ we mean κ), $h_1\in(\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ such that $f_1\le j_2(F^u_{\langle\mu_0^\alpha\rangle})(\alpha)$, $h_1\le j_2(h^u)(\mu_0^0,\kappa)$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'')=f_1$, $j_2(h_1''^*)(\kappa)=h_1$. Now set

$$S''=\pi^{-1}_{\beta'',\alpha}T^u_{\langle\mu_0^\alpha\rangle}\qquad F''=F^u_{\langle\mu_0^\alpha\rangle}\circ\pi_{\beta'',\alpha}\qquad H''=H^u_{\langle\mu_0^0\rangle}.$$

If there's

$$\sigma\Vdash q\cup\{\langle 0,\langle\nu_0^0,\mu_0^0\rangle,\langle f_0^{0,0},f_1'^*\rangle,\langle h_0^{0,0},h_1'^*\rangle\rangle,\langle\beta',\langle\nu_0^{\beta'},\mu_0^{\beta'}\rangle,S',F',H'\rangle\}\le^*$$
$$(u_0)_{\langle\alpha,\langle\mu_0^\alpha\rangle\rangle}\cup\{\langle 0,\langle\nu_0^0,\mu_0^0\rangle,\langle f_0^{0,0},f_1''^*\rangle,\langle h_0^{0,0},h_1''^*\rangle,\langle\beta'',\langle\nu_0^{\beta''},\mu_0^{\beta''}\rangle,S'',F'',H''\rangle\}$$

then

$$p_{0,0,0}=q-(u_0)_{\langle\alpha,\langle\mu_0^\alpha\rangle\rangle}\qquad f_1^{0,0,0^*}=f_1'^*\qquad h_1^{0,0,0^*}=h_1'^*\qquad S^{0,0,0}=S'\qquad F^{0,0,0}=F'$$
$$H^{0,0,0}=H'\qquad\beta_{0,0,0}=\beta'$$

else

$$p_{0,0,0}=\emptyset\quad f_1^{0,0,0^*}=f_1''^*\qquad h_1^{0,0,0^*}=h_1''^*\qquad S^{0,0,0}=S''\qquad F^{0,0,0}=F''$$
$$H^{0,0,0}=H''\qquad\beta_{0,0,0}=\beta''$$

Now suppose we have $\langle p_{0,0,\bar\rho},S^{0,0,\bar\rho},F^{0,0,\bar\rho},H^{0,0,\bar\rho},f_1^{0,0,\bar\rho^*},h_1^{0,0,\bar\rho^*},\beta_{0,0,\bar\rho}\mid\bar\rho<\rho\rangle$. By construction $\langle j_2(f_1^{0,0,\bar\rho^*})(\beta_{0,0,\bar\rho})\mid\bar\rho<\rho\rangle$ is an anti-chain and $\langle j_2(h_1^{0,0,\bar\rho^*})(\kappa)\mid\bar\rho<\rho\rangle$ is a decreasing sequence. If the anti-chain is maximal then the induction on ρ is finished. So suppose it's not a maximal anti-chain.

Pick $f_1\in(\mathrm{Col}(\mu_0^{0+b(1)},<\kappa)\times\prod_{1\le j\le b(1)}\mathrm{C}(\mu_0^{0+j},\mu_{0+e(1,j)}^{+d(1,j)}))_{M_2}$, $f_1\le j_2(F^u_{\langle\mu_0^\alpha\rangle})(\alpha)$ which is incompatible with $\langle j_2(f_1^{0,0,\bar\rho^*})(\beta_{0,0,\bar\rho})\mid\bar\rho<\rho\rangle$ and $h_1\in(\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{0,0,\bar\rho^*})(\kappa)\mid\bar\rho<\rho\rangle$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'')=f_1$, $j_2(h_1''^*)(\kappa)=h_1$.

If $\rho=\bar\rho+1$ then set

$$S''=\pi^{-1}_{\beta'',\beta_{0,0,\bar\rho}}S^{0,0,\bar\rho}\quad F''=F^{0,0,\bar\rho}\circ\pi_{\beta'',\beta_{0,0,\bar\rho}}\qquad H''=H^{0,0,\bar\rho}\qquad p''=p_{0,0,\bar\rho}.$$

Otherwise

$$S''=\bigcap_{\bar\rho<\rho}\pi^{-1}_{\beta'',\beta_{0,0,\bar\rho}}S^{0,0,\bar\rho}\qquad\forall\bar\rho<\rho\quad F''\le F^{0,0,\bar\rho}\circ\pi_{\beta'',\beta_{0,0,\bar\rho}}\quad\forall\bar\rho<\rho\quad H''\le H^{0,0,\bar\rho}$$
$$p''=\bigcup_{\bar\rho<\rho}p_{0,0,\bar\rho}.$$

This last union might cause a problem for large enough ρ. Namely

$$p''\cup(u_0)_{\langle\alpha,<\mu_0^\alpha\rangle\rangle}\cup\{\langle 0,\langle\nu_0^0,\mu_0^0\rangle,\langle f_0^{0,0},f_1''^*\rangle,\langle h_0^{0,0},h_1''^*\rangle\rangle,\langle\beta'',\langle\nu_0^{\beta''},\mu_0^{\beta''}\rangle,S'',F'',H''\rangle\}$$

might not be a condition. (in p'' there are too many coordinates which can be enlarged together). The solution is to shrink S'' so all these continuations won't be possible. Let's set

$C_{0,0,\rho} = \{\eta \text{ inaccessible} | \rho > \eta > \mu_0^0\}$. This set is bounded hence $C_{0,0,\rho} \notin U_0$ so we can shrink S'' to $S'' - \pi^{-1}_{\beta'',0}(C_{0,0,\rho})$.
If there's

$$\sigma \| \; q \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,0}, f_1'^{*}\rangle, \langle h_0^{0,0}, h_1'^{*}\rangle\rangle, \langle \beta', \langle \nu_0^{\beta}, \mu_0^{\beta}\rangle, S', F', H'\rangle\} \leq^{*}$$
$$p'' \cup (u_0)_{\langle \alpha, <\mu_0^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,0}, f_1''^{*}\rangle, \langle h_0^{0,0}, h_1''^{*}\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_0^{\beta''}\rangle, S'', F'', H''\rangle\}$$

then

$$p_{0,0,\rho} = q - (u_0)_{\langle \alpha, \langle \mu_0^\alpha\rangle\rangle} \quad f_1^{0,0,\rho^*} = f_1'^{*} \quad h_1^{0,0,\rho^*} = h_1'^{*} \quad S^{0,0,\rho} = S' \quad F^{0,0,\rho} = F'$$
$$H^{0,0,\rho} = H' \quad \beta_{0,0,\rho} = \beta'$$

else

$$p_{0,0,\rho} = p'' \; f_1^{0,0,\rho^*} = f_1''^{*} \quad h_1^{0,0,\rho^*} = h_1''^{*} \quad S^{0,0,\rho} = S'' \quad F^{0,0,\rho} = F''$$
$$H^{0,0,\rho} = H'' \quad \beta_{0,0,\rho} = \beta''$$

When the induction is finished we have $\langle j_2(f_1^{0,0,\rho^*})(\beta_{0,0,\rho}) | \rho < \rho_0\rangle$ a maximal anti-chain below $j_2(F^u_{\langle \mu_0^\alpha\rangle})(\alpha)$ and $\langle j_2(h_1^{0,0,\rho^*})(\kappa) | \rho < \rho_0\rangle$ a decreasing sequence.

We'll continue with the general case. Assume $\langle j_2(f_1^{0,\bar\zeta,\rho^*})(\beta_{0,\bar\zeta,\rho}) | \rho < \rho_{\bar\zeta}\rangle$ is a maximal anti-chain below $j_2(F^u_{<\mu_0^\alpha>})(\alpha)$ for all $\bar\zeta < \zeta$ and $\langle j_2(h_1^{0,\bar\zeta,\rho^*})(\kappa) | \rho < \rho_{\bar\zeta}, \bar\zeta < \zeta\rangle$ is a decreasing sequence.

Pick $f_1 \in (\mathrm{Col}(\mu_0^{0+b(1)}, <\kappa) \times \prod_{1 \leq j \leq b(1)} \mathrm{C}(\mu_0^{0+j}, \mu_{0+e(1,j)}^{+d(1,j)}))_{M_2}$, $f_1 \leq j_2(F^u_{\langle \mu_0^\alpha\rangle})(\alpha)$ and $h_1 \in (\mathrm{Col}(\kappa, \kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{0,\bar\zeta,\rho^*})(\kappa) | \rho < \rho_{\bar\zeta}, \bar\zeta < \zeta\rangle$ and β'', $f_1''^{*}$, $h_1''^{*}$ such that $j_2(f_1''^{*})(\beta'') = f_1$, $j_2(h_1''^{*})(\kappa) = h_1$.
Set

$$S'' = \bigcap_{\substack{\bar\zeta<\zeta \\ \rho<\rho_{\bar\zeta}}} \pi^{-1}_{\beta'',\beta_{0,\bar\zeta,\rho}} S^{0,\bar\zeta,\rho} \quad \forall \bar\zeta < \zeta \;\; \rho < \rho_{\bar\zeta} \quad F'' \leq F^{0,\bar\zeta,\rho} \circ \pi_{\beta'',\beta_{0,\bar\zeta,\rho}}$$
$$\forall \bar\zeta < \zeta \;\; \rho < \rho_{\bar\zeta} \quad H'' \leq H^{0,\bar\zeta,\rho}$$
$$p'' = \bigcup_{\substack{\bar\zeta<\zeta \\ \rho<\rho_{\bar\zeta}}} p_{0,\bar\zeta,\rho} \qquad C_{0,\zeta} = \{\eta \text{ inaccessible} | \zeta > \eta > \mu_0^0\}$$

and shrink S'' to $S'' - \pi^{-1}_{\beta'',0} C_{0,\zeta}$.

If there's

$$\sigma \| \; q \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,\zeta}, f_1'^{*}\rangle, \langle h_0^{0,\zeta}, h_1'^{*}\rangle\rangle, \langle \beta', \langle \nu_0^{\beta}, \mu_0^{\beta}\rangle, S', F', H'\rangle\} \leq^{*}$$
$$p'' \cup (u_0)_{\langle \alpha, <\mu_0^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,\zeta}, f_1''^{*}\rangle, \langle h_0^{0,\zeta}, h_1''^{*}\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_0^{\beta''}\rangle, S'', F'', H''\rangle\}$$

then

$$p_{0,\zeta,0} = q - (u_0)_{\langle \alpha, \langle \mu_0^\alpha\rangle\rangle} \quad f_1^{0,\zeta,0^*} = f_1'^{*} \quad h_1^{0,\zeta,0^*} = h_1'^{*} \quad S^{0,\zeta,0} = S' \quad F^{0,\zeta,0} = F'$$
$$H^{0,\zeta,0} = H' \quad \beta_{0,\zeta,0} = \beta'$$

else

$$p_{0,\zeta,0} = p'' \; f_1^{0,\zeta,0^*} = f_1''^* \quad h_1^{0,\zeta,0^*} = h_1''^* \quad S^{0,\zeta,0} = S'' \quad F^{0,\zeta,0} = F''$$
$$H^{0,\zeta,0} = H'' \quad \beta_{0,\zeta,0} = \beta''$$

Now suppose we have $\langle p_{0,\zeta,\bar\rho}, S^{0,\zeta,\bar\rho}, F^{0,\zeta,\bar\rho}, H^{0,\zeta,\bar\rho}, f_1^{0,\zeta,\bar\rho^*}, h_1^{0,\zeta,\bar\rho^*}, \beta_{0,\zeta,\bar\rho} | \bar\rho < \rho\rangle$. By construction $\langle j_2(f_1^{0,\zeta,\bar\rho^*})(\beta_{0,\zeta,\bar\rho}) | \bar\rho < \rho\rangle$ is an anti-chain and $\langle j_2(h_1^{0,\zeta,\bar\rho^*})(\kappa) | \bar\rho < \rho\rangle$ is a decreasing sequence. If the anti-chain is maximal below $j_2(F^u_{<\mu_0^\alpha>})(\alpha)$ then the induction on ρ is finished. So suppose it's not a maximal anti-chain.
Pick $f_1 \in (\mathrm{Col}(\mu_0^{0+b(1)}, <\kappa) \times \prod_{1\le j\le b(1)} \mathrm{C}(\mu_0^{0+j}, \mu_{0+e(1,j)}^{+d(1,j)}))_{M_2}$, $f_1 \le j_2(F^u_{\langle\mu_0^\alpha\rangle})(\alpha)$ which is incompatible with $\langle j_2(f_1^{0,\zeta,\bar\rho^*})(\beta_{0,\zeta,\bar\rho}) | \bar\rho < \rho\rangle$ and $h_1 \in (\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{0,\zeta,\bar\rho^*})(\kappa) | \bar\rho < \rho\rangle$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'') = f_1$, $j_2(h_1''^*)(\kappa) = h_1$.
If $\rho = \bar\rho + 1$ then set

$$S'' = \pi^{-1}_{\beta'',\beta_{0,\zeta,\bar\rho}} S^{0,\zeta,\bar\rho} \quad F'' = F^{0,\zeta,\bar\rho} \circ \pi_{\beta'',\beta_{0,\zeta,\bar\rho}} \quad H'' = H^{0,\zeta,\bar\rho} \quad p'' = p_{0,\zeta,\bar\rho}.$$

Otherwise

$$S'' = \bigcap_{\bar\rho<\rho} \pi^{-1}_{\beta'',\beta_{0,\zeta,\bar\rho}} S^{0,\zeta,\bar\rho} \qquad \forall\bar\rho<\rho \quad F'' \le F^{0,\zeta,\bar\rho} \circ \pi_{\beta'',\beta_{0,\zeta,\bar\rho}} \quad \forall\bar\rho<\rho \quad H'' \le H^{0,\zeta,\bar\rho}$$
$$p'' = \bigcup_{\bar\rho<\rho} p_{0,\zeta,\bar\rho}.$$

This last union might cause a problem for large enough ρ. Namely

$$p'' \cup (u_0)_{\langle\alpha,\langle\mu_0^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,\zeta}, f_1''^*\rangle, \langle h_0^{0,\zeta}, h_1''^*\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_0^{\beta''}\rangle, S'', F'', H''\rangle\}$$

might not be a condition. (in p'' there are too many coordinates which can be enlarged together). The solution is to shrink S'' so all these continuations won't be possible. Let's set
$C_{0,\zeta,\rho} = \{\eta \text{ inaccessible} | \rho > \eta > \mu_0^0\}$. This set is bounded hence $C_{0,\zeta,\rho} \notin U_0$ so we can shrink S'' to $S'' - \pi^{-1}_{\beta'',0}(C_{0,\zeta,\rho})$.
If there's

$$\sigma \Vdash q \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,\zeta}, f_1'^*\rangle, \langle h_0^{0,\zeta}, h_1'^*\rangle\rangle, \langle \beta', \langle \nu_0^{\beta'}, \mu_0^{\beta'}\rangle, S', F', H'\rangle\} \le^*$$
$$p'' \cup (u_0)_{\langle\alpha,\langle\mu_0^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_0^0\rangle, \langle f_0^{0,\zeta}, f_1''^*\rangle, \langle h_0^{0,\zeta}, h_1''^*\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_0^{\beta''}\rangle, S'', F'', H''\rangle\}$$

then

$$p_{0,\zeta,\rho} = q - (u_0)_{\langle\alpha,\langle\mu_0^\alpha\rangle\rangle} \quad f_1^{0,\zeta,\rho^*} = f_1'^* \quad h_1^{0,\zeta,\rho^*} = h_1'^* \quad S^{0,\zeta,\rho} = S' \quad F^{0,\zeta,\rho} = F'$$
$$H^{0,\zeta,\rho} = H' \quad \beta_{0,\zeta,\rho} = \beta'$$

else

$$p_{0,\zeta,\rho} = p'' \; f_1^{0,\zeta,\rho^*} = f_1''^* \quad h_1^{0,\zeta,\rho^*} = h_1''^* \quad S^{0,\zeta,\rho} = S'' \quad F^{0,\zeta,\rho} = F''$$
$$H^{0,\zeta,\rho} = H'' \quad \beta_{0,\zeta,\rho} = \beta''$$

When the induction on ρ is finished we have $\langle j_2(f_1^{0,\zeta,\rho^*})(\beta_{0,\zeta,\rho}) | \rho < \rho_\zeta \rangle$ a maximal anti-chain below $j_2(F^u_{\langle \mu_0^\alpha \rangle})(\alpha)$ and $\langle j_2(h_1^{0,\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta \rangle$ a decreasing sequence.

When the induction on ζ is finished we have $\langle j_2(f_1^{0,\zeta,\rho^*})(\beta_{0,\zeta,\rho}) | \rho < \rho_\zeta \rangle$ a maximal anti-chain below $j_2(F^u_{<\mu_0^\alpha>})(\alpha)$ for all $\zeta < \zeta_0$ and $\langle j_2(h_1^{0,\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta, \zeta < \zeta_0 \rangle$ a decreasing sequence.

We'll continue with the general case. We have $\langle \langle f_0^{0,\zeta}, h_0^{0,\zeta} \rangle | \zeta < \zeta_0 \rangle$ a maximal anti-chain below $j_2(F^u_{\langle \mu_0^\alpha \rangle})(\alpha)$ for all **Error! Objects cannot be created from editing field codes.** for all **Error! Objects cannot be created from editing field codes.** and $\langle j_2(h_1^{\bar{\xi},\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta, \zeta < \zeta_{\bar{\xi}} \rangle$ a decreasing sequence for all $\bar{\xi} < \xi$.

If $\xi = \bar{\xi} + 1$ then set

$$u_\xi = u_{\bar{\xi}} \cup \bigcup_{\zeta < \zeta_{\bar{\xi}}, \rho < \rho_\zeta} p_{\bar{\xi},\zeta,\rho},\ C_\xi = \bigcup_{\zeta < \zeta_{\bar{\xi}}} C_{\bar{\xi},\zeta} \cup \bigcup_{\zeta < \zeta_{\bar{\xi}}, \rho < \rho_\zeta} C_{\bar{\xi},\zeta,\rho}.$$

Else set

$$u_\xi = \bigcup_{\bar{\xi} < \xi} u_{\bar{\xi}},\ C_\xi = \bigcup_{\bar{\xi} < \xi} C_{\bar{\xi}}.$$

Pick $\mu_\xi^\alpha = \min^{\prec} \{ \mu \in T^u \mid [\kappa] - \pi^{-1}_{\alpha,0} C_\xi | \forall \bar{\xi} < \xi\, \mu \neq \mu_{\bar{\xi}}^\alpha \}$.

Let $\langle \langle f_0^{\xi,\zeta}, h_0^{\xi,\zeta} > | \zeta < \zeta_\xi \rangle$ be enumeration of conditions from $[\mathrm{Col}(\nu_0^{0+b(0)}, < \mu_\xi^0) \times \prod_{1 \le j \le b(0)} \mathrm{C}(\nu_0^{0+j}, \mu^{0+d(0,j)}_{\xi+e(0,j)})] \times \mathrm{Col}(\mu_\xi^0, \mu_\xi^{0+})$

which are stronger than $\langle f_0^{u^*}(\mu_\xi^\alpha), h_0^{u^*}(\mu_\xi^0) \rangle$.

Pick $f_1 \in (\mathrm{Col}(\mu_\xi^{0+b(1)}, < \kappa) \times \prod_{1 \le j \le b(1)} \mathrm{C}(\mu_0^{0+j}, \mu^{+d(1,j)}_{\xi+e(1,j)}))_{M_2}$ (Here also by $\mu_{\xi+e(1,j)}$ when $e(1,j) = 1$ we mean κ), $h_1 \in (\mathrm{Col}(\kappa, \kappa^+))_{M_2}$ such that $f_1 \le j_2(F^u_{\langle \mu_\xi^\alpha \rangle>})(\alpha)$, $h_1 \le j_2(h)(\mu_\xi^0, \kappa)$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'') = f_1$, $j_2(h_1''^*)(\kappa) = h_1$. Now set

$$S'' = \pi^{-1}_{\beta'',\alpha} T^u_{<\mu_\xi^\alpha>} - \pi^{-1}_{\beta'',0} C_\xi \quad F'' = F^u_{<\mu_\xi^\alpha>} \circ \pi_{\beta'',\alpha} \qquad H'' = H^u_{\langle \mu_\xi^0 \rangle}$$

If there's

$$\sigma \Vdash q \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0 \rangle, \langle f_0^{\xi,0}, f_1'^* \rangle, \langle h_0^{\xi,0}, h_1'^* \rangle \rangle, \langle \beta', \langle \nu_0^{\beta'}, \mu_\xi^{\beta'} \rangle, S', F', H' \rangle \} \le^* (u_\xi)_{\langle \alpha, \langle \mu_\xi^\alpha \rangle \rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0 \rangle, \langle f_0^{\xi,0}, f_1''^* \rangle, \langle h_0^{\xi,0}, h_1''^* \rangle \rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''} \rangle, S'', F'', H'' \rangle \}$$

then

$$p_{\xi,0,0} = q - (u_\xi)_{\langle \alpha, \langle \mu_\xi^\alpha \rangle \rangle} \qquad f_1^{\xi,0,0^*} = f_1'^* \quad h_1^{\xi,0,0^*} = h_1'^* \quad S^{\xi,0,0} = S' \quad F^{\xi,0,0} = F' \quad H^{\xi,0,0} = H' \quad \beta_{\xi,0,0} = \beta'$$

else

$$p_{\xi,0,0} = \emptyset \quad f_1^{\xi,0,0^*} = f_1''^* \quad h_1^{\xi,0,0^*} = h_1''^* \quad S^{\xi,0,0} = S'' \quad F^{\xi,0,0} = F'' \quad H^{\xi,0,0} = H'' \quad \beta_{\xi,0,0} = \beta''$$

Now suppose we have $\langle p_{\xi,0,\bar{\rho}}, S^{\xi,0,\bar{\rho}}, F^{\xi,0,\bar{\rho}}, H^{\xi,0,\bar{\rho}}, f_1^{\xi,0,\bar{\rho}^*}, h_1^{\xi,0,\bar{\rho}^*}, \beta_{\xi,0,\bar{\rho}} | \bar{\rho} < \rho \rangle$. By construction $\langle j_2(f_1^{\xi,0,\bar{\rho}^*})(\beta_{\xi,0,\bar{\rho}}) | \bar{\rho} < \rho \rangle$ is an anti-chain and $\langle j_2(h_1^{\xi,0,\bar{\rho}^*})(\kappa) | \bar{\rho} < \rho \rangle$ is a

decreasing sequence. If the anti-chain is maximal then the induction on ρ is finished. So suppose it's not a maximal anti-chain.
Pick $f_1 \in (\mathrm{Col}(\mu_\xi^{0+b(1)}, <\kappa) \times \prod_{1\le j\le b(1)} \mathrm{C}(\mu_0^{0+j}, \mu_{\xi+e(1,j)}^{+d(1,j)}))_{M_2}$ such that $f_1 \le j_2(F^u_{\langle\mu_\xi^\alpha\rangle})(\alpha)$ which is incompatible with $\langle j_2(f_1^{\xi,0,\bar\rho^*})(\beta_{\xi,0,\bar\rho}) | \bar\rho<\rho\rangle$ and $h_1 \in (\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{\xi,0,\bar\rho^*})(\kappa) | \bar\rho<\rho\rangle$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'') = f_1$, $j_2(h_1''^*)(\kappa) = h_1$.
If $\rho = \bar\rho + 1$ then set

$$S'' = \pi^{-1}_{\beta'',\beta_{\xi,0,\bar\rho}} S^{\xi,0,\bar\rho} \quad F'' = F^{\xi,0,\bar\rho} \circ \pi_{\beta'',\beta_{\xi,0,\bar\rho}} \quad H'' = H^{\xi,0,\bar\rho} \quad p'' = p_{\xi,0,\bar\rho}.$$

Otherwise

$$S'' = \bigcap_{\bar\rho<\rho} \pi^{-1}_{\beta'',\beta_{\xi,0,\bar\rho}} S^{\xi,0,\bar\rho} \quad \forall\bar\rho<\rho \;\; F'' \le F^{\xi,0,\bar\rho} \circ \pi_{\beta'',\beta_{\xi,0,\bar\rho}} \quad \forall\bar\rho<\rho \;\; H'' \le H^{\xi,0,\bar\rho}$$

$$p'' = \bigcup_{\bar\rho<\rho} p_{\xi,0,\bar\rho}.$$

This last union might cause a problem for large enough ρ. Namely

$$p'' \cup (u_\xi)_{\langle\alpha,\mu_\xi^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0\rangle, \langle f_0^{\xi,0}, f_1''^*\rangle, \langle h_0^{\xi,0}, h_1''^*\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''}\rangle, S'', F'', H''\rangle\}$$

might not be a condition. (in p'' there are too many coordinates which can be enlarged together). The solution is to shrink S'' so all these continuations won't be possible. Let's set
$C_{\xi,0,\rho} = \{\eta \text{ inaccessible} | \rho > \eta > \mu_\xi^0\}$. This set is bounded hence $C_{\xi,0,\rho} \notin U_0$ so we can shrink S'' to $S'' - \pi^{-1}_{\beta'',0}(C_{\xi,0,\rho})$.
If there's

$$\sigma \Vert q \cup \{\langle 0, \nu_0^0, \mu_\xi^0\rangle, \langle f_0^{\xi,0}, f_1'^*\rangle, \langle h_0^{\xi,0}, h_1'^*\rangle\rangle, \langle \beta', \langle \nu_0^{\beta'}, \mu_\xi^{\beta'}\rangle, S', F', H'\rangle\} \le^*$$
$$p'' \cup (u_\xi)_{\langle\alpha,\langle\mu_\xi^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0\rangle, \langle f_0^{\xi,0}, f_1''^*\rangle, \langle h_0^{\xi,0}, h_1''^*\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''}\langle, S'', F'', H''\rangle\}$$

then

$$p_{\xi,0,\rho} = q - (u_\xi)_{\langle\alpha,\langle\mu_\xi^\alpha\rangle\rangle} \quad f_1^{\xi,0,\rho^*} = f_1'^* \quad h_1^{\xi,0,\rho^*} = h_1'^* \quad S^{\xi,0,\rho} = S' \quad F^{\xi,0,\rho} = F'$$
$$H^{\xi,0,\rho} = H' \quad \beta_{\xi,0,\rho} = \beta'$$

else

$$p_{\xi,0,\rho} = p'' \quad f_1^{\xi,0,\rho^*} = f_1''^* \quad h_1^{\xi,0,\rho^*} = h_1''^* \quad S^{\xi,0,\rho} = S'' \quad F^{\xi,0,\rho} = F''$$
$$H^{\xi,0,\rho} = H'' \quad \beta_{\xi,0,\rho} = \beta''$$

When the induction is finished we have $\langle j_2(f_1^{\xi,0,\rho^*})(\beta_{\xi,0,\rho}) | \rho<\rho_0\rangle$ a maximal anti-chain below $j_2(F^u_{<\mu_\xi^\alpha>})(\alpha)$ and $\langle j_2(h_1^{\xi,0,\rho^*})(\kappa) | \rho<\rho_0\rangle$ is a decreasing sequence.

We'll continue with the general case. Assume $\langle j_2(f_1^{\xi,\bar\zeta,\rho^*})(\beta_{\xi,\bar\zeta,\rho}) | \rho<\rho_{\bar\zeta}\rangle$ is a maximal anti-chain below $j_2(F^u_{\langle\mu_\xi^\alpha\rangle})(\alpha)$ for all $\bar\zeta<\zeta$ and $\langle j_2(h_1^{\xi,\bar\zeta,\rho^*})(\kappa) | \rho<\rho_{\bar\zeta}, \bar\zeta<\zeta\rangle$ is a decreasing sequence.

Pick $f_1 \in (\mathrm{Col}(\mu_\xi^{0+b(1)}, <\kappa) \times \prod_{1\le j\le b(1)} \mathrm{C}(\mu_0^{0+j}, \mu_{\xi+e(1,j)}^{+d(1,j)}))_{M_2}$, $f_1 \le j_2(F^u_{\langle \mu_\xi^\alpha\rangle})(\alpha)$ and $h_1 \in (\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{\xi,\bar\zeta,\rho^*})(\kappa) | \rho < \rho_{\bar\zeta}, \bar\zeta < \zeta\rangle$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'') = f_1$, $j_2(h_1''^*)(\kappa) = h_1$.
Set

$$S'' = \bigcap_{\substack{\bar\zeta<\zeta\\ \rho<\rho_{\bar\zeta}}} \pi^{-1}_{\beta'',\beta_{\xi,\bar\zeta,\rho}} S^{\xi,\bar\zeta,\rho} \quad \forall\bar\zeta<\zeta\ \ \rho<\rho_{\bar\zeta} \quad F'' \le F^{\xi,\bar\zeta,\rho} \circ \pi_{\beta'',\beta_{\xi,\bar\zeta,\rho}}$$

$$\forall\bar\zeta<\zeta\ \ \rho<\rho_{\bar\zeta} \quad H'' \le H^{\xi,\bar\zeta,\rho}$$

$$p'' = \bigcup_{\substack{\bar\zeta<\zeta\\ \rho<\rho_{\bar\zeta}}} p_{\xi,\bar\zeta,\rho} \qquad C_{\xi,\zeta} = \{\eta \text{ inaccessible} | \rho > \eta > \mu_\xi^0\}$$

and shrink S'' to $S'' - \pi^{-1}_{\beta'',0} C_{\xi,\zeta}$.

If there's

$$\sigma \| q \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0\rangle, \langle f_0^{\xi,\zeta}, f_1'^*\rangle, \langle h_0^{\xi,\zeta}, h_1'^*\rangle\rangle, \langle \beta', \langle \nu_0^{\beta'}, \mu_\xi^{\beta'}\rangle, S', F', H'\rangle\} \le^*$$
$$p'' \cup (u_\xi)_{\langle \alpha, \langle \mu_\xi^\alpha\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0\rangle, \langle f_0^{\xi,\zeta}, f_1''^*\rangle, \langle h_0^{\xi,\zeta}, h_1''^*\rangle\rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''}\rangle, S'', F'', H''\rangle\}$$

then

$$p_{\xi,\zeta,0} = q - (u_\xi)_{\langle\alpha,\langle\mu_\xi^\alpha\rangle\rangle} \qquad f_1^{\xi,\zeta,0^*} = f_1'^* \qquad h_1^{\xi,\zeta,0^*} = h_1'^* \qquad S^{\xi,\zeta,0} = S' \qquad F^{\xi,\zeta,0} = F'$$
$$H^{\xi,\zeta,0} = H' \qquad \beta_{\xi,\zeta,0} = \beta'$$

else

$$p_{\xi,\zeta,0} = p'' \ f_1^{\xi,\zeta,0^*} = f_1''^* \qquad h_1^{\xi,\zeta,0^*} = h_1''^* \qquad S^{\xi,\zeta,0} = S'' \qquad F^{\xi,\zeta,0} = F''$$
$$H^{\xi,\zeta,0} = H'' \qquad \beta_{\xi,\zeta,0} = \beta''$$

Now suppose we have $\langle p_{\xi,\zeta,\bar\rho}, S^{\xi,\zeta,\bar\rho}, F^{\xi,\zeta,\bar\rho}, H^{\xi,\zeta,\bar\rho}, f_1^{\xi,\zeta,\bar\rho^*}, h_1^{\xi,\zeta,\bar\rho^*}, \beta_{\xi,\zeta,\bar\rho} | \bar\rho < \rho\rangle$. By construction $\langle j_2(f_1^{\xi,\zeta,\bar\rho^*})(\beta_{\xi,\zeta,\bar\rho}) | \bar\rho<\rho\rangle$ is an anti-chain and $\langle j_2(h_1^{\xi,\zeta,\bar\rho^*})(\kappa) | \bar\rho < \rho\rangle$ is a decreasing sequence. If the anti-chain is maximal below $j_2(F^u_{\langle\mu_\xi^\alpha\rangle})(\alpha)$ then the induction on ρ is finished. So suppose it's not a maximal anti-chain.
Pick $f_1 \in (\mathrm{Col}(\mu_\xi^{0+b(1)}, <\kappa) \times \prod_{1\le j\le b(1)} \mathrm{C}(\mu_0^{0+j}, \mu_{\xi+e(1,j)}^{+d(1,j)}))_{M_2}$, $f_1 \le j_2(F^u_{\langle\mu_\xi^\alpha\rangle})(\alpha)$ which is incompatible with $\langle j_2(f_1^{\xi,\zeta,\bar\rho^*})(\beta_{\xi,\zeta,\bar\rho}) | \bar\rho<\rho\rangle$ and $h_1 \in (\mathrm{Col}(\kappa,\kappa^+))_{M_2}$ which is stronger than $\langle j_2(h_1^{\xi,\zeta,\bar\rho^*})(\kappa) | \bar\rho<\rho\rangle$ and β'', $f_1''^*$, $h_1''^*$ such that $j_2(f_1''^*)(\beta'') = f_1$, $j_2(h_1''^*)(\kappa) = h_1$.
If $\rho = \bar\rho + 1$ then set

$$S'' = \pi^{-1}_{\beta'',\beta_{\xi,\zeta,\bar\rho}} S^{\xi,\zeta,\bar\rho} \ \ F'' = F^{\xi,\zeta,\bar\rho} \circ \pi_{\beta'',\beta_{\xi,\zeta,\bar\rho}} \qquad H'' = H^{\xi,\zeta,\bar\rho} \qquad p'' = p_{\xi,\zeta,\bar\rho}.$$

Otherwise

$$S'' = \bigcap_{\bar\rho<\rho} \pi^{-1}_{\beta'',\beta_{\xi,\zeta,\bar\rho}} S^{\xi,\zeta,\bar\rho} \qquad \forall\bar\rho<\rho \quad F'' \le F^{\xi,\zeta,\bar\rho} \circ \pi_{\beta'',\beta_{\xi,\zeta,\bar\rho}} \quad \forall\bar\rho<\rho \quad H'' \le H^{\xi,\zeta,\bar\rho}$$

$$p'' = \bigcup_{\bar\rho<\rho} p_{\xi,\zeta,\bar\rho}.$$

This last union might cause a problem for large enough ρ. Namely

$$p'' \cup (u)_{\langle \alpha, \langle \mu_\xi^\alpha \rangle \rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0 \rangle, \langle f_0^{\xi,\zeta}, f_1''^* \rangle, \langle h_0^{\xi,\zeta}, h_1''^* \rangle \rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''} \rangle, S'', F'', H'' \rangle \}$$

might not be a condition. (in p'' there are too many coordinates which can be enlarged together). The solution is to shrink S'' so all these continuations won't be possible. Let's set

$C_{\xi,\zeta,\rho} = \{\eta \text{ inaccessible} | \rho > \eta > \mu_\xi^0\}$. This set is bounded hence $C_{\xi,\zeta,\rho} \notin U_0$ so we can shrink S'' to $S'' - \pi_{\beta'',0}^{-1}(C_{\xi,\zeta,\rho})$.

If there's

$$\sigma \| q \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0 \rangle, \langle f_0^{\xi,\zeta}, f_1'^* \rangle, \langle h_0^{\xi,\zeta}, h_1'^* \rangle \rangle, \langle \beta', \langle \nu_0^{\beta'}, \mu_\xi^{\beta'} \rangle, S', F', H' \rangle \} \le^*$$
$$p'' \cup (u_\xi)_{\langle \alpha, \langle \mu_\xi^\alpha \rangle \rangle} \cup \{\langle 0, \langle \nu_0^0, \mu_\xi^0 \rangle, \langle f_0^{\xi,\zeta}, f_1''^* \rangle, \langle h_0^{\xi,\zeta}, h_1''^* \rangle \rangle, \langle \beta'', \langle \nu_0^{\beta''}, \mu_\xi^{\beta''} \rangle, S'', F'', H'' \rangle \}$$

then

$$p_{\xi,\zeta,\rho} = q - (u_\xi)_{\langle \alpha, \langle \mu_\xi^\alpha \rangle \rangle} \qquad f_1^{\xi,\zeta,\rho^*} = f_1'^* \qquad h_1^{\xi,\zeta,\rho^*} = h_1'^* \qquad S^{\xi,\zeta,\rho} = S' \qquad F^{\xi,\zeta,\rho} = F'$$
$$H^{\xi,\zeta,\rho} = H' \qquad \beta_{\xi,\zeta,\rho} = \beta'$$

else

$$p_{\xi,\zeta,\rho} = p'' \qquad f_1^{\xi,\zeta,\rho^*} = f_1''^* \qquad f_1^{\xi,\zeta,\rho^*} = h_1''^* \qquad S^{\xi,\zeta,\rho} = S'' \qquad F^{\xi,\zeta,\rho} = F''$$
$$H^{\xi,\zeta,\rho} = H'' \qquad \beta_{\xi,\zeta,\rho} = \beta''$$

When the induction on ρ is finished we have $\langle j_2(f_1^{\xi,\zeta,\rho^*})(\beta_{\xi,\zeta,\rho}) | \rho < \rho_\zeta \rangle$ a maximal anti-chain below $j_2(F^u_{<\mu_\xi^\alpha>})(\alpha)$ and $\langle j_2(h_1^{\xi,\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta \rangle$ a decreasing sequence.

When the induction on ζ is finished we have $\langle j_2(f_1^{\xi,\zeta,\rho^*})(\beta_{\xi,\zeta,\rho}) | \rho < \rho_\zeta \rangle$ is a maximal anti-chain below $j_2(F^u_{\langle \mu_\xi^\alpha \rangle})(\alpha)$ for all $\zeta < \zeta_\xi$ and $\langle j_2(h_1^{\xi,\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta,\ \zeta < \zeta_\xi \rangle$ a decreasing sequence.

When the induction on ξ is finished we have $\langle j_2(f_1^{\xi,\zeta,\rho^*})(\beta_{\xi,\zeta,\rho}) | \rho < \rho_\zeta \rangle$ is a maximal anti-chain below $j_2(F^u_{\langle \mu_\xi^\alpha \rangle})(\alpha)$ for all $\zeta < \zeta_\xi$ for all $\xi < \kappa$ and $\langle j_2(h_1^{\xi,\zeta,\rho^*})(\kappa) | \rho < \rho_\zeta,\ \zeta < \zeta_\xi \rangle$ a decreasing sequence for all $\xi < \kappa$.

Set $C = \{\mu_\xi^0 | \xi < \kappa\}$.

We claim that $C \in U_0$. If not we'll define a regressive function on $R: \pi_{\alpha,0}(T^u | [\kappa]) - C \to \kappa$ as follows: By our construction if $\mu \in \pi_{\alpha,0}(T^u | [\kappa]) - C$ then there's a unique $\xi < \kappa$ such that $\mu_\xi^0 < \mu < \mu_{\xi+1}^0$ so set $R(\mu) = \mu_\xi^0$. Hence there's $\xi < \kappa$ and $A \subseteq \pi_{\alpha,0}(T^u | [\kappa]) - C$, $A \in U_0$ such that $\forall \mu \in A \ \ R(\mu) = \mu_\xi^0$. This can happen if $\zeta_\xi = \kappa$ or if there's $\zeta < \zeta_\xi$ such that $\rho_\zeta = \kappa$. By construction neither of this cases can happen so he have a contradiction.

Let's set

$$p_1 = \bigcup_{\xi < \kappa} u_\xi \qquad S'^1 = \pi_{\beta_1,\alpha}^{-1} T^u \cap \pi_{\beta_1,0}^{-1} C$$

$S^1 | [\kappa] = S'^1 | [\kappa]$

$S^1_{\langle\nu\rangle} = S'^1_{\langle\nu\rangle} \cap \bigcap_{\substack{\zeta<\zeta_\xi \\ \rho<\rho_\zeta}} \pi^{-1}_{\beta_1,\beta_{\xi,\zeta,\rho}} S^{\xi,\zeta,\rho}$ where $\pi_{\beta_1,\alpha}(\nu) = \mu^\alpha_\xi$.

$F^1(\nu_1,\nu_2) = F^u \circ \pi_{\beta_1,\alpha}(\nu_1,\nu_2)$

$F^1_{\langle\nu_1\rangle}(\nu_2,\dots,\nu_n) \le F^{\xi,\zeta,\rho} \circ \pi_{\beta_1,\beta_{\xi,\zeta,\rho}}(\nu_2,\dots,\nu_n)$ where $\pi_{\beta_1,\alpha}(\nu_1) = \mu^\alpha_\xi$.

$H^1_{\langle\nu^0_1\rangle}(\nu^0_2,\dots,\nu^0_l) \le H^{\xi,\zeta,\rho}(\nu^0_2,\dots,\nu^0_l)$ where $\nu^0_1 = \pi_{\alpha,0}(\mu^\alpha_\xi)$.

We can't just take $H^1(\nu^0_1,\nu^0_2) \le h_1^{\xi,\zeta,\rho^*}(\nu^0_2)$ where $\nu^0_1 = \pi_{\alpha,0}(\mu^0_\xi)$ as we need $j_2(H^1)(\kappa,\kappa_1)$ to be in the generic.

So we'll show what we have gained for a close enough condition.

Take $\bar{h}(\nu^0_1,\nu^0_2) \le h_1^{\xi,\zeta,\rho^*}(\nu^0_2)$ where $\nu^0_1 = \pi_{\alpha,0}(\mu^\alpha_\xi)$.

Take $\langle \nu^{\beta_1}_1 \rangle \in S^1$, so there's $\xi < \kappa$ such that $\mu^\alpha_\xi = \pi_{\beta_1,\alpha}(\nu^{\beta_1}_1)$. Assume

$$\begin{aligned}\sigma \Vdash q \cup \{\langle 0, \langle \nu^0_0, \nu^0_1\rangle, \langle f'_0, f_1'^*\rangle, \langle h'_0, h_1'^*\rangle\rangle, \langle \beta', \langle \nu^{\beta'}_0, \nu^{\beta'}_1\rangle, S', F', H'\rangle\} \le \\ (p_1)_{\langle\beta_1,\langle\nu^{\beta_1}_1\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \nu^0_1\rangle, \langle f^{u^*}_0(\mu^\alpha_\xi), F^1_{\langle\nu^{\beta_1}_1\rangle}\rangle, \langle h^{u^*}_0(\nu^0_1), \bar{h}(\nu^0_1)\rangle\rangle, \\ \langle \beta_1, \langle \nu^{\beta_1}_0, \nu^{\beta_1}_1\rangle, S^1_{\langle\nu^{\beta_1}_1\rangle}, F^1_{\langle\nu^{\beta_1}_1\rangle}, H^1_{\langle\nu^0_1\rangle}\rangle\}\end{aligned}$$

By definition we have $f'_0 \le f^{u^*}_0(\mu^\alpha_\xi)$, $h'_0 \le h^{u^*}_0(\nu^0_1)$, $f_1'^* \le F^1_{<\nu^{\beta_1}_1>} \circ \pi_{\beta',\beta_1}$, $h_1'^* \le \bar{h}(\nu^0_1)$.

By construction we can find $\zeta < \zeta_\xi$, $\rho < \rho_\zeta$ such that $\langle f'_0, h'_0\rangle = \langle f_0^{\xi,\zeta}, h_0^{\xi,\zeta}\rangle$ and $j_2(f_1'^*)(\beta') \Vert j_2(f_1^{\xi,\zeta,\rho^*})(\beta_{\xi,\zeta,\rho})$. Hence we can find $f_1''^* \le f_1'^*, f_1^{\xi,\zeta,\rho^*}$. As $h_1'^* \le \bar{h}_{\langle\nu^0_1\rangle}$ and $\bar{h}_{\langle\nu^0_1\rangle} \le h_1^{\xi,\zeta,\rho^*}$ we get that $h_1'^* \le h_1^{\xi,\zeta,\rho^*}$. So we get

$$\begin{aligned}\sigma \Vdash q \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1''^*\rangle, \langle h_0^{\xi,\zeta}, h_1'^*\rangle\rangle, \langle \beta', \langle \nu^{\beta'}_0, \nu^{\beta'}_1\rangle, S', F', H'\rangle\} \le^* \\ (p_1)_{\langle\beta_1,\langle\nu^{\beta_1}_1\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1^{\xi,\zeta,\rho^*} \circ \pi_{\beta_1,\beta_{\xi\zeta\rho}}\rangle, \langle h_0^{\xi,\zeta}, h_1^{\xi,\zeta,\rho^*}\rangle\rangle, \\ \langle \beta_1, \langle \nu^{\beta_1}_0, \nu^{\beta_1}_1\rangle, S^1_{\langle\nu^{\beta_1}_1\rangle}, F^1_{\langle\nu^{\beta_1}_1\rangle}, H^1_{\langle\nu^0_1\rangle}\rangle\} \le^* \\ p_{\xi,\zeta,\rho} \cup (u_\xi)_{\langle\beta_1,\langle\nu^{\beta_1}_1\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1^{\xi,\zeta,\rho^*}\rangle, \langle h_0^{\xi,\zeta}, h_1^{\xi,\zeta,\rho^*}\rangle\rangle, \\ \langle \beta_{\xi,\zeta,\rho}, \langle \nu_0^{\beta_{\xi\zeta\rho}}, \mu_\xi^{\beta_{\xi\zeta\rho}}\rangle, S^{\xi,\zeta,\rho}, F^{\xi,\zeta,\rho}, H^{\xi,\zeta,\rho}\rangle\}\end{aligned}$$

By noting that $(u_\xi)_{\langle\beta_1,\langle\nu^{\beta_1}_1\rangle\rangle} = (u_\xi)_{\langle\alpha,\langle\mu^\alpha_\xi\rangle\rangle}$ we get that

$$\begin{aligned}\sigma \Vdash q \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1''^*\rangle, \langle h_0^{\xi,\zeta}, h_1'^*\rangle\rangle, \langle \beta', \langle \nu^{\beta'}_0, \nu^{\beta'}_1\rangle, S', F', H'\rangle\} \le^* \\ p_{\xi,\zeta,\rho} \cup (u_\xi)_{\langle\alpha,<\mu^\alpha_\xi\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1^{\xi,\zeta,\rho^*}\rangle, \langle h_0^{\xi,\zeta}, h_1^{\xi,\zeta,\rho^*}\rangle\rangle, \\ \langle \beta_{\xi,\zeta,\rho}, \langle \nu_0^{\beta_{\xi\zeta\rho}}, \mu_\xi^{\beta_{\xi\zeta\rho}}\rangle, S^{\xi,\zeta,\rho}, F^{\xi,\zeta,\rho}, H^{\xi,\zeta,\rho}\rangle\}\end{aligned}$$

hence by construction

$$\begin{aligned}p_{\xi,\zeta,\rho} \cup (u_\xi)_{\langle\alpha,<\mu^\alpha_\xi\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1^{\xi,\zeta,\rho^*}\rangle, \langle h_0^{\xi,\zeta}, h_1^{\xi,\zeta,\rho^*}\rangle\rangle, \\ \langle \beta_{\xi,\zeta,\rho}, \langle \nu_0^{\beta_{\xi\zeta\rho}}, \mu_\xi^{\beta_{\xi\zeta\rho}}\rangle, S^{\xi,\zeta,\rho}, F^{\xi,\zeta,\rho}, H^{\xi,\zeta,\rho}\rangle\} \Vert \sigma\end{aligned}$$

which give us

$$\begin{aligned}(p_1)_{\langle\beta_1,\langle\nu^{\beta_1}_1\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_\xi\rangle, \langle f_0^{\xi,\zeta}, f_1^{\xi,\zeta,\rho^*} \circ \pi_{\beta_1,\beta_{\xi\zeta\rho}}\rangle, \langle h_0^{\xi,\zeta}, \bar{h}_{\langle\nu^0_1\rangle}\rangle\rangle, \\ \langle \beta_1, \langle \nu^{\beta_1}_0, \nu^{\beta_1}_1\rangle, S^1_{\langle\nu^{\beta_1}_1\rangle}, F^1_{\langle\nu^{\beta_1}_1\rangle}, H^1_{\langle\nu^0_1\rangle}\rangle\} \Vert \sigma\end{aligned}$$

by setting $f_1'''^{*} = f_1^{\xi,\zeta,\rho^*} \circ \pi_{\beta_1,\beta_{\xi,\zeta,\rho}}$ we get

$$(p_1)_{\langle\beta_1,\nu_1^{\beta_1}\rangle} \cup \{\langle 0,\langle \nu_0^0,\nu_1^0\rangle,\langle f_0',f_1'''^{*}\rangle,\langle h_1',\overline{h}_{\langle\nu_1^0\rangle}\rangle\rangle, \langle\beta_1,\langle \nu_0^{\beta_1},\nu_1^{\beta_1}\rangle,S^1_{\langle\nu_1^{\beta_1}\rangle},F^1_{\langle\nu_1^{\beta_1}\rangle},H^1_{\langle\nu_1^1\rangle}\rangle\} \Vdash \sigma$$

We'll summarize what we proved:

For any $j_2(\widetilde{h})(\kappa,\kappa_1) \le j_2(H^u)(\kappa,\kappa_1)$ we can find $j_2(\overline{h})(\kappa,\kappa_1) \le j_2(\widetilde{h})(\kappa,\kappa_1)$, p_1, S^1, F^1, H^1 such that if

$$\sigma \Vdash q \cup \{\langle 0,\langle\nu_0^0,\nu_1^0\rangle,\langle f_0',f_1'^{*}\rangle,\langle h_0',h_1'^{*}\rangle\rangle,\langle\beta',\langle\nu_0^{\beta'},\nu_1^{\beta'}\rangle,S',F',H'\rangle\} \le$$
$$(p_1)_{\langle\beta_1,\nu_1^{\beta_1}\rangle} \cup \{\langle 0,\langle\nu_0^0,\nu_1^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta_1,\alpha}(\nu_1^{\beta_1}),F^1_{\langle\nu_1^{\beta_1}\rangle}\rangle,\langle h_0^{u^*}(\nu_1^0),\overline{h}(\nu_1^0)\rangle\rangle, \langle\beta_1,\langle\nu_0^{\beta_1},\nu_1^{\beta_1}\rangle,S^1_{\langle\nu_1^{\beta_1}\rangle},F^1_{\langle\nu_1^{\beta_1}\rangle},H^1_{\langle\nu_1^0\rangle}\rangle\}$$

then there's $f_1'''^{*} \Vdash f_1'^{*}$ such that

$$(p_1)_{\langle\beta_1,\nu_1^{\beta_1}\rangle} \cup \{\langle 0,\langle\nu_0^0,\nu_1^0\rangle,\langle f_0',f_1'''^{*}\rangle,\langle h_1',\overline{h}(\nu_1^0)\rangle\rangle, \langle\beta_1,\langle\nu_0^{\beta_1},\nu_1^{\beta_1}\rangle,S^1_{\langle\nu_1^{\beta_1}\rangle},F^1_{\langle\nu_1^{\beta_1}\rangle},H^1_{\langle\nu_1^0\rangle}\rangle\} \Vdash \sigma.$$

As we've been working in V^2 this means we can take the $i_2(\overline{h})(\kappa,i(\kappa))$ from the generic filter J_1'. So now we can set $H^1(\nu_1^0,\nu_2^0) = \overline{h}(\nu_1^0,\nu_2^0)$ and when we go back to V^1 we have: If

$$\sigma \Vdash q \cup \{\langle 0,\langle \nu_0^0,\nu_1^0\rangle,\langle f_0',f_1'^{*}\rangle,\langle h_0',h_1'^{*}\rangle\rangle,\langle\beta',\langle\nu_0^{\beta'},\nu_1^{\beta'}\rangle,S',F',H'\rangle\} \le$$
$$p_1 \cup \{\langle 0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta_1,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta_1,\langle\nu_0^{\beta_1}\rangle,S^1,F^1,H^1\rangle\}$$

then there's $f_1'''^{*} \Vdash f_1'^{*}$ such that

$$(p_1)_{\langle\beta_1,\langle\nu_1^{\beta_1}\rangle\rangle} \cup \{\langle 0,\langle\nu_0^0,\nu_1^0\rangle,\langle f_0',f_1'''^{*}\rangle,\langle h_0',H^1_{<\nu_1^0>}\rangle\rangle, \langle\beta_1,\langle\nu_0^{\beta_1},\nu_1^{\beta_1}\rangle,S^1_{\langle\nu_1^{\beta_1}\rangle},F^1_{\langle\nu_1^{\beta_1}\rangle},H^1_{\langle\nu_1^0\rangle}\rangle\} \Vdash \sigma.$$

and therefore there's $f_1''^{*} \le f_1'^{*}, f_1'''^{*}$ such that $\forall\beta'' \ge \beta'$

$$(p_1)_{\langle\beta_1,\langle\nu_1^{\beta_1}\rangle\rangle} \cup \{\langle 0,\langle\nu_0^0,\nu_1^0\rangle,\langle f_0',f_1''^{*}\circ\pi_{\beta'',\beta'}\rangle,\langle h_0',H^1_{<\nu_1^0>}\rangle\rangle, \langle\beta_1,\langle\nu_0^{\beta_1},\nu_1^{\beta_1}\rangle\rangle, \langle\beta'',\langle\nu_0^{\beta''},\nu_1^{\beta''}\rangle,\pi^{-1}_{\beta'',\beta_1}S^1_{\langle\nu_1^{\beta_1}\rangle},F^1_{\langle\nu_1^{\beta_1}\rangle}\circ\pi_{\beta'',\beta_1},H^1_{\langle\nu_1^0\rangle}\rangle\} \Vdash \sigma.$$

With this we finished with the 1st level.

We'll continue into the higher levels. As the proof for the 2nd level is a degenerate version of the higher levels we'll show how to continue with the 3rd level. So we're assuming that we have

$$p_2 \cup \{\langle 0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta_2,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta_2,\langle\nu_0^{\beta_2}\rangle,S^2,F^2,H^2\rangle\} \le^*$$
$$p_1 \cup \{\langle 0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta_1,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,<\beta_1,<\nu_0^{\beta_1}>,S^1,F^1,H^1>\}$$

such that if

$$\sigma \Vdash q \cup \{\langle 0,\langle\nu_0^0,\nu_1^0,\nu_2^0\rangle,\langle f_0',f_1',f_2'^{*}\rangle,\langle h_0',h_1',h_2'^{*}\rangle\rangle,\langle\beta',\langle\nu_0^{\beta'},\nu_1^{\beta'},\nu_2^{\beta'}\rangle,S',F',H'\rangle\} \le$$
$$p_2 \cup \{\langle 0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta_1,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta_2,\langle\nu_0^{\beta_2}\rangle,S^2,F^2,H^2\rangle\}$$

then there's $f_2'''^{*} \Vdash f_2'^{*}$ such that

$$(p_2)_{\langle\beta_2,\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle\rangle} \cup \{\langle 0,\langle\nu_0^0,\nu_1^0,\nu_2^0\rangle,\langle f_0',f_1',f_2'''^{*}\rangle,\langle h_0',H^2_{\langle\nu_1^0\rangle},H^2_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\rangle,$$

$$\langle\beta_2,\langle\nu_0^{\beta_2},\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle,S^2_{\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle},F^2_{\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle},H^2_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\}\Vdash\sigma.$$

and therefore there's $f_2''^* \leq f_2'^*, f_2'''^*$ such that $\forall\beta''\geq\beta'$

$$(p_2)_{\langle\beta_2,\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle\rangle}\cup\{\langle 0,\langle\nu_0^0,\nu_1^0,\nu_2^0\rangle,\langle f_0',f_1',f_2^{\prime *}\circ\pi_{\beta'',\beta'}\rangle,\langle h_0',H^2_{\langle\nu_1^0\rangle},H^2_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\rangle,$$
$$\langle\beta_2,\langle\nu_0^{\beta_2},\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle\rangle,$$
$$\langle\beta'',\langle\nu_0^{\beta''},\nu_1^{\beta''},\nu_2^{\beta''}\rangle,\pi^{-1}_{\beta'',\beta_2}S^2_{\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle},F^2_{\langle\nu_1^{\beta_2},\nu_2^{\beta_2}\rangle}\circ\pi_{\beta'',\beta_2},H^2_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\}\Vdash\sigma.$$

Let's go into the 3rd level.
We'll well order $S^2|[\kappa]^2$ such that $\langle\nu_{0,1},\nu_{0,2}\rangle \prec \langle\nu_{1,1},\nu_{1,2}\rangle\Rightarrow\nu_{0,2}^0\leq\nu_{1,2}^0$.
Set $\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle=\min^{\prec}S^2|[\kappa]^2$. Consider the condition

$$(p_2)_{\langle\beta_2,\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle\rangle}\cup\{\langle 0,\langle\nu_0^0,\mu_{0,1}^0,\mu_{0,2}^0\rangle,$$
$$\langle f_0^{u*}\circ\pi_{\beta_2,\alpha}(\mu_{0,1}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2}\rangle}(\mu_{0,2}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle}\rangle,$$
$$\langle h_0^{u*}(\mu_{0,1}^0),H^2_{\langle\mu_{0,1}^0\rangle}(\mu_{0,2}^0),H^2_{\langle\mu_{0,1}^0,\mu_{0,2}^0\rangle}\rangle\rangle,$$
$$\langle\beta_2,\langle\nu_0^{\beta_2},\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle,S^2_{\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle},F^2_{\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle},H^2_{\langle\mu_{0,1}^0,\mu_{0,2}^0\rangle}\rangle\}$$

This condition is in P_3. Starting with this condition, working in V^4 and doing the same work as we have done for the 1st level we can find a direction extension of it

$$p'\cup\{\langle 0,\langle\nu_0^0,\mu_{0,1}^0,\mu_{0,2}^0\rangle,$$
$$\langle f_0^{u*}\circ\pi_{\beta_2,\alpha}(\mu_{0,1}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2}\rangle}(\mu_{0,2}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle}\circ\pi_{\gamma_0,\beta_2}\rangle,$$
$$\langle h_0^{u*}(\mu_{0,1}^0),H^2_{\langle\mu_{0,1}^0\rangle}(\mu_{0,2}^0),H^2_{\langle\mu_{0,1}^0,\mu_{0,2}^0\rangle}\rangle\rangle,$$
$$\langle\gamma_0,\langle\nu_0^{\gamma_0},\mu_{0,1}^{\gamma_0},\mu_{0,2}^{\gamma_0}\rangle,S^{3,0},F^{3,0},H^{3,0}\rangle\}$$

such that if

$$\sigma\Vdash q\cup\{\langle 0,\langle\nu_0^0,\mu_{0,1}^0,\mu_{0,2}^0,\nu_3^0\rangle,$$
$$\langle f_0',f_1',f_2',f_3'^*\rangle,$$
$$\langle h_0',h_1',h_2',h_3'^*\rangle\rangle,$$
$$\langle\beta',\langle\nu_0^{\beta'},\mu_{0,1}^{\beta'},\mu_{0,2}^{\beta'},\nu_3^{\beta'}\rangle,T',F',H'\rangle\}\leq$$
$$p'\cup\{\langle 0,\langle\nu_0^0,\mu_{0,1}^0,\mu_{0,2}^0\rangle,$$
$$\langle f_0^{u*}\circ\pi_{\beta_2,\alpha}(\mu_{0,1}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2}\rangle}(\mu_{0,2}^{\beta_2}),F^2_{\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle}\circ\pi_{\gamma_0,\beta_0}\rangle,$$
$$\langle h_0^{u*}(\mu_{0,1}^0),H^2_{\langle\mu_{0,1}^0\rangle}(\mu_{0,2}^0),H^2_{\langle\mu_{0,1}^0,\mu_{0,2}^0\rangle}\rangle\rangle,$$
$$\langle\gamma_0,\langle\nu_0^{\gamma_0},\mu_{0,1}^{\gamma_0},\mu_{0,2}^{\gamma_0}\rangle,S^{3,0},F^{3,0},H^{3,0}\rangle\}$$

Then there's $f_3'''^*\Vdash f_3'^*$ such that

$$p'_{\langle\gamma_0,\langle\nu_3^{\gamma_0}\rangle\rangle}\cup\{\langle 0,\langle\nu_0^0,\mu_{0,1}^0,\mu_{0,2}^0,\nu_3^0\rangle,$$
$$\langle f_0',f_1',f_2',f_3'''^*\rangle,$$
$$\langle h_0',h_1',h_2',H^{3,0}_{\langle\nu_3^0\rangle}\rangle\rangle,$$
$$\langle\gamma_0,\langle\nu_0^{\gamma_0},\mu_{0,1}^{\gamma_0},\mu_{0,2}^{\gamma_0},\nu_3^{\gamma_0}\rangle,S^{3,0}_{\langle\nu_3^{\gamma_0}\rangle},F^{3,0}_{\langle\nu_3^{\gamma_0}\rangle},H^{3,0}_{\langle\nu_3^0\rangle}\rangle\}\Vdash\sigma$$

Set

$$p_{3,0}=p_2\cup(p'-(p_2)_{\langle\beta_2,\langle\mu_{0,1}^{\beta_2},\mu_{0,2}^{\beta_2}\rangle\rangle})$$
$$C_0=\{\langle\nu_1,\nu_2\rangle\in S^2\mid\nu_2^0>\mu_{0,2}^0\quad\nu_1\text{ or }\nu_2\text{ aren't legal for }p_{3,0}\}$$

Let's continue with the general case. Hence we have $\langle p_{3,\zeta}, \gamma_\zeta, C_\zeta, \langle \mu^{\beta_2}_{\zeta,1}, \mu^{\beta_2}_{\zeta,2}\rangle \mid \zeta < \xi\rangle$, $\xi < \kappa$.
If ξ is limit then $p'' = \bigcup_{\zeta<\xi} p_{3,\zeta}$ else $\xi = \zeta + 1$ and we set $p'' = p_{3,\zeta}$.
Set

$$C'' = \bigcup_{\zeta<\xi} C_\zeta$$
$$\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle = \min^{\mathrm{p}}(S^2 \restriction [\kappa]^2 - C'')$$
$$\beta'' \geq \gamma_\zeta \text{ for all } \zeta < \xi$$
$$S'' = \pi^{-1}_{\beta'',\beta_2}(S^2_{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle} - C'')$$
$$F'' = F^2_{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle} \circ \pi_{\beta'',\beta_2}$$
$$H'' = H^2_{\langle \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle}$$

Doing the same work we have done for the 1st level, this time in V^4 we can find

$$\begin{aligned}
&p' \cup \{\langle 0, \langle \nu^0_0, \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle,\\
&\qquad \langle f_0^{u^*} \circ \pi_{\beta_2,\alpha}(\mu^{\beta_2}_{\xi,1}), F^2_{\langle \mu^{\beta_2}_{\xi,1}\rangle}(\mu^{\beta_2}_{\xi,2}), F^2_{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle} \circ \pi_{\gamma_\xi,\beta_2}\rangle,\\
&\qquad \langle h_0^{u^*}(\mu^0_{\xi,1}), H^2_{\langle \mu^0_{\xi,1}\rangle}(\mu^0_{\xi,2}), H^2_{\langle \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle}\rangle\rangle,\\
&\quad \langle \gamma_\xi, \langle \nu_0^{\gamma_\xi}, \mu^{\gamma_\xi}_{\xi,1}, \mu^{\gamma_\xi}_{\xi,2}\rangle, T^{3,\xi}, F^{3,\xi}, H^{3,\xi}\rangle\} \leq^*\\
&(p'')_{\langle \beta_2, \langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle,\\
&\qquad \langle f_0^{u^*} \circ \pi_{\beta_2,\alpha}(\mu^{\beta_2}_{\xi,1}), F^2_{\langle \mu^{\beta_2}_{\xi,1}\rangle}(\mu^{\beta_2}_{\xi,2}), F^2_{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle}\rangle,\\
&\qquad \langle h_0^{u^*}(\mu^0_{\xi,1}), H^2_{\langle \mu^0_{\xi,1}\rangle}(\mu^0_{\xi,2}), H^2_{\langle \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle}\rangle\rangle,\\
&\quad \langle \beta'', \langle \nu_0^{\beta''}, \mu^{\beta''}_{\xi,1}, \mu^{\beta''}_{\xi,2}\rangle, S'', F'', H''\rangle\}
\end{aligned}$$

such that if

$$\begin{aligned}
&\sigma \parallel q \cup \{\langle 0, \langle \nu^0_0, \mu^0_{\xi,1}, \mu^0_{\xi,2}, \nu^0_3\rangle,\\
&\qquad \langle f'_0, f'_1, f'_2, f'^*_3\rangle,\\
&\qquad \langle h'_0, h'_1, h'_2, h'^*_3\rangle\rangle,\\
&\quad \langle \beta', \langle \nu_0^{\beta'}, \mu^{\beta'}_{0,1}, \mu^{\beta'}_{0,2}, \nu_3^{\beta'}\rangle, T', F', H'\rangle\} \leq\\
&p' \cup \{\langle 0, \langle \nu^0_0, \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle,\\
&\qquad \langle f_0^{u^*} \circ \pi_{\beta_2,\alpha}(\mu^{\beta_2}_{\xi,1}), F^2_{\langle \mu^{\beta_2}_{\xi,1}\rangle}(\mu^{\beta_2}_{\xi,2}), F^2_{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle} \circ \pi_{\gamma_\xi,\beta_2}\rangle,\\
&\qquad \langle h_0^{u^*}(\mu^0_{\xi,1}), H^2_{\langle \mu^0_{\xi,1}\rangle}(\mu^0_{\xi,2}), H^2_{\langle \mu^0_{\xi,1}, \mu^0_{\xi,2}\rangle}\rangle\rangle,\\
&\quad \langle \gamma_\xi, \langle \nu_0^{\gamma_\xi}, \mu^{\gamma_\xi}_{0,1}, \mu^{\gamma_\xi}_{0,2}\rangle, T^{3,\xi}, F^{3,\xi}, H^{3,\xi}\rangle\}
\end{aligned}$$

Then there's $f'''^*_3 \parallel f'^*_3$ such that

$$\begin{aligned}
&(p')_{\langle \gamma_\xi, \langle \nu_3^{\gamma_\xi}\rangle\rangle} \cup \{\langle 0, \langle \nu^0_0, \mu^0_{\xi,1}, \mu^0_{\xi,2}, \nu^0_3\rangle,\\
&\qquad \langle f'_0, f'_1, f'_2, f'''^*_3\rangle,\\
&\qquad \langle h'_0, h'_1, h'_2, H^{3,\xi}_{\langle \mu^0_3\rangle}\rangle\rangle,\\
&\qquad \langle \gamma_\xi, \langle \nu_0^{\gamma_\xi}, \mu^{\gamma_\xi}_{0,1}, \mu^{\gamma_\xi}_{0,2}, \nu_3^{\gamma_\xi}\rangle, S^{3,\xi}_{\langle \nu_3^{\gamma_\xi}\rangle}, F^{3,\xi}_{\langle \nu_3^{\gamma_\xi}\rangle}, H^{3,\xi}_{\langle \nu^0_3\rangle}\rangle\} \parallel \sigma
\end{aligned}$$

We set

$$p_{3,\xi} = p'' \cup (p' - (p'')_{\langle \beta_2, \langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2} \rangle\rangle})$$

$$C_\xi = \{\langle \nu_1, \nu_2 \rangle \in S^2 \mid \nu_2^0 > \mu^0_{\xi,2} \text{ and } \nu_1 \text{ or } \nu_2 \text{ aren't legal for } p_{3,\xi}\}$$

When the induction is finished we have $\langle p_{3,\xi}, \gamma_\xi, C_\xi, \langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle \mid \xi < \kappa\rangle$. We'll combine everything into one condition. Let

$$p_3 = \bigcup_{\xi<\kappa} p_{3,\xi}$$

$$\beta_3 \geq \gamma_\xi \text{ for all } \xi < \kappa$$

$$C = \pi^{-1}_{\beta_3,\beta_2} \{\langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle \mid \xi < \kappa\}$$

$$S^3 \restriction [\kappa]^2 = C$$

$$S^3_{\langle \nu_1, \nu_2\rangle} = (\pi^{-1}_{\beta_3,\beta_{3,\xi}} S^{3,\xi}) \cap C \text{ where } \pi_{\beta_3,\beta_2}(\langle \nu_1, \nu_2 \rangle) = \langle \mu^{\beta_2}_{\xi,1}, \nu^{\beta_2}_{\xi,2}\rangle$$

$$F^3 \restriction [\kappa]^2 = F^2 \restriction [\kappa]^2 \circ \pi_{\beta_3,\beta_2}$$

$$F^3_{\langle \nu_1, \nu_2 \rangle} = F^{3,\xi} \text{ where } \pi_{\beta_3,\beta_2}(\langle \nu_1, \nu_2\rangle) = \langle \mu^{\beta_2}_{\xi,1}, \nu^{\beta_2}_{\xi,2}\rangle$$

$$H^3 \restriction [\kappa]^2 = H^2 \restriction [\kappa]^2$$

$$H^3_{\langle \nu_1, \nu_2\rangle} \leq H^{3,\xi} \text{ for all } \xi \text{ such that } \langle \nu_1^0, \nu_2^0 \rangle = \langle \mu^0_{\xi,1}, \nu^0_{\xi,2}\rangle$$

We get that

$$p_3 \cup \{\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta_3,\alpha}\rangle, \langle h_0^{u^*}\rangle\rangle, \langle \beta_3, \langle \nu_0^{\beta_3}\rangle, S^3, F^3, H^3\rangle\} \leq^*$$
$$p_2 \cup \{\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta_2,\alpha}\rangle, \langle h_0^{u^*}\rangle\rangle, \langle \beta_2, \langle \nu_0^{\beta_2}\rangle, S^2, F^2, H^2\rangle\}$$

Suppose now that

$$\sigma \restriction q \cup \{\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0, \nu_3^0\rangle, \langle f_0', f_1', f_2', f_3'^*\rangle, \langle h_0', h_1', h_2', h_3'^*\rangle\rangle,$$
$$\langle \beta', \langle \nu_0^{\beta'}, \nu_1^{\beta'}, \nu_2^{\beta'}, \nu_3^{\beta'}\rangle, S', F', H'\rangle\} \leq$$
$$p_3 \cup \{\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta_3,\alpha}\rangle, \langle h_0^{u^*}\rangle\rangle, \langle \beta_3, \langle \nu_0^{\beta_3}\rangle, S^3, F^3, H^3\rangle\}$$

By definition we have

$$\sigma \restriction q \cup \{\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0, \nu_3^0\rangle, \langle f_0', f_1', f_2', f_3'^*\rangle, \langle h_0', h_1', h_2', h_3'^*\rangle\rangle,$$
$$\langle \beta', \langle \nu_0^{\beta'}, \nu_1^{\beta'}, \nu_2^{\beta'}, \nu_3^{\beta'}\rangle, S', F', H'\rangle\} \leq$$
$$(p_3)_{\langle \beta_3, \langle \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0, \nu_3^0\rangle,$$
$$\langle f_0^{u^*} \circ \pi_{\beta_3,\alpha}(\nu_1^{\beta_3}), F^3_{\langle \nu_1^{\beta_3}\rangle}(\nu_2^{\beta_3}), F^3_{\langle \nu_1^{\beta_3}, \nu_2^{\beta_3}\rangle}(\nu_3^{\beta_3}), F^3_{\langle \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle}\rangle,$$
$$\langle h_0^{u^*}(\nu_1^0), H^3_{\langle \nu_1^0\rangle}(\nu_2^0), H^3_{\langle \nu_1^0, \nu_2^0\rangle}(\nu_3^0), H^3_{\langle \nu_1^0, \nu_2^0, \nu_3^0\rangle}\rangle\rangle,$$
$$\langle \beta_3, \langle \nu_0^{\beta_3}, \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle, S^3_{\langle \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle}, F^3_{\langle \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle}, H^3_{\langle \nu_1^0, \nu_2^0, \nu_3^0\rangle}\rangle\}$$

By construction of S^3 there's $\xi < \kappa$ such that $\pi_{\beta_3,\beta_2}(\langle \nu_1^{\beta_3}, \nu_2^{\beta_3}\rangle) = \langle \mu^{\beta_2}_{\xi,1}, \mu^{\beta_2}_{\xi,2}\rangle$ hence we have

$$\sigma \restriction q \cup \{\langle 0, \langle \nu_0^0, \mu^0_{\xi,1}, \mu^0_{\xi,2}, \nu_3^0\rangle, \langle f_0', f_1', f_2', f_3'^*\rangle, \langle h_0', h_1', h_2', h_3'^*\rangle\rangle,$$
$$\langle \beta', \langle \nu_0^{\beta'}, \nu_1^{\beta'}, \nu_2^{\beta'}, \nu_3^{\beta'}\rangle, S', F', H'\rangle\} \leq$$
$$(p_3)_{\langle \beta_3, \langle \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle\rangle} \cup \{\langle 0, \langle \nu_0^0, \mu^0_{\xi,1}, \mu^0_{\xi,2}, \nu_3^0\rangle,$$
$$\langle f_0^{u^*} \circ \pi_{\beta_2,\alpha}(\mu^{\beta_2}_{\xi,1}), F^2_{\langle \mu^{\beta_2}_{\xi,1}\rangle}(\mu^{\beta_2}_{\xi,2}), F^{3,\xi}(\nu_3^{\beta_3}), F^{3,\xi}_{\langle \nu_3^{\beta_3}\rangle}\rangle,$$
$$\langle h_0^{u^*}(\mu^0_{\xi,1}), H^2_{\langle \mu^0_{\xi,1}\rangle}(\mu^0_{\xi,2}), H^{3,\xi}(\nu_3^0), H^{3,\xi}_{\langle \nu_3^0\rangle}\rangle\rangle,$$
$$\langle \beta_3, \langle \nu_0^{\beta_3}, \nu_1^{\beta_3}, \nu_2^{\beta_3}, \nu_3^{\beta_3}\rangle, S^{3,\xi}_{\langle \nu_3^{\beta_3}\rangle}, F^{3,\xi}_{\langle \nu_3^{\beta_3}\rangle}, H^{3,\xi}_{\langle \nu_3^0\rangle}\rangle\} \leq$$

$$(p_{3,\xi})_{\langle\beta_2,\langle\mu^{\beta_2}_{\xi,1},\mu^{\beta_2}_{\xi,2}\rangle\rangle}\cup\{\langle 0,\langle \nu^0_0,\mu^0_{\xi,1},\mu^0_{\xi,2}\rangle,$$
$$\langle f_0^{u^*}\circ\pi_{\beta_2,\alpha}(\mu^{\beta_2}_{\xi,1}),F^2_{\langle\mu^{\beta_2}_{\xi,1}\rangle}(\mu^{\beta_2}_{\xi,2}),F^2_{\langle\mu^{\beta_2}_{\xi,1},\mu^{\beta_2}_{\xi,2}\rangle}\circ\pi_{\gamma_\xi,\beta_2}\rangle,$$
$$\langle h_0^{u^*}(\mu^0_{\xi,1}),H^2_{\langle\mu^0_{\xi,1}\rangle}(\mu^0_{\xi,2}),H^2_{\langle\mu^0_{\xi,1},\mu^0_{\xi,2}\rangle}\rangle\rangle,$$
$$\langle\gamma_\xi,\langle\nu^{\gamma_\xi}_0,\mu^{\gamma_\xi}_{\xi,1},\mu^{\gamma_\xi}_{\xi,2}\rangle,S^{3,\xi},F^{3,\xi},H^{3,\xi}\rangle\}$$

So from the induction we get that there's $f_3'''^* \Vdash f_3'^*$ such that

$$(p_{3,\xi})_{\langle\beta_2,\langle\mu^{\beta_2}_{\xi,1},\mu^{\beta_2}_{\xi,2},\nu^{\beta_2}_3\rangle\rangle}\cup\{\langle 0,\langle \nu^0_0,\mu^0_{\xi,1},\mu^0_{\xi,2},\nu^0_3\rangle,$$
$$\langle f_0',f_1',f_2',f_3'''^*\rangle,$$
$$\langle h_0',h_1',h_2',H^{3,\xi}_{\langle\nu^0_3\rangle}\rangle\rangle,$$
$$\langle\gamma_\xi,\langle\nu^{\gamma_\xi}_0,\mu^{\gamma_\xi}_{\xi,1},\mu^{\gamma_\xi}_{\xi,2},\nu^{\gamma_\xi}_3\rangle,S^{3,\xi}_{\langle\nu^{\gamma_\xi}_3\rangle},F^{3,\xi}_{\langle\nu^{\gamma_\xi}_3\rangle},H^{3,\xi}_{\langle\nu^0_3\rangle}\rangle\}\Vdash\sigma$$

hence

$$(p_3)_{\langle\beta_3,\langle\mu^{\beta_3}_{\xi,1},\mu^{\beta_3}_{\xi,2},\nu^{\beta_3}_3\rangle\rangle}\cup\{\langle 0,\langle \nu^0_0,\mu^0_{\xi,1},\mu^0_{\xi,2},\nu^0_3\rangle,$$
$$\langle f_0',f_1',f_2',f_3'''^*\rangle,$$
$$\langle h_0',h_1',h_2',H^3_{\langle\mu^0_{\xi,1},\mu^0_{\xi,2}\nu^0_3\rangle}\rangle\rangle,$$
$$\langle\beta_3,\langle\nu^{\beta_3}_0,\mu^{\beta_3}_{\xi,1},\mu^{\beta_3}_{\xi,2},\nu^{\beta_3}_3\rangle,$$
$$S^3_{\langle\mu^{\beta_3}_{\xi,1},\mu^{\beta_3}_{\xi,2},\nu^{\beta_3}_3\rangle},F^3_{\langle\mu^{\beta_3}_{\xi,1},\mu^{\beta_3}_{\xi,2},\nu^{\beta_3}_3\rangle},H^3_{\langle\mu^0_{\xi,1},\mu^0_{\xi,2},\nu^0_3\rangle}\rangle\}\Vdash\sigma$$

After working on all levels we'll have $<p_n,\beta_n,S^n,F^n,H^n|n<\omega>$. We combine it all into one condition by setting

$$p=\bigcup_{n<\omega}p_n \qquad \beta\geq\beta_n \quad S=\bigcap_{n<\omega}\pi^{-1}_{\beta,\beta_n}S^n \qquad F|[\kappa]^n=F^n|[\kappa]^n$$
$$H|[\kappa]^n=H^n|[\kappa]^n$$

giving us the required

$$p\cup\{<0,<\nu^0_0>,<f_0^{u^*}\circ\pi_{\beta,\alpha}>,<h_0^{u^*}>>,<\beta,<\nu^\beta_0>,S,F,H>\}.$$

◊

Lemma 4.24.2: There are p^*, β^*, S^*, F^*, H^* such that

$$p^*\cup\left\{\langle 0,\langle\nu^0_0\rangle,\langle f_0^{u^*}\circ\pi_{\beta^*,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta^*,<\nu^{\beta^*}_0>,S^*,F^*,H^*\rangle\right\}\leq^*$$
$$p\cup\left\{\langle 0,\langle\nu^0_0\rangle,\langle f_0^{u^*}\circ\pi_{\beta,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta,\langle\nu^\beta_0\rangle,S,F,H\rangle\right\}$$

and such that if

$$\sigma\Vdash q\cup\left\{\langle 0,\langle\nu^0_0,\ldots,\nu^0_n\rangle,\langle f_0,\ldots,f_{n-1},f_n'^*\rangle,\langle h_0,\ldots h_{n-1},h_n'^*\rangle\rangle,\right.$$
$$\left.\langle\delta,\langle\nu^\delta_0,\ldots,\nu^\delta_n\rangle,T',F',H'\rangle\right\}\leq$$
$$p^*\cup\left\{\langle 0,\langle\nu^0_0\rangle,\langle f_0^{u^*}\circ\pi_{\beta^*,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\langle\beta^*,\langle\nu^\gamma_0\rangle,S^*,F^*,H^*\rangle\right\}$$

then $q \,\|\, \sigma$.

Proof: Set $\beta_n=j_n(\beta)$. In M_2 define

$D'_{\langle\rangle}=\{\langle f_1,j_{2,3}(f_2^*)(\gamma),h_1\rangle|\exists f_0\leq j_2(f_0^{u^*})(\alpha)\;\; h_0\leq j_2(h_0^{u^*})(\kappa)$

$$
\begin{aligned}
& f_1 \le j_2(F)(\beta,\beta_1) \quad j_{2,3}(f_2^*)(\gamma) \le j_{2,3}(j_2(F)_{\langle\beta,\beta_1\rangle})(\beta_2) \quad h_1 \le j_2(H)(\kappa,\kappa_1) \\
& \forall \gamma' \ge \gamma \\
& (j_2(p))_{\langle \beta_2, \langle \beta,\beta_1\rangle\rangle} \cup \\
& \quad \{\langle j_2(0), \langle \nu_0^0, \kappa, \kappa_1\rangle, \langle f_0, f_1, f_2^* \circ j_2(\pi)_{\gamma,\beta_2}\rangle, < h_0, h_1, j_2(H)_{<\kappa,\kappa_1>} >>, \\
& \quad \langle \beta_2, < \nu_0^\beta, \beta, \beta_1\rangle\rangle, \\
& \quad \langle \gamma', \langle \nu_0^{\gamma'}, \nu_1^{\gamma'}, \nu_2^{\gamma'}\rangle, \\
& \qquad j_2(\pi)^{-1}_{\gamma,\beta_2} j_2(S)_{\langle\beta,\beta_1\rangle}, j_2(F)_{\langle\beta,\beta_1\rangle} \circ j_2(\pi)_{\gamma,\beta_2}, j_2(H)_{\langle\kappa,\kappa_1\rangle}\rangle\} \| j_2(\sigma)\}
\end{aligned}
$$

$$
\begin{aligned}
D_{\langle\rangle} = \{\langle f_1, f_2, h_1\rangle \mid \langle f_1, f_2, h_1\rangle \in D'_{\langle\rangle} \ \vee \\
\forall \langle f_1', f_2', h_1'\rangle \in D'_{\langle\rangle} \ \langle f_1', f_2', h_1'\rangle \not\le \langle f_1, f_2, h_1\rangle\}
\end{aligned}
$$

Back in V take f_1, f_2, h_1 such that

$$\langle j_2(F)(\beta,\beta_1), j_3(F)(\beta,\beta_1,\beta_2), j_2(H)(\kappa,\kappa_1)\rangle \ge < f_1, f_2, h_1 > \in D_{<>} \cap G_1$$

which is possible since

$$\langle j_2(F)(\beta,\beta_1), j_3(F)(\beta,\beta_1,\beta_2), j_2(H)(\kappa,\kappa_1)\rangle \in G_1.$$

(Recall from the previous section that we arrived to the model M_2 by forcing with a κ_2^+-closed forcing and note that the forcing the dense set $D_{\langle\rangle}$ resides in has κ_2-c.c. hence indeed the generic we have built meet this dense set)

Now take appropriate β^0, $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{h}_1$ such that $f_1 = j_2(\tilde{f}_1)(\beta^0, \beta_1^0)$, $f_2 = j_3(\tilde{f}_2)(\beta_1^0, \beta_2^0)$, $h_1 = j_2(\tilde{h}_1)(\kappa, \kappa_1)$ where $\beta_n^0 = j_n(\beta^0)$.

Let $S'^0 = \pi^{-1}_{\beta^0,\beta} S$.

$$
\begin{aligned}
& F'^0(\mu_1, \mu_2) = \tilde{f}_1(\mu_1, \mu_2) \\
& F'^0(\mu_1, \mu_2, \mu_3) = \tilde{f}_2(\mu_2, \mu_3) \\
& F'^0(\mu_1, \mu_2, \mu_3, \ldots) = F \circ \pi_{\beta^0,\beta}(\mu_1, \mu_2, \mu_3, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
& H'^0(\nu_1^0, \nu_2^0) = \tilde{h}_1(\nu_1^0, \nu_2^0) \\
& H'^0(\nu_1^0, \nu_2^0, \ldots) = H(\nu_1^0, \nu_2^0, \ldots)
\end{aligned}
$$

$$
\begin{aligned}
A_0 = \{ & \langle \nu_1, \nu_2\rangle \in S'^0 \mid [\kappa]^2 \mid \exists \langle f_0, f_1\rangle \le \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha}(\nu_1), F'^0(\nu_1, \nu_2)\rangle \\
& \exists \gamma \ \exists f_2^* \le F'^0_{<\nu_1,\nu_2>}(-) \\
& \exists \langle h_0, h_1\rangle \le \langle h_0^{u^*}(\nu_1^0), H'^0(\nu_1^0, \nu_2^0)\rangle \\
& \forall \gamma' \ge \gamma \\
& (p)_{\langle\beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2)\rangle\rangle} \cup \\
& \quad \{\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0\rangle, \langle f_0, f_1, f_2^* \circ \pi_{\gamma,\gamma}\rangle, \langle h_0, h_1, H'^0_{<\nu_1,\nu_2>}\rangle, \\
& \quad \langle \beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2)\rangle\rangle, \\
& \quad \langle \gamma', \langle \nu_0^{\gamma'}, \nu_1^{\gamma'}, \nu_2^{\gamma'}\rangle, \pi^{-1}_{\gamma,\beta^0} S'^0_{\langle \nu_1,\nu_2\rangle}, F'^0_{\langle\nu_1,\nu_2\rangle} \circ \pi_{\gamma,\beta^0}, H'^0_{\langle \nu_1^0, \nu_2^0\rangle}\rangle\} \| \sigma\}
\end{aligned}
$$

$A_1 = S'^0 \mid [\kappa]^2 - A_0$

Set $S^0 = S'^0 \mid A_i$ where $A_i \in U^2_{\beta^0}$., and let $F^0 = F'^0 \mid S^0$, $H^0 = H'^0 \mid \pi_{\beta^0,0}(S^0)$.

We'll see what we gained with the condition

$p \cup \{\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha} \rangle, \langle h_0^{u^*} \rangle \rangle, \langle \beta, \langle \nu_0^\beta \rangle \rangle, \langle \beta^0, \langle \nu_0^{\beta^0} \rangle, S^0, F^0, H^0 \rangle \}$.

Assume that we have

$$\begin{aligned}\sigma \| q \cup \{ &\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0 \rangle, \langle f_0', f_1', f_2'^* \rangle, \langle h_0', h_1', h_2'^* \rangle \rangle, \\ &\langle \gamma, \langle \nu_0^\gamma, \nu_1^\gamma, \nu_2^\gamma \rangle, T', F', H' \rangle \} \le \\ p \cup \{ &\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha} \rangle, \langle h_0^{u^*} \rangle \rangle, \langle \beta, \langle \nu_0^\beta \rangle \rangle, \langle \beta^0, \langle \nu_0^{\beta^0} \rangle, S^0, F^0, H^0 \rangle \}\end{aligned}$$

As

$$\begin{aligned}&p \cup \{ \langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha} \rangle, \langle h_0^{u^*} \rangle \rangle, \langle \beta, \langle \nu_0^\beta \rangle \rangle, \langle \beta^0, \langle \nu_0^{\beta^0} \rangle, S^0, F^0, H^0 \rangle \} \le^* \\ &p \cup \{ \langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta,\alpha} \rangle, \langle h_0^{u^*} \rangle \rangle, \langle \beta, \langle \nu_0^\beta \rangle, S, F, H \rangle \}\end{aligned}$$

then

$$\begin{aligned}\sigma \| q \cup \{ &\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0 \rangle, \langle f_0', f_1', f_2'^* \rangle, \langle h_0', h_1', h_2'^* \rangle \rangle, \\ &\langle \gamma, \langle \nu_0^\gamma, \nu_1^\gamma, \nu_2^\gamma \rangle, T', F', H' \rangle \} \le \\ p \cup \{ &\langle 0, \langle \nu_0^0 \rangle, \langle f_0^{u^*} \circ \pi_{\beta,\alpha} \rangle, \langle h_0^{u^*} \rangle \rangle, \langle \beta, \langle \nu_0^\beta \rangle, S, F, H \rangle \}\end{aligned}$$

hence by previous lemma there's $f_2''^* \le f_2'^*$ such that $\forall \gamma' \ge \gamma$

$$\begin{aligned}(p)_{\langle \beta, \langle \nu_1^\beta, \nu_2^\beta \rangle \rangle} \cup \{ &\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0 \rangle, \langle f_0', f_1', f_2''^* \circ \pi_{\gamma',\gamma} \rangle, \langle h_0', h_1', H_{\langle \nu_1^0, \nu_2^0 \rangle}(-) \rangle \rangle, \\ &\langle \beta, \langle \nu_0^\beta, \nu_1^\beta, \nu_2^\beta \rangle \rangle, \\ &\langle \gamma', \langle \nu_0^{\gamma'}, \nu_1^{\gamma'}, \nu_2^{\gamma'} \rangle, \pi_{\gamma',\beta}^{-1} S_{\langle \nu_1^\beta, \nu_2^\beta \rangle}, F_{\langle \nu_1^\beta, \nu_2^\beta \rangle} \circ \pi_{\gamma',\beta}, H_{\langle \nu_1^0, \nu_2^0 \rangle} \rangle \} \| \sigma\end{aligned}$$

by our assumption there's $\langle \nu_1, \nu_2 \rangle \in S^0$ such that $\nu_1^\beta = \pi_{\beta^0,\beta}(\nu_1)$, $\nu_2^\beta = \pi_{\beta^0,\beta}(\nu_2)$ hence $\forall \gamma' \ge \gamma$

$$\begin{aligned}(p)_{\langle \beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2) \rangle \rangle} \cup \{ &\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0 \rangle, \langle f_0', f_1', f_2''^* \circ \pi_{\gamma',\gamma} \rangle, \langle h_0', h_1', H'_{\langle \nu_1^0, \nu_2^0 \rangle} \rangle, \\ &\langle \beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2) \rangle \rangle, \\ &\langle \gamma', \langle \nu_0^{\gamma'}, \nu_1, \nu_2 \rangle, \pi_{\gamma',\beta^0}^{-1} S'^0_{\langle \nu_1, \nu_2 \rangle}, F'^0_{\langle \nu_1, \nu_2 \rangle} \circ \pi_{\gamma',\beta^0}, H'^0_{\langle \nu_1^0, \nu_2^0 \rangle} \rangle \} \| \sigma.\end{aligned}$$

This means that $S^0 | [\kappa]^2 = A_0$ and so for each $\langle \nu_1, \nu_2 \rangle \in S^0$ we have

$$\begin{aligned}&\langle f_0(\nu_1, \nu_2), f_1(\nu_1, \nu_2) \rangle \le \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha}(\nu_1), F'^0(\nu_1, \nu_2) \rangle \\ &\gamma(\nu_1, \nu_2) \;\; f_2^*(\nu_1, \nu_2) \le F'^0_{\langle \nu_1, \nu_2 \rangle}(-) \\ &< h_0(\nu_1, \nu_2), h_1(\nu_1, \nu_2) > \le < h_0^{u^*}(\nu_1^0), H'^0(\nu_1^0, \nu_2^0) >\end{aligned}$$

such that $\forall \gamma' \ge \gamma(\nu_1, \nu_2)$

$$\begin{aligned}(p)_{\langle \beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2) \rangle \rangle} \cup \\ \{ &\langle 0, \langle \nu_0^0, \nu_1^0, \nu_2^0 \rangle, \langle f_0(\nu_1, \nu_2), f_1(\nu_1, \nu_2), f_2^*(\nu_1, \nu_2) \circ \pi_{\gamma',\gamma(\nu_1,\nu_2)} \rangle, \\ &\qquad \langle h_0(\nu_1, \nu_2), h_1(\nu_1, \nu_2), H'^0_{\langle \nu_1^0, \nu_2^0 \rangle} \rangle \rangle, \\ &\langle \beta, \langle \nu_0^\beta, \pi_{\beta^0,\beta}(\nu_1), \pi_{\beta^0,\beta}(\nu_2) \rangle \rangle, \\ &\langle \gamma', \langle \nu_0^{\gamma'}, \nu_1, \nu_2 \rangle, \pi_{\gamma',\beta^0}^{-1} S'^0_{\langle \nu_1, \nu_2 \rangle}, F'^0_{\langle \nu_1, \nu_2 \rangle} \circ \pi_{\gamma',\beta^0}, H'^0_{\langle \nu_1^0, \nu_2^0 \rangle} \rangle \} \| \sigma \}\end{aligned}$$

thus we have $\forall \gamma' \ge j_2(\gamma)(\beta^0, \beta_1^0)$

$$\begin{aligned}&(j_2(p))_{\langle \beta_2, \langle \beta, \beta_1 \rangle \rangle} \cup \\ &\quad \{ \langle 0, \langle \nu_0^0, \kappa, \kappa_1 \rangle, \\ &\qquad \langle j_2(f_0)(\beta^0, \beta_1^0), j_2(f_1)(\beta^0, \beta_1^0), j_2(f_2^*)(\beta^0, \beta_1^0) \circ j_2(\pi)_{\gamma', j_2(\gamma)(\beta^0, \beta_1^0)} \rangle,\end{aligned}$$

$$\langle j_2(h_0)(\beta^0,\beta_1^0), j_2(h_1)(\beta^0,\beta_1^0), j_2(H'^0)_{\langle\kappa,\kappa_1\rangle}(-)\rangle\rangle,$$
$$\langle\beta_2,\langle \nu_0^\beta,\beta,\beta_1\rangle\rangle,$$
$$\langle\gamma',\langle \nu_0^{\gamma'},\nu_1^{\gamma'},\nu_1^{\gamma'}\rangle,$$
$$j_2(\pi)^{-1}_{\gamma,\beta_2^0} j_2(S'^0)_{\langle\beta^0,\beta_1^0\rangle}, j_2(F'^0)_{\langle\beta^0,\beta_1^0\rangle}\circ j_2(\pi)_{\gamma,\beta_2^0}, j_2(H'^0)_{\langle\kappa,\kappa_1\rangle}\rangle\}\Vdash j_2(\sigma).$$

This means that

$$\langle j_2(f_1)(\beta^0,\beta_1^0), j_{2,3}(j_2(f_2^*)(\beta^0,\beta_1^0))(j_2(\gamma)(\beta^0,\beta_1^0)), j_2(h_1)(\beta^0,\beta_1^0)\rangle\in D'_{\langle\rangle}.$$

By construction

$$\langle j_2(f_1)(\beta^0,\beta_1^0), j_{2,3}(j_2(f_2^*)(\beta^0,\beta_1^0))(j_2(\gamma)(\beta^0,\beta_1^0)), j_2(h_1)(\beta^0,\beta_1^0)\rangle\le$$
$$\langle j_2(F'^0)(\beta^0,\beta_1^0), j_{2,3}(j_2(F'^0)(\beta^0,\beta_1^0))(\beta_2^0), j_2(H'^0)(\kappa,\kappa_1)\rangle\in D_{\langle\rangle}$$

hence

$$\langle j_2(F'^0)(\beta^0,\beta_1^0), j_{2,3}(j_2(F'^0)(\beta^0,\beta_1^0))(\beta_2^0), j_2(H'^0)(\kappa,\kappa_1)\rangle\in D'_{\langle\rangle}.$$

Literally this means there's $\langle f_0,h_0\rangle\le\langle j_2(f_0^{u^*})(\alpha), j_2(h_0^{u^*})(\kappa)\rangle$, δ such that $\forall\delta'\ge\delta$

$$(j_2(p))_{\langle\beta_2,\langle\beta,\beta_1\rangle\rangle}\cup$$
$$\{\langle j_2(0),\langle\nu_0^0,\kappa,\kappa_1\rangle,\langle f_0, j_2(F'^0)(\beta^0,\beta_1^0), j_2(F'^0)(\beta^0,\beta_1^0)\circ j_2(\pi)_{\delta',\beta_2^0}(-)\rangle,$$
$$\langle h_0, j_2(H'^0)(\kappa,\kappa_1), j_2(H'^0)(\kappa,\kappa_1)(-)\rangle\rangle,$$
$$\langle\beta_2,\langle\nu_0^\beta,\beta,\beta_1\rangle\rangle,$$
$$\langle\delta',\langle\nu_0^{\delta'},\nu_1^{\delta'},\nu_2^{\delta'}\rangle,$$
$$j_2(\pi)^{-1}_{\delta',\beta_2}\circ j_2(S)_{\langle\beta,\beta_1\rangle}, j_2(F)_{\langle\beta,\beta_\#\rangle}\circ j_2(\pi)_{\delta',\beta_2}, j_2(H)_{\langle\kappa,\kappa_1\rangle}\rangle\}\Vdash j_2(\sigma)$$

Choosing a large enough δ, setting $j_2(f_0^*)(\delta)=f_0$, $S''^0=\pi^{-1}_{\delta,\beta^0}S^0$, $F''^0=F^0\circ\pi_{\delta,\beta^0}$, $H''^0=H^0$ and allowing $j_2(h_0^*)(\kappa)\le h_0$ we get

$$(j_2(p))_{\langle\delta_2,\langle\delta,\delta_1\rangle\rangle}\cup$$
$$\{\langle j_2(0),\langle\nu_0^0,\kappa,\kappa_1\rangle,\langle j_2(f_0^*)(\delta), j_2(F''^0)(\delta,\delta_1), j_2(F''^0)(\delta,\delta_1)(-)\rangle,$$
$$\langle j_2(h_0^*)(\kappa), j_2(H''^0)(\kappa,\kappa_1), j_2(H''^0)(\kappa,\kappa_1)(-)\rangle\rangle,$$
$$\langle\beta_2,\langle\nu_0^\beta,\beta,\beta_1\rangle\rangle$$
$$\langle\delta_2,\langle\nu_0^\delta,\delta,\delta_1\rangle, j_2(S''^0)_{\langle\delta,\delta_1\rangle}, j_2(F''^0)_{\langle\delta,\delta_1\rangle}, j_2(H''^0)_{\langle\kappa,\kappa_1\rangle}\rangle\}\Vdash j_2(\sigma)$$

Set

$$B_0=\{\langle\nu_1,\nu_2\rangle\in S''^0 \mid (p)_{<\delta,<\nu_1,\nu_2>>}\cup$$
$$\{\langle 0,\langle\nu_0^0,\nu_1^0,\nu_2^0\rangle,\langle f_0^*(\nu_1), F''^0(\nu_1,\nu_2), F''^0(\nu_1,\nu_2)(-)\rangle,$$
$$\langle h_0^*(\nu_1^0), H''^0(\nu_1^0,\nu_2^0), H''^0_{\langle\nu_1^0,\nu_2^0\rangle}(-)\rangle\rangle,$$
$$\langle\beta,\langle\nu_0^\beta,\pi_{\delta,\beta}(\nu_1),\pi_{\delta,\beta}(\nu_2)\rangle\rangle,$$
$$\langle\delta,\langle\nu_0^\delta,\nu_1,\nu_2\rangle, S''^0_{\langle\nu_1,\nu_2\rangle}, F''^0_{\langle\nu_1,\nu_2\rangle}, H''^0_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\}\Vdash\sigma\}$$

$$B_1=\{<\nu_1,\nu_2>\in S^0 \mid (p)_{\langle\delta,\langle\nu_1,\nu_2\rangle\rangle}\cup$$
$$\{\langle 0,\langle\nu_0^0,\nu_1^0,\nu_2^0\rangle,\langle f_0^*(\nu_1), F''^0(\nu_1,\nu_2), F''^0(\nu_1,\nu_2)(-)\rangle,$$
$$\langle h_0^*(\nu_1^0), H''^0(\nu_1^0,\nu_2^0), H''^0_{\langle\nu_1^0,\nu_2^0\rangle}(-)\rangle\rangle,$$
$$\langle\beta,\langle\nu_0^\beta,\pi_{\gamma,\beta}(\nu_1),\pi_{\gamma,\beta}(\nu_2)\rangle\rangle,$$
$$\langle\delta,\langle\nu_0^\delta,\nu_1,\nu_2\rangle, S''^0_{\langle\nu_1,\nu_2\rangle}, F''^0_{\langle\nu_1,\nu_2\rangle}, H''^0_{\langle\nu_1^0,\nu_2^0\rangle}\rangle\}\Vdash\neg\sigma\}$$

Take $i\in 2$ such that $B_i\in U_\delta^2$ and restrict S''^0 to B_i. By definition we get

$$\sigma \Vdash p \cup \{\langle 0,\langle \nu_0^0\rangle,\langle f_0^*\rangle,\langle h_0^*\rangle,\langle \beta,\langle \nu_0^\beta\rangle\rangle,\langle \delta,\langle \nu_0^\delta\rangle, S''^0, F''^0, H''^0\rangle\} \le^*$$
$$p \cup \{\langle 0,\langle \nu_0^0\rangle,\langle f_0^{u^*}\rangle,< h_0^{u^*}>,\langle \beta,\langle \nu_0^\beta\rangle, S, F, H>\}$$

as needed.

We'll sketch how to continue into the next level.

In M_2 define for each $\langle \nu_1,\nu_2\rangle \in S^0$

$$D'_{\langle \nu_1,\nu_2\rangle} = \{\langle f_3, j_{2,3}(f_4^*)(\gamma), h_3\rangle \mid$$
$$\exists \langle f_0, f_1, f_2\rangle \le \langle f_0^{u^*} \circ \pi_{\beta^0,\alpha}(\nu_1), F^0(\nu_1,\nu_2), j_2(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0)\rangle$$
$$\exists \langle h_0, h_1, h_2\rangle \le \langle h_0^{u^*}(\nu_1^0), H^0(\nu_1^0,\nu_2^0), j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})(\kappa)\rangle$$
$$f_3 \le j_2(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0,\beta_1^0) \quad j_{2,3}(f_4^*)(\gamma) \le j_{2,3}(j_2(F^0_{\langle \nu_1,\nu_2\rangle})_{\langle \beta^0,\beta_1^0\rangle})(\beta_2^0)$$
$$h_1 \le j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})(\kappa,\kappa_1) \quad \forall \gamma' \ge \gamma$$
$$(j_2(p))_{\langle \beta_2^0,\langle \nu_1,\nu_2,\beta^0,\beta_1^0\rangle\rangle} \cup$$
$$\{\langle j_2(0),\langle \nu_0^0,\nu_1^0,\nu_2^0,\kappa,\kappa_1\rangle,\langle f_0,f_1,f_2,f_3,f_4^* \circ \pi_{\gamma',\gamma}\rangle,$$
$$\langle h_0,h_1,h_2, j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})_{\langle \kappa,\kappa_1\rangle}(-)\rangle\rangle,$$
$$\langle \beta_2,\langle \nu_0^\beta,\pi_{\beta_2^0,\beta_2}(\nu_1),\pi_{\beta_2^0,\beta_2}(\nu_2),\beta,\beta_1\rangle\rangle$$
$$\langle \beta_2^0,\langle \nu_0^\beta,\nu_1,\nu_2,\beta^0,\beta_1^0\rangle\rangle$$
$$\langle \gamma',\langle \nu_0^{\gamma'},\nu_1^{\gamma'},\nu_2^{\gamma'},\nu_3^{\gamma'},\nu_4^{\gamma'}\rangle, j_2(\pi)^{-1}_{\gamma',\beta_2^0} j_2(S^0_{\langle \nu_1,\nu_2\rangle})_{\langle \beta^0,\beta_1^0\rangle},$$
$$j_2(F^0_{\langle \nu_1,\nu_2\rangle})_{\langle \beta^0,\beta_1^0\rangle} \circ j_2(\pi)_{\gamma',\beta_2^0}, j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})_{\langle \kappa,\kappa_1\rangle} >\} \Vdash j_2(\sigma)\}$$
$$D_{\langle \nu_1,\nu_2\rangle} = \{\langle f_3,f_4,h_3\rangle \mid \langle f_3,f_4,h_3\rangle \in D'_{\langle \nu_1,\nu_2\rangle} \ \vee$$
$$\forall \langle f_3',f_4',h_3'\rangle \in D'_{\langle \nu_1,\nu_2\rangle} \ \langle f_3',f_4',h_3'\rangle \not\le \langle f_3,f_4,h_3\rangle\}$$

Back in V take $f_3^{\langle \nu_1,\nu_2\rangle}$, $f_4^{\langle \nu_1,\nu_2\rangle}$, $h_3^{\langle \nu_1,\nu_2\rangle}$ such that

$$\langle j_2(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0,\beta_1^0), j_3(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0,\beta_1^0,\beta_2^0), j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})(\kappa,\kappa_1)\rangle \ge$$
$$\langle f_3^{\langle \nu_1,\nu_2\rangle}, f_4^{\langle \nu_1,\nu_2\rangle}, h_3^{\langle \nu_1,\nu_2\rangle} > \in D_{\langle \nu_1,\nu_2\rangle} \cap G_3$$

which is possible since

$$\langle j_2(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0,\beta_1^0), j_3(F^0_{\langle \nu_1,\nu_2\rangle})(\beta^0,\beta_1^0,\beta_2^0), j_2(H^0_{\langle \nu_1^0,\nu_2^0\rangle})(\kappa,\kappa_1)\rangle \in G_3.$$

Now take appropriate β^1, $\tilde{f}_3^{\langle \nu_1,\nu_2\rangle}$, $\tilde{f}_4^{\langle \nu_1,\nu_2\rangle}$, $\tilde{h}_3^{\langle \nu_1,\nu_2\rangle}$ such that

$$f_3^{\langle \nu_1,\nu_2\rangle} = j_2(\tilde{f}_3^{\langle \nu_1,\nu_2\rangle})(\beta^1,\beta_1^1),$$
$$f_4^{\langle \nu_1,\nu_2\rangle} = j_3(\tilde{f}_4^{\langle \nu_1,\nu_2\rangle})(\beta_1^1,\beta_2^1),$$
$$h_3^{\langle \nu_1,\nu_2\rangle} = j_2(\tilde{h}_3^{\langle \nu_1,\nu_2\rangle})(\kappa,\kappa_1)$$

where $\beta_n^1 = j_n(\beta^1)$. We can take one such β^1 for all $\langle \nu_1,\nu_2\rangle \in S^0$ due to the κ^+ directness of our extender.

Let $S'^1 = \pi^{-1}_{\beta^1,\beta^0} S^0$.

$$F'^1(\nu_1,\nu_2) = F^0 \circ \pi_{\beta^1,\beta^0}(\nu_1,\nu_2)$$
$$F'^1(\nu_1,\nu_2,\nu_3) = F^0 \circ \pi_{\beta^1,\beta^0}(\nu_1,\nu_2,\nu_3)$$
$$F'^1(\nu_1,\nu_2,\nu_3,\nu_4) = \tilde{f}_3^{\langle \nu_1,\nu_2\rangle}(\nu_3,\nu_4)$$

$$F'^1(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)=\tilde{f}_4^{\langle\nu_1,\nu_2\rangle}(\nu_4,\nu_5)$$
$$F'^1(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\ldots)=F^0\circ\pi_{\beta^1,\beta^0}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\ldots)$$

$$H'^1(\nu_1^0,\nu_2^0)=H^0(\nu_1^0,\nu_2^0)$$
$$H'^1(\nu_1^0,\nu_2^0,\nu_3^0)=H^0(\nu_1^0,\nu_2^0,\nu_3^0)$$
$$H'^1(\nu_1^0,\nu_2^0,\nu_3^0,\nu_4^0)=\tilde{h}_3(\nu_3^0,\nu_4^0)$$
$$H'^1(\nu_1^0,\nu_2^0,\nu_3^0,\nu_4^0,\ldots)=H^0(\nu_1^0,\nu_2^0,\nu_3^0,\nu_4^0,\ldots)$$

$$\begin{aligned}
A_0^{\langle\nu_1,\nu_2\rangle}=\{\langle\nu_3,\nu_4\rangle\in S'^1_{\langle\nu_1,\nu_2\rangle}\restriction[\kappa]^2\mid\ &\\
&\exists\langle f_0,f_1,f_2,f_3\rangle\le\langle f_0^{u^*}\circ\pi_{\beta^1,\alpha}(\nu_1),F'^1(\nu_1,\nu_2),F'^1_{\langle\nu_1,\nu_2\rangle}(\nu_3),F'^1_{\langle\nu_1,\nu_2\rangle}(\nu_3,\nu_4)\rangle\\
&\exists\gamma\ \ \exists f_4^*\le F'^1_{\langle\nu_1,\nu_2,\nu_3,\nu_4\rangle}(-)\\
&\exists\langle h_0,h_1,h_2,h_3\rangle\le\langle h_0^{u^*}(\nu_1^0),H'^1(\nu_1^0,\nu_2^0),H'^1_{\langle\nu_1^0,\nu_2^0\rangle}(\nu_3^0),H'^1_{\langle\nu_1^0,\nu_2^0\rangle}(\nu_3^0,\nu_4^0)\rangle\\
&\forall\gamma'\ge\gamma\\
&(p)_{\langle\beta,\langle\nu_0^\beta,\pi_{\beta^1,\beta}(\nu_1),\pi_{\beta^1,\beta}(\nu_2),\pi_{\beta^1,\beta}(\nu_3),\pi_{\beta^1,\beta}(\nu_4)\rangle\rangle}\cup\\
&\quad\{<0,<\nu_0^0,\nu_1^0,\nu_2^0,\nu_3^0,\nu_4^0>,\langle f_0,f_1,f_2,f_3,f_4^*\circ\pi_{\gamma',\gamma}\rangle,\\
&\qquad\qquad\langle h_0,h_1,h_2,h_3,H'^1_{\langle\nu_1,\nu_2,\nu_3,\nu_4\rangle}(-)\rangle,\\
&\quad\langle\beta,\langle\nu_0^\beta,\pi_{\beta^1,\beta}(\nu_1),\pi_{\beta^1,\beta}(\nu_2),\pi_{\beta^1,\beta}(\nu_3),\pi_{\beta^1,\beta}(\nu_4)\rangle\rangle,\\
&\quad\langle\beta^0,\langle\nu_0^{\beta^0},\pi_{\beta^1,\beta^0}(\nu_1),\pi_{\beta^1,\beta^0}(\nu_2),\pi_{\beta^1,\beta^0}(\nu_3),\pi_{\beta^1,\beta^0}(\nu_4)\rangle\rangle,\\
&\quad\langle\gamma',\langle\nu_0^{\gamma'},\nu_1^{\gamma'},\nu_2^{\gamma'},\nu_3^{\gamma'},\nu_4^{\gamma'}\rangle,\\
&\qquad\qquad\pi^{-1}_{\gamma',\beta^1}S'^1_{\langle\nu_1,\nu_2,\nu_3,\nu_4\rangle},F'^1_{\langle\nu_1,\nu_2,\nu_3,\nu_4\rangle}\circ\pi_{\gamma',\beta^1},H'^1_{\langle\nu_1^0,\nu_2^0,\nu_3^0,\nu_4^0\rangle}\rangle\}\Vert\sigma\}
\end{aligned}$$

$$A_1^{\langle\nu_1,\nu_2\rangle}=S'^1_{\langle\nu_1,\nu_2\rangle}\restriction[\kappa]^2-A_0^{\langle\nu_1,\nu_2\rangle}$$

Set $S^1_{\langle\nu_1,\nu_2\rangle}=S'^1_{\langle\nu_1,\nu_2\rangle}\restriction A_i^{\langle\nu_1,\nu_2\rangle}$ where $A_i^{\langle\nu_1,\nu_2\rangle}\in U^2_{\beta^1}$., and let $F^1_{\langle\nu_1,\nu_2\rangle}=F'^1_{\langle\nu_1,\nu_2\rangle}\restriction S^1_{\langle\nu_1,\nu_2\rangle}$, $H^1_{\langle\nu_1^0,\nu_2^0\rangle}=H'^1_{\langle\nu_1^0,\nu_2^0\rangle}\restriction\pi_{\beta^1,0}S^1_{\langle\nu_1,\nu_2\rangle}$.

The new condition is

$$\begin{aligned}
p\cup\{&\langle 0,\langle\nu_0^0\rangle,\langle f_0^{u^*}\circ\pi_{\beta^1,\alpha}\rangle,\langle h_0^{u^*}\rangle\rangle,\\
&\langle\beta,\langle\nu_0^\beta\rangle\rangle,\langle\beta^0,\langle\nu_0^{\beta^0}\rangle\rangle,\\
&\langle\beta^1,\langle\nu_0^{\beta^1}\rangle,S^1,F^1,H^1\rangle\}
\end{aligned}$$

and by working on all levels the lemma is proved.◊

Let $G\subseteq P$ be a generic filter. As P has the κ^{++}-c.c. all the cardinals $\ge\kappa^{++}$ remain cardinals in $V[G]$. We show know that κ^+ is also preserved.

Claim 4.25: κ^+ remains a cardinal in $V[G]$.

Proof: Suppose κ^+ had been collapsed. As $\mathrm{cf}_{V[G]}\kappa = \omega$ we must have that $\mathrm{cf}_{V[G]}\kappa^+ = \mu < \kappa$. Let $p' \Vdash$"$\tilde{g} : \hat{\mu} \to \hat{\kappa}^+$ an unbounded function" in $V[G]$. Take $p \le p'$ such that $\max p^0 > \mu$. Using the same method as in the first sub-lemma of the Prikry condition proof, there's

$$q_0 \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha_0,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha_0, q_0^{\alpha_0}, S^0, F^0, H^0\rangle\} \le^* p$$

such that if there's $\zeta < \kappa^+$, r such that

$$\tilde{g}(\hat{0}) = \hat{\zeta} \Vdash r \le q_0 \cup \{\langle 0, p^0, f^p, h^p\rangle, \langle \alpha_0, q_0^{\alpha_0}, S^0, f^{p^*} \circ \pi_{\alpha_0,\mathrm{mc}(p)}, F^0, h^{p^*}, H^0\rangle\}$$

then $\exists f''^* \le f^{r^*}$, such that

$$(q_0)_{\langle \alpha_0, r^{\alpha}\rangle} \cup \{\langle 0, r^0, f^r, f''^*, h^r, H^0_{r^0 - q_0^0}(-)\rangle,$$
$$\langle \mathrm{mc}(r), r^{\mathrm{mc}}, \pi^{-1}_{\mathrm{mc}(r),\alpha_0} S^0, F^0_{r^{\alpha_0} - q_0^{\alpha_0}} \circ \pi_{\mathrm{mc}(r),\alpha_0}, H^0_{r^0 - q_0^0}\rangle\} \Vdash \tilde{g}(\hat{0}) = \hat{\zeta}\,.$$

Now, assume we have $\lambda < \mu$ and

$$\langle q_\xi \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha_\xi,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha_\xi, q_\xi^{\alpha_\xi}, S^\xi, F^\xi, H^\xi\rangle\} \mid \xi < \lambda\rangle$$

Choose $q_\lambda \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha_\lambda,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha_\lambda, q_\lambda^{\alpha_\lambda}, S^\lambda, F^\lambda, H^\lambda\rangle\}$ which is $\le^*$ from all these satisfying the same thing for $\tilde{g}(\hat{\lambda})$. After finishing the induction we have

$$\{q_\xi \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha_\xi,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha_\xi, q_\xi^{\alpha_\xi}, S^\xi, F^\xi, H^\xi\rangle\} \mid \xi < \mu\rangle$$

Take $q \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha, q^\alpha, S, F, H\rangle\}$ which is $\le^*$ from all these. Suppose now that there are r, $\xi < \mu$, $\zeta < \kappa^+$ such that

$$\tilde{g}(\hat{\xi}) = \hat{\zeta} \Vdash r \le q \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha,\mathrm{mc}(p)}, h^{p^*}\rangle, \langle \alpha, q^\alpha, S, F, H\rangle\}$$

then

$$\tilde{g}(\hat{\xi}) = \hat{\zeta} \Vdash r \le q_\xi \cup \{\langle 0, p^0, f^p, f^{p^*} \circ \pi_{\alpha_\xi,\mathrm{mc}(p)}, h^p, h^{p^*}\rangle, \langle \alpha_\xi, q_\xi^{\alpha_\xi}, S^\xi, F^\xi, H^\xi\rangle\}$$

so from the construction we have $f''^* \le f^{r^*}$ such that

$$(q_\xi)_{\langle \alpha_\xi, r^{\alpha_\xi}\rangle} \cup \{\langle 0, r^0, f^r, f''^*, h^r, H^\xi_{r^0 - q_\xi^0}(-)\rangle,$$
$$\langle \mathrm{mc}(r), r^{\mathrm{mc}}, \pi^{-1}_{\mathrm{mc}(r),\alpha_\xi} S^\xi, F^\xi_{r^{\alpha_\xi} - q_\xi^{\alpha_\xi}} \circ \pi_{\mathrm{mc}(r),\alpha_\xi}, H^\xi_{r^0 - q_\xi^0}\rangle \Vdash \tilde{g}(\hat{\xi}) = \hat{\zeta}$$

from which we get that

$$(q)_{\langle \alpha, r^\alpha\rangle} \cup \{\langle 0, r^0, f^r, f''^*, h^r, H_{r^0 - q^0}(-)\rangle,$$

$$\langle \mathrm{mc}(r), r^{\mathrm{mc}}, \pi^{-1}_{\mathrm{mc}(r),\alpha} S, F_{r^\alpha - q^\alpha} \circ \pi_{\mathrm{mc}(r),\alpha}, H_{r^0 - q^0} \rangle\} \Vdash \tilde{g}(\hat{\xi}) = \hat{\zeta}$$

This means that the value of $\tilde{g}(\hat{\xi})$ is decided by condition of this form. However, condition of this form can force at most κ different values and so these values are bounded in κ^+. This it true for all $\xi < \mu$ and as $\mu < \kappa$ we get that $\tilde{g}[G]$ is bounded in κ^+. Contradiction.◊

Let $\langle \tau_n | n < \omega \rangle = \bigcup \{ p^0 | p \in G \}$. Denseness arguments give us that this sequence is unbounded, that $\forall n < \omega$ the cardinals between $\tau_n^{+b(n)+1}$ and τ_{n+1} are collapsed and that $\forall 0 < n < \omega$ τ_n^+ is collapsed. 4.19 shows that the other cardinals aren't collapsed and so κ remains a cardinal also. Denseness argument show that $2^{\tau_n^{+k}} = \tau_{n+e(n,k)}^{+d(n,k)}$ and because τ_n^+ is collapsed for $n > 1$ we get in fact $2^{\tau_n} = \tau_{n+e(n,1)}^{+d(n,1)}$. Another denseness argument will show that $(2^\kappa)_{V[G]} \geq \kappa^{+m}$. Noting that $V[G] \vDash$ "$2^\kappa = \kappa^\omega$" and using the same technique used to show that κ^+ isn't collapsed we get that $(2^\kappa)_{V[G]} \leq |P| = \kappa^{+m}$ we get that $(2^\kappa)_{V[G]} = \kappa^{+m}$. We finish the proof by forcing in $V[G]$ with $\mathrm{Col}(\aleph_0, \tau_0^+)$.

Acknowledgment.

The 2nd author would like to thank J. Cummings for several talks on elementary embeddings and their continuation to generic extensions.

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