PRIKRY ON EXTENDERS, REVISITED

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Abstract. The extender based forcing of Gitik and Magidor is generalized to yield, given any extender $j:V \to M$ with critical point κ, a cardinal preserving generic extension with no new bounded subset of κ in which $\mathrm{cf}(\kappa) = \omega$ and $\kappa^\omega = |j(\kappa)|$.

Assuming a superstrong cardinal exists, the forcing notion is used to construct a model in which the added Prikry sequences are a scale in the normal Prikry sequence.

In addition, several ways to produce generic filter over an iterated ultrapower are presented.

1. Introduction

In [5] an extender based forcing notion was introduced which yields a cardinal preserving generic extension with no new bounded subset of κ in which $\mathrm{cf}(\kappa) = \omega$ and $\kappa^\omega = \lambda$ for some $\lambda < j(\kappa)$, given an extender $j:V \to M$ with critical point κ. In the case that $\sup\{j(f)(\kappa) \mid f:\kappa \to \kappa\} < \lambda < j(\kappa)$ this construction required a preliminary forcing to add a function $f:\kappa \to \kappa$ such that $j(f)(\kappa) > \lambda$.

In this work we present a modification of the above mentioned forcing which eliminates the requirement for a function f such that $\lambda < j(f)(\kappa)$. We use this forcing to construct from a superstrong extender $j:V \to M$ a model satisfying $2^\kappa = j(\kappa)$, and having a Prikry sequence G^κ in κ together with a scale $\langle G^\lambda \mid \lambda < j(\kappa)\rangle$ in $\prod G^\kappa/D$, such that $\mathrm{tcf}(\prod G^\lambda/D) = \lambda$ for each regular $\lambda < j(\kappa)$, where D is the cofinite filter on ω.

In order to show this we calculate the tcf of the added Prikry sequences. This calculation builds on Sharon's calculation [10] of the tcf of the sequences added in the forcing of [5], which in turn followed on the calculation in [6] of the tcf of sequences in Prikry's original forcing [9], and in the model of Magidor [8].

In [3, 1] it was shown how to construct, working inside V, a generic filter for the Prikry forcing over an iterated ultrapower of V. In [2] it was shown how to construct, working inside V, a generic filter for the Radin forcing over an iterated ultrapower of V. In [7] the sequences generated along the ω-iterations, and their relation to the extender based forcing were considered. Along these lines we have tried to construct, working in V, a generic filter for the extender based forcing over the ω-iterated ultrapower of V. To get a full result we had to use either some form of supercompactness or to Cohen blow-up the power of κ^+.

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The structure of this work is as follows: In section 2 we give some basic definitions used in later sections (e.g., extenders, trees). In section 3 a detailed presentation of the forcing notion is given. This section culminates with the theorem

Theorem (3.32)**.** *Assume GCH,* $j : V \to M \supset M^\kappa$ *and* $\operatorname{crit}(j) = \kappa$*. Then there is cardinal preserving generic extension in which* $\kappa^\omega = |j(\kappa)|$, $\operatorname{cf} \kappa = \omega$*, and there are no new bounded subsets of* κ*.*

In section 4 we start from a superstrong cardinal and get the following.

Theorem (4.15)**.** *Assume GCH,* $j : V \to M \supset M^\kappa$*,* $M \supset V_{j(\kappa)}$ *and* $\operatorname{crit} j = \kappa$*. Then there is a cardinal preserving generic extension in which* $\operatorname{cf} \kappa = \omega$*,* $\kappa^\omega = j(\kappa)$*, and such that there is a Prikry sequence* G^κ *in* κ *and a scale* $\langle G^\lambda \mid \lambda < j(\kappa)\rangle$ *of length* $j(\kappa)$ *in* $\prod G^\kappa/D$ *such that* $\operatorname{tcf}(\prod G^\lambda/D) = \lambda$ *for each regular cardinal* $\lambda < j(\kappa)$*.*

In section 5 we show several ways (mainly because we do not know the 'right' way, if it exists at all) to generate a generic filter in V over the ω iterate of V.

This work is largely self contained, however knowledge of [5] will make it much easier. The notation we use is standard. We assume fluency with forcing, large cardinals, extenders, and some basic pcf theory.

2. Preliminaries

2.1. **Elementary embeddings and Extenders.**

The extenders we use here were used in [5] where they were called nice system. A simplification appears in [4].

Definition 2.1. Let $j : V \to M$ be an elementary embedding and $\operatorname{crit}(j) = \kappa$.

(1) The generators[1] of j are defined by induction as

$\kappa_0 = \operatorname{crit}(j)$,

$\kappa_\xi = \min\{\lambda \in \mathrm{On} \mid \forall \xi' < \xi\ \forall \mu \in \mathrm{On}\ \forall f : \mu \to \mathrm{On}\ j(f)(\kappa_{\xi'}) \neq \lambda\}$.

If the induction terminates, then we have a set of generators for j:

$$\mathfrak{g}(j) = \{\kappa_\xi \mid \xi < \xi^*\}.$$

(2) For $\alpha, \beta \in \mathrm{On}$ we say $\alpha <_{\mathrm{j}} \beta$ if
  (a) $\alpha < \beta$.
  (b) There are $\mu \in \mathrm{On}$ and $f : \mu \to \mathrm{On}$ such that $j(f)(\beta) = \alpha$.

(3) Assume $\alpha \in \mathrm{On}$ and $\lambda \in \mathrm{On}$ is minimal such that $j(\lambda) > \alpha$. We set

$$E(\alpha) = \{A \subseteq \lambda \mid \alpha \in j(A)\}.$$

It is well know that $E(\alpha)$ is a κ-complete ultrafilter over λ.

Definition 2.2. Let $j : V \to M \supset M^\kappa$ be an elementary embedding such that $\operatorname{crit}(j) = \kappa$ and $\mathfrak{g}(j) \subset j(\kappa)$. The extender E derived from j is the system

$$E = \langle \langle E(\alpha) \mid \alpha \in j(\kappa) \setminus \kappa\rangle, \langle \pi_{\beta,\alpha} \mid \alpha, \beta \in j(\kappa) \setminus \kappa,\ \alpha <_{\mathrm{j}} \beta\rangle\rangle.$$

where

(1) $E(\alpha) = \{A \subseteq \kappa \mid \alpha \in j(A)\}$.

[1] The definition of generators differs slightly from the usual one since we use only one ordinal to index our extenders.

(2) For $\alpha, \beta \in j(\kappa) \setminus \kappa$ such that $\alpha <_{\mathrm{j}} \beta$ the function $\pi_{\beta,\alpha} : \kappa \to \kappa$ is such that $j(\pi_{\beta,\alpha})(\beta) = \alpha$. (Note that $\alpha <_{\mathrm{j}} \beta$ means there are many such functions. Any one of them will do as $\pi_{\beta,\alpha}$.)

We assume that it is known how to reconstruct j from E, i.e., j is the natural embedding $j : V \to \mathrm{Ult}(V, E)$. We will use $<_{\mathrm{E}}$ and $\mathrm{dom}\, E$ as synonyms for $<_{\mathrm{j}}$ and $j(\kappa) \setminus \kappa$ respectively.

Claim 2.3. *Assume* $j : V \to M \supseteq M^{\kappa}$, $\mathrm{crit}(j) = \kappa$ *and* $\mathfrak{g}(j) \subseteq j(\kappa)$. *Then* $<_{\mathrm{j}} \restriction j(\kappa)$ *is* κ^{+}*-directed.*

Proof. Let $X \in [j(\kappa)]^{\leq \kappa}$. We need to find $\beta < j(\kappa)$ such that $\forall \alpha \in X\ \beta >_{\mathrm{j}} \alpha$.

Let us fix a function $e : \kappa \xrightarrow{\text{onto}} [\kappa]^{<\kappa}$ such that for each $A \in [\kappa]^{<\kappa}$, $e^{-1}A$ is unbounded in κ. Of course, $j(e) : j(\kappa) \xrightarrow{\text{onto}} ([j(\kappa)]^{<j(\kappa)})_M$.

Let $\mu = \sup X$. Since $X \in M$ we get $\mu < j(\kappa)$. Since $X \in ([j(\kappa)]^{<j(\kappa)})_M$ there are $\beta \geq \mu$ and a function g such that $j(e)(\beta) = X$ and $j(g)(\beta) = j''\,\mathrm{ot}(X)$. We show that $\beta \geq_{\mathrm{j}} \alpha$ for all $\alpha \in X$. So, let $\alpha \in X$.

We let $\xi = \mathrm{ot}(X \cap \alpha)$. Then we set $\forall \nu < \kappa\ g_{\xi}(\nu) = \mathrm{ot}(g(\nu) \cap \xi)$. Thus $j(g_{\xi})(\beta) = \mathrm{ot}(j(g)(\beta) \cap j(\xi)) = \mathrm{ot}(j''\,\mathrm{ot}(X) \cap j(\xi)) = \xi$. We set $\forall \nu < \kappa\ f(\nu) = min\{\gamma \in e(\nu) \mid \mathrm{ot}(e(\nu) \cap \gamma) = g_{\xi}(\nu))\}$. Then $j(f)(\beta) = \min\{\gamma \in X \mid \mathrm{ot}(X \cap \gamma) = \xi\} = \alpha$. □

In this paper we use only elementary embeddings with a set of generators. (I.e., $\mathfrak{g}(j)$ is bounded by some ordinal). Hence an elementary embedding j is definable from a set parameter, so terms of the forms $j(j)$ have definite meaning. We assume the theory of iterating elementary embeddings is known and give the basic definitions and propositions we need in order to get to the ω-iterate of V.

Definition 2.4. Assume $j : V \to M$ is an elementary embedding. We define by induction for each $n < \omega$

$$\begin{aligned} &j_{0,1} = j,\ M_0 = V, \\ &j_{n+1,n+2} = j(j_{n,n+1}) : M_{n+1} \to M_{n+2}. \end{aligned}$$

We 'complete' the list of j's by setting $\forall n < m < \omega$

$$\begin{aligned} &j_{n,n} = \mathrm{id}, \\ &j_{n,m} = j_{m-1,m} \circ \cdots \circ j_{n,n+1}, \\ &j_n = j_{0,n}. \end{aligned}$$

Proposition 2.5. *Assume* $j : V \to M \supset M^{\kappa}$, $\mathrm{crit}(j) = \kappa$, $\mathfrak{g}(j) \subset j(\kappa)$, $0 < n < \omega$ *and* $\tau \in j_n(j(\kappa) \setminus \kappa)$. *Then there is* $\tau^* \in j(\kappa)$ *such that* $j_n(\tau^*) >_{j_{n,n+1}} \tau$.

Proof. The proof is done by induction on n.

- $n = 1$: We choose $f : \kappa \to j(\kappa)$ and $\alpha \in j(\kappa) \setminus \kappa$ such that $j(f)(\alpha) = \tau$. Since $<_{\mathrm{j}} \restriction j(\kappa)$ is κ^{+}-directed, there is $\tau^* \in j(\kappa)$ such that $\forall \nu < \kappa\ \tau^* >_{\mathrm{j}} f(\nu)$. Hence $j(\tau^*) >_{j_{1,2}} j(f)(\alpha) = \tau$.
- $n > 1$: Assume $\tau \in j_n(j(\kappa) \setminus \kappa)$ Since $j_{n-1,n}(j_n(\kappa) \setminus j_{n-1}(\kappa)) = j_n(j(\kappa) \setminus \kappa)$, we get $\tau \in j_{n-1,n}(j_n(\kappa) \setminus j_{n-1}(\kappa))$. By the case $n = 1$ applied in M_{n-1} there is $\tau'^* \in j_n(\kappa)$ such that $j_{n-1,n}(\tau'^*) >_{j_{n,n+1}} \tau$. By the induction hypothesis, there is $\tau^* \in j(\kappa)$ such that $j_{n-1}(\tau^*) >_{j_{n-1,n}} \tau'^*$. So

$$j_n(\tau^*) = j_{n-1,n}(j_{n-1}(\tau^*)) >_{j_{n,n+1}} j_{n-1,n}(\tau'^*) >_{j_{n,n+1}} \tau.$$

□

Corollary 2.6. *Assume* $j : V \to M \supset M^\kappa$, $\mathrm{crit}(j) = \kappa$, $\mathfrak{g}(j) \subset j(\kappa)$, $n < \omega$ *and* $x \in M_n$. *Then there are* $f : [\kappa]^n \to V$ *and* $\alpha \in j(\kappa)$ *such that* $x = j_n(f)(\alpha, j(\alpha), \dots, j_{n-1}(\alpha))$.

2.2. **Trees and U-trees.**

Definition 2.7. $[\kappa]^{<\omega} = \{\langle \nu_0, \dots, \nu_k \rangle \mid k < \omega,\ \nu_0 < \cdots < \nu_k < \kappa\}$.

Definition 2.8. A set $T \subseteq [\kappa]^{<\omega}$ ordered by end-extension and closed under initial segments is called a tree.

Definition 2.9. Assume $T \subseteq [\kappa]^{<\omega}$ is a tree. Then for each $k < \omega$

1. $\forall \langle \nu_0, \dots, \nu_k \rangle \in T$ $\mathrm{Suc}_T(\nu_0, \dots, \nu_k) = \{\nu < \kappa \mid \langle \nu_0, \dots, \nu_k, \nu \rangle \in T\}$.
2. $\mathrm{Lev}_k(T) = T \cap [\kappa]^{k+1}$.

Definition 2.10. Assume $T \subseteq [\kappa]^{<\omega}$ is a tree, $k < \omega$ and $\langle \nu_0, \dots, \nu_{k-1} \rangle \in T$. Then

$$T_{\langle \nu_0, \dots, \nu_{k-1} \rangle} = \{\langle \nu_k, \dots, \nu_n \rangle \in [\kappa]^{<\omega} \mid \langle \nu_0, \dots, \nu_{k-1}, \nu_k, \dots, \nu_n \rangle \in T\}.$$

Note the degenerate case $T_{\langle\rangle} = T$.

Definition 2.11. Assume $T \subseteq [\kappa]^{<\omega}$ is a tree, $k < \omega$ and $A \subseteq [\kappa]^{k+1}$. Then

$$T \restriction A = \{\langle \nu_0, \dots, \nu_n \rangle \in T \mid n < \omega,\ \langle \nu_0, \dots, \nu_k \rangle \in A\}.$$

Definition 2.12. Assume $T \subseteq [\kappa]^{<\omega}$ is a tree and $\pi : \kappa \to \kappa$. Then

$$\pi^{-1}(T) = \{\langle \nu_0, \dots, \nu_k \rangle \in [\kappa]^{<\omega} \mid k < \omega,\ \langle \pi(\nu_0), \dots, \pi(\nu_k) \rangle \in T\}.$$

Definition 2.13. Let F be a function such that $\mathrm{dom}\, F \subseteq [\kappa]^{<\omega}$ is a tree, $k < \omega$ and $\langle \nu_0, \dots, \nu_{k-1} \rangle \in \mathrm{dom}\, F$. Then $F_{\langle \nu_0, \dots, \nu_{k-1} \rangle}$ is the function defined as follows:

1. $\mathrm{dom}(F_{\langle \nu_0, \dots, \nu_{k-1} \rangle}) = (\mathrm{dom}\, F)_{\langle \nu_0, \dots, \nu_{k-1} \rangle}$.
2. $F_{\langle \nu_0, \dots, \nu_{k-1} \rangle}(\nu_k, \dots, \nu_n) = F(\nu_0, \dots, \nu_{k-1}, \nu_k, \dots, \nu_n)$.

Note the degenerate case $F_{\langle\rangle} = F$.

Definition 2.14. Let F be a function such that $\mathrm{dom}\, F \subseteq [\kappa]^{<\omega}$ is a tree and $\pi : \kappa \to \kappa$. Then $\pi^{-1}(F)$ is the function defined as follows:

1. $\mathrm{dom}\, \pi^{-1}(F) = \pi^{-1}(\mathrm{dom}\, F)$.
2. $\forall \langle \nu_0, \dots, \nu_k \rangle \in \mathrm{dom}\, \pi^{-1}(F)$ $(\pi^{-1}(F))(\nu_0, \dots, \nu_k) = F(\pi(\nu_0), \dots, \pi(\nu_k))$.

From now until the end of the section we assume U is a κ-complete ultrafilter on κ.

Definition 2.15. A tree $T \subseteq [\kappa]^{<\omega}$ is a U-tree if

1. $\mathrm{Lev}_0(T) \in U$.
2. $\forall k < \omega$ $\forall \langle \nu_0, \dots, \nu_k \rangle \in T$ $\mathrm{Suc}_T(\nu_0, \dots, \nu_k) \in U$.

We recall the definition of filter product in order to define powers of U.

Definition 2.16. We define powers of U by induction as follows:

1. For $k = 1$: $U^1 = U$.
2. For $1 < k < \omega$: $\forall A \subseteq [\kappa]^k$, $A \in U^k \iff$

$$\{\langle \nu_0, \dots, \nu_{k-2} \rangle \in [\kappa]^{k-1} \mid \{\nu_{k-1} \in \kappa \mid \langle \nu_0, \dots, \nu_{k-2}, \nu_{k-1} \rangle \in A\} \in U\} \in U^{k-1}.$$

Note that we identify κ with $[\kappa]^1$ in this definition. Recall that U^k is a κ-complete ultrafilter on $[\kappa]^k$.

The following is straightforward.

Proposition 2.17. *Assume $T \subseteq [\kappa]^{<\omega}$ and $T^\xi \subseteq [\kappa]^{<\omega}$ are U-trees for each $\xi < \zeta$, where $\zeta < \kappa$. Then*

(1) $\forall k < \omega \ \operatorname{Lev}_k(T) \in U^{k+1}$.
(2) $\forall k < \omega \ \forall \langle \nu_0, \ldots, \nu_{k-1} \rangle \in T \ T_{\langle \nu_0, \ldots, \nu_{k-1} \rangle}$ *is a U-tree.*
(3) $\forall k < \omega \ (A \in U^{k+1} \implies T \restriction A$ *is a U-tree).*
(4) $\bigcap_{\xi < \zeta} T^\xi$ *is a U-tree.*

3. $\mathbb{P}_E$-Forcing

In this section we give a detailed presentation of the Prikry on extender forcing notion. We assume the existence of an elementary embedding $j : V \to M \supset M^\kappa$ such that $j(\kappa) \supset \mathfrak{g}(j)$ and $\operatorname{crit}(j) = \kappa$. We assume the GCH. Let E be the extender derived from j. Recall $\operatorname{dom} E = j(\kappa) \setminus \kappa$.

Definition 3.1. Assume $d \subseteq \operatorname{dom} E$ and $|d| \leq \kappa$. Then

$$\operatorname{mc}(d) = \min\{\alpha \in \operatorname{dom} E \mid \forall \beta \in d \ \alpha \geq_{\mathrm{E}} \beta\}.$$

Note that there is $h : \kappa \to \kappa$ such that $j(h)(\operatorname{mc}(d)) = j'' d$.

Definition 3.2. We define the forcing notion $\mathbb{P}^*_E$ as follows.

$$\mathbb{P}^*_E = \{f : d \to [\kappa]^{<\omega} \mid d \subseteq \operatorname{dom} E, \ |d| \leq \kappa, \ \kappa \in d, \ \operatorname{mc}(d) \in d\}.$$

The partial order $\leq^*$ on $\mathbb{P}^*_E$ is defined by: $f \leq^* g \iff f \supseteq g$. (Note that $\mathbb{P}^*_E$ is the Cohen forcing for adding $|j(\kappa)|$ subsets to κ^+.)

Definition 3.3. Assume $f \in \mathbb{P}^*_E$. Then $\operatorname{mc}(f) = \operatorname{mc}(\operatorname{dom} f)$.

The Prikry on extender forcing is defined as follows.

Definition 3.4. A condition p in $\mathbb{P}_E$ is of the form $\langle f, F \rangle$ where

(1) $f \in \mathbb{P}^*_E$.
(2) $F : T \to [\operatorname{dom} f]^{<\kappa}$ is such that for each $k < \omega$
  (a) T is an $E(\operatorname{mc}(f))$-tree.
  (b) $\forall \langle \nu_0, \ldots, \nu_{k-1} \rangle \in T \ j(F_{\langle \nu_0, \ldots, \nu_{k-1} \rangle})(\operatorname{mc}(f)) = j'' \operatorname{dom} f$.
  (c) $\forall \langle \nu_0, \ldots, \nu_{k-1}, \nu \rangle \in T \ \kappa \in F_{\langle \nu_0, \ldots, \nu_{k-1} \rangle}(\nu)$.
  (d) $\forall \langle \nu_0, \ldots, \nu_{k-1}, \nu \rangle \in T \ \big(|F_{\langle \nu_0, \ldots, \nu_{k-1} \rangle}(\nu)| \leq \pi_{\operatorname{mc}(f), \kappa}(\nu)\big)$.
  (e) $\forall \langle \nu_0, \ldots, \nu_{k-1}, \nu_k \rangle \in T \ \big(F(\nu_0, \ldots, \nu_{k-1}) \subseteq F(\nu_0, \ldots, \nu_{k-1}, \nu_k)\big)$.
  (f) $\forall \beta \in \operatorname{dom} f \ \forall \langle \nu_0, \ldots, \nu_k \rangle \in T$
$$f(\beta)^\frown \langle \pi_{\operatorname{mc}(f), \beta}(\nu_i) \mid i \leq k, \ \beta \in F(\nu_0, \ldots, \nu_i) \rangle \in [\kappa]^{<\omega}.$$
  (Recall that $[\kappa]^{<\omega}$ is the set of finite *increasing* sequences in κ).

We write $\operatorname{supp} p$, $\operatorname{mc}(p)$, f^p, F^p, and $\operatorname{dom} p$, for $\operatorname{dom} f$, $\operatorname{mc}(f)$, f, F, and T, respectively.

Proposition 3.5. *Assume $\langle f, F \rangle$ satisfy* (1), (2a)*, and* (2b) *of definition 3.4, then there is a function F^* such that $F \restriction \operatorname{dom} F^* = F^*$ and $\langle f, F^* \rangle \in \mathbb{P}_E$.*

Proof. We fix demands (2c), (2d) and (2f) as follows. We note that for each $n<\omega$ and $\langle \nu_0,\dots,\nu_{n-1}\rangle \in \operatorname{dom} F$,

$$\begin{aligned}
& j(F_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\operatorname{mc}(f)) = j'' \operatorname{dom} f,\\
& j(\kappa) \in j'' \operatorname{dom} f,\\
& |j'' \operatorname{dom} f| \leq \kappa,
\end{aligned}$$

and

$$\forall \beta \in j'' \operatorname{dom} f \ \big(\max j(f)(\beta) < j(\pi)_{j(\operatorname{mc}(f)),\beta}(\operatorname{mc}(f))\big).$$

Łoś theorem yields that for each $n<\omega$ and $\langle \nu_0,\dots,\nu_{n-1}\rangle \in \operatorname{dom} F$,

$$\begin{aligned}
\{\nu_{n-1} < \nu < \kappa \mid \kappa \in F_{\langle \nu_0,\dots,\nu_{n-1}\rangle}(\nu),\ |F_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\nu)| \leq \pi_{\operatorname{mc}(f),\kappa}(\nu),\\
\forall \beta \in F_{\langle \nu_0,\dots,\nu_{n-1}\rangle}(\nu)\ \big(\max f(\beta) < \pi_{\operatorname{mc}(f),\beta}(\nu)\big)\} \in E(\operatorname{mc}(f)).
\end{aligned}$$

So we shrink $\operatorname{dom} F$ to these sets, working up $\operatorname{dom} F$ level by level. Thus we get for each $n<\omega$ and $\langle \nu_0,\dots,\nu_{n-1},\nu\rangle \in \operatorname{dom} F$,

$$\begin{aligned}
& \kappa \in F_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\nu),\\
& |F_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\nu)| \leq \pi_{\operatorname{mc}(f),\kappa}(\nu),
\end{aligned}$$

and

$$\forall \beta \in F_{\langle \nu_0,\dots,\nu_{n-1}\rangle}(\nu)\ \big(\max f(\beta) < \pi_{\operatorname{mc}(f),\beta}(\nu)\big).$$

Demand (2e) is fixed by taking unions along branches. That is for each $n<\omega$, and $\langle \nu_0,\dots,\nu_n\rangle \in \operatorname{dom} F$ we set

$$F^*(\nu_0,\dots,\nu_n) = \bigcup_{k\leq n} F(\nu_0,\dots,\nu_k).$$

Thus $\langle f, F^*\rangle \in \mathbb{P}_E$. □

Definition 3.6. Let $p,q \in \mathbb{P}_E$. We say that p is a Prikry extension of q ($p \leq^* q$) if

(1) $\operatorname{supp} p \supseteq \operatorname{supp} q$.
(2) $f^p \restriction \operatorname{supp} q = f^q$.
(3) $\operatorname{dom} p \subseteq \pi^{-1}_{\operatorname{mc}(p),\operatorname{mc}(q)}(\operatorname{dom} q)$.
(4) $\forall k>0\ \forall \langle \nu_0,\dots,\nu_{k-1},\nu\rangle \in \operatorname{dom} p\ \forall \beta \in (\pi^{-1}_{\operatorname{mc}(p),\operatorname{mc}(q)}(F^q))(\nu_0,\dots,\nu_{k-1},\nu)$

$$\pi_{\operatorname{mc}(p),\beta}(\nu) = \pi_{\operatorname{mc}(q),\beta}(\pi_{\operatorname{mc}(p),\operatorname{mc}(q)}(\nu)).$$

(5) $\forall k>0\ \forall \langle \nu_0,\dots,\nu_k\rangle \in \operatorname{dom} p$

$$F^p(\nu_0,\dots,\nu_k) \supseteq (\pi^{-1}_{\operatorname{mc}(p),\operatorname{mc}(q)}(F^q))(\nu_0,\dots,\nu_k),$$

and

$$F^p(\nu_0,\dots,\nu_k) \setminus (\pi^{-1}_{\operatorname{mc}(p),\operatorname{mc}(q)}(F^q))(\nu_0,\dots,\nu_k) \subseteq \operatorname{supp} p \setminus \operatorname{supp} q.$$

Definition 3.7. Let $q \in \mathbb{P}_E$ and $\langle \nu\rangle \in \operatorname{dom} q$. We define $q_{\langle\nu\rangle} \in \mathbb{P}_E$ to be p where

(1) $\operatorname{supp} p = \operatorname{supp} q$.
(2) $\forall \beta \in \operatorname{supp} p\ f^p(\beta) = \begin{cases} f^q(\beta) ^\frown \langle \pi_{\operatorname{mc}(q),\beta}(\nu)\rangle & \text{if } \beta \in F^q(\nu).\\ f^q(\beta) & \text{if } \beta \notin F^q(\nu).\end{cases}$
(3) $F^p = F^q_{\langle\nu\rangle}$.

For $k > 0$ we define $q_{\langle\nu_0,\dots,\nu_k\rangle}$ recursively, setting $q_{\langle\nu_0,\dots,\nu_k\rangle} = (q_{\langle\nu_0,\dots,\nu_{k-1}\rangle})_{\langle\nu_k\rangle}$.

Definition 3.8. Let $p, q \in \mathbb{P}_E$. We say that p is a 1-point extension of q ($p \leq^1 q$) if there is $\langle\nu\rangle \in \operatorname{dom} q$ such that $p \leq^* q_{\langle\nu\rangle}$.

Definition 3.9. Let $p, q \in \mathbb{P}_E$ and $n < \omega$. We say that p is an n-point extension of q ($p \leq^n q$) if there are $p^n, \dots, p^0$ such that

$$p = p^n \leq^1 \dots \leq^1 p^0 = q.$$

Definition 3.10. Let $p, q \in \mathbb{P}_E$. We say that p is an extension of q ($p \leq q$) if there is an $n < \omega$ such that $p \leq^n q$.

Thus we have the forcing notions $\mathbb{P}_E = \langle\mathbb{P}_E, \leq\rangle$ and $\mathbb{P}^*_E = \langle\mathbb{P}^*_E, \leq^*\rangle$.
The following is immediate from the definition of the forcing notion.

Claim 3.11. *Let $p, q \in \mathbb{P}_E$ be such that $p \leq q$. Then there are $k < \omega$ and $\langle\nu_0, \dots, \nu_{k-1}\rangle \in \operatorname{dom} q$ such that $p \leq^* q_{\langle\nu_0,\dots,\nu_{k-1}\rangle}$.*

Definition 3.12. Let G be $\mathbb{P}_E$-generic. Then

$$\forall\alpha \in \operatorname{dom} E\ G^\alpha = \bigcup\{f^p(\alpha) \mid p \in G,\ \alpha \in \operatorname{supp} p\}.$$

We write $\underset{\sim}{G}^\alpha$ for the $\mathbb{P}_E$-name of G^α.

Given a condition $\langle f, F\rangle$ and $g \leq^* f$, the pair $\langle g, \pi^{-1}_{\mathrm{mc}(g),\mathrm{mc}(f)}(F)\rangle$ might not be a condition, and if it was it might not satisfy $\langle f, \pi^{-1}_{\mathrm{mc}(g),\mathrm{mc}(f)}(F)\rangle \leq^* \langle f, F\rangle$. (The transitivity $\pi_{\mathrm{mc}(g),\beta} = \pi_{\mathrm{mc}(f),\beta} \circ \pi_{\mathrm{mc}(g),\mathrm{mc}(f)}$ may be violated on some ν's and β's). The following lemma shows that by removing a measure zero set from $\operatorname{dom} \pi^{-1}_{\mathrm{mc}(g),\mathrm{mc}(f)}(F)$, we get a condition and the transitivity.

Claim 3.13. *Assume $p \in \mathbb{P}_E$ and $f \in \mathbb{P}^*_E$ are such that $f \leq^* f^p$. Then there is $q \leq^* p$ such that $f^q = f$.*

Proof. Let $h : \kappa \to \mathcal{P}(\kappa)$ be such that $j(h)(\mathrm{mc}(f)) = j'' \operatorname{dom} f$. We define a function F' with domain $\pi^{-1}_{\mathrm{mc}(f),\mathrm{mc}(p)}(\operatorname{dom} p)$ as follows. For each $n < \omega$ and each $\langle\nu_0, \dots, \nu_{n-1}\rangle \in \operatorname{dom} F'$

$$F'(\nu_0, \dots, \nu_{n-1}) = (\pi^{-1}_{\mathrm{mc}(f),\mathrm{mc}(p)}(F^p))(\nu_0, \dots, \nu_{n-1}) \cup h(\nu_{n-1}).$$

The pair $\langle f, F'\rangle$ might not be a condition since demands (2d), (2e), and (2f), in definition 3.4 might be violated. We construct F'' from F' using 3.5, thus getting $\langle f, F''\rangle \in \mathbb{P}_E$. The obstacle to $\langle f, F''\rangle \leq^* p$ is the transitivity demand (4) of definition 3.6. We shrink $\operatorname{dom} F''$ as follows to ensure it. We observe that for each $\beta \in \operatorname{supp} p$,

$$j(\pi_{\mathrm{mc}(f),\beta})(\mathrm{mc}(f)) = \beta,$$
$$j(\pi_{\mathrm{mc}(p),\beta})(j(\pi_{\mathrm{mc}(f),\mathrm{mc}(p)})(\mathrm{mc}(f))) = \beta.$$

So in M we have

$$\forall\beta \in j'' \operatorname{supp} p\ j(\pi)_{j(\mathrm{mc}(f)),\beta}(\mathrm{mc}(f)) = j(\pi)_{j(\mathrm{mc}(p)),\beta}(j(\pi_{\mathrm{mc}(f),\mathrm{mc}(p)})(\mathrm{mc}(f))),$$

and for each $n < \omega$, and $\langle\nu_0, \dots, \nu_{n-1}\rangle \in \operatorname{dom} F^p$,

$$j(F^p_{\langle\nu_0,\dots,\nu_{n-1}\rangle})(\mathrm{mc}(p)) = j'' \operatorname{supp} p.$$

Applying Łoś theorem to the last two equations yields that for each $n < \omega$ and $\langle \nu_0, \dots, \nu_{n-1} \rangle \in \operatorname{dom} F''$,

$$A_{\langle \nu_0,\dots,\nu_{n-1}\rangle} = \{\nu < \kappa \mid \forall \beta \in (\pi^{-1}_{\operatorname{mc}(f),\operatorname{mc}(p)}(F^p))_{\langle \nu_0,\dots,\nu_{n-1}\rangle}(\nu) \\ \pi_{\operatorname{mc}(f),\beta}(\nu) = \pi_{\operatorname{mc}(p),\beta}(\pi_{\operatorname{mc}(f),\operatorname{mc}(p)}(\nu))\} \in E(\operatorname{mc}(f)).$$

So, let F be F'' shrunken to these sets, namely

$$\operatorname{Lev}_0(\operatorname{dom} F) = \operatorname{Lev}_0(\operatorname{dom} F'') \cap A_{\langle\rangle},$$

and

$$\forall n < \omega\ \forall \langle \nu_0, \dots, \nu_n \rangle \in \operatorname{dom} F\ \ \operatorname{Suc}_{\operatorname{dom} F}(\nu_0, \dots, \nu_n) = \\ \operatorname{Suc}_{\operatorname{dom} F''}(\nu_0, \dots, \nu_n) \cap A_{\langle \nu_0,\dots,\nu_n\rangle}.$$

Thus transitivity has been restored and we have $\langle f, F \rangle \leq^* p$. □

Corollary 3.14. *Let $q \in \mathbb{P}_E$ and $\alpha \in \operatorname{dom} E$. Then there is $p \leq^* q$ with $\alpha \in \operatorname{supp} p$.*

From the above propositions we see that for all $\alpha \in \operatorname{dom} E$, G^α is not empty. In fact using density arguments we get:

Proposition 3.15. *Let G be $\mathbb{P}_E$-generic. Then in $V[G]$:*
1. *$\operatorname{ot} G^\alpha = \omega$.*
2. *G^α is unbounded in κ.*
3. *$\alpha \neq \beta \implies G^\alpha \neq G^\beta$.*

Lemma 3.16. *Assume $\zeta < \kappa$, $\forall \xi < \zeta\ p^\xi \in \mathbb{P}_E$, and $\forall \xi_1, \xi_2 < \zeta\ \operatorname{supp} p^{\xi_1} = \operatorname{supp} p^{\xi_2}$. Then*

$$\{\langle \nu_0, \dots, \nu_n \rangle \in \bigcap_{\xi<\zeta} T^{p^\xi} \mid n < \omega,\ \forall \xi_1, \xi_2 < \zeta\ F^{p^{\xi_1}}(\nu_0, \dots, \nu_n) = F^{p^{\xi_2}}(\nu_0, \dots, \nu_n)\}$$

is an $E(\alpha)$-tree, where α is the common value of $\operatorname{mc}(p^\xi)$.

Proof. The claim follows by Łoś theorem from the fact that for each $n < \omega$ and $\langle \nu_0, \dots, \nu_{n-1} \rangle \in \bigcap_{\xi<\zeta} T^{p^\xi}$,

$$\forall \xi_1, \xi_2 < \zeta\ j(F^{p^{\xi_1}}_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\operatorname{mc}(p^{\xi_1})) = j'' \operatorname{supp} p^{\xi_1} = \\ j'' \operatorname{supp} p^{\xi_2} = j(F^{p^{\xi_2}}_{\langle \nu_0,\dots,\nu_{n-1}\rangle})(\operatorname{mc}(p^{\xi_2})),$$

and the existence of $\alpha \in \operatorname{dom} E$ such that $\forall \xi < \zeta\ \operatorname{mc}(p^\xi) = \alpha$. □

Proposition 3.17. *$\mathbb{P}_E$ satisfies the κ^{++}-cc.*

Proof. Assume $X \subseteq \mathbb{P}_E$ and $|X| = \kappa^{++}$. Since for each $p \in X$ we have $|\operatorname{supp} p| \leq \kappa$, we can assume that $\{\operatorname{supp} p \mid p \in X\}$ forms a Δ-system. That is, there is $d \in [\operatorname{dom} E]^{\leq\kappa}$ such that $\forall p, q \in X\ \operatorname{supp} p \cap \operatorname{supp} q = d$. Since $|d| \leq \kappa$ we have $|\{f \mid f : d \to [\kappa]^{<\omega}\}| \leq \kappa^+$, so we can assume that $\forall p, q \in X\ f^p \restriction d = f^q \restriction d$.

Let us fix two conditions $p, q \in X$. Let $f = f^p \cup f^q$. Then $f : \operatorname{supp} p \cup \operatorname{supp} q \to [\kappa]^{<\omega}$. By 3.13 there are $r^1 \leq^* p$ and $r^2 \leq^* q$ such that $f^{r^1} = f^{r^2} = f$. By 3.16 there is an $E(\operatorname{mc}(f))$-tree T such that $F^{r^1} \restriction T = F^{r^2} \restriction T$. We set $F = F^{r^1} \restriction T$. Then $\langle f, F \rangle \leq^* r^1, r^2$, thus $\langle f, F \rangle \leq^* p, q$. □

Up to this point we know that in a $\mathbb{P}_E$-generic extension we have
1. $\operatorname{cf} \kappa = \omega$.

(2) $2^\kappa = |j(\kappa)|$.
(3) Cardinals above κ^+ are preserved.

In order to see that no damage happens below κ we use the Prikry ordering.

Proposition 3.18. *$\langle \mathbb{P}_E, \leq^* \rangle$ is κ-closed.*

Proof. Assume $\zeta < \kappa$ and $\langle p^\xi \mid \xi < \zeta \rangle \subseteq \mathbb{P}_E$ are such that $\forall \xi_2 < \xi_1 < \zeta \; p^{\xi_1} \leq^* p^{\xi_2}$.
By the definition of $\leq^*$ we have

$$\forall \xi_1 < \xi_2 < \zeta \; f^{p^{\xi_2}} \restriction \operatorname{supp} p^{\xi_1} = f^{p^{\xi_1}},$$

hence $f = \cup \{ f^{p^\xi} \mid \xi < \zeta \} : \bigcup_{\xi < \zeta} \operatorname{supp} p^\xi \to [\kappa]^{<\omega}$. For each $\xi < \zeta$, by 3.13, there is $r^\xi \leq^* p^\xi$ such that $f^{r^\xi} = f$. By 3.16 there is T, an $E(f)$-tree, such that $\forall \xi_1, \xi_2 < \zeta$ $F^{\xi_1} \restriction T = F^{\xi_2} \restriction T$. We set $F = F^{r^0} \restriction T$ (that is, the common value of $F^\xi \restriction T$.) Then $\forall \xi < \zeta \; \langle f, F \rangle \leq^* r^\xi$, thus $\forall \xi < \zeta \; \langle f, F \rangle \leq^* p^\xi$. □

We show that forcing with $\langle \mathbb{P}_E, \leq^* \rangle$ is the same as forcing with the Cohen forcing for adding $|j(\kappa)|$ subsets to κ^+.

Lemma 3.19. *Assume G^* is $\langle \mathbb{P}_E, \leq^* \rangle$-generic and $p \in G^*$. If $q \in \mathbb{P}_E$ is such that $f^q = f^p$ then $q \in G^*$.*

Proof. Assume $p, q \in \mathbb{P}_E$ are such that $f^p = f^q$. Pick any $r \leq^* q$. Since $f^r \leq^* f^p$, by 3.13 there is $s \leq^* p$ such that $f^s = f^r$. By 3.16 there is T, an $E(\operatorname{mc}(f^s))$-tree, such that $F^s \restriction T = F^r \restriction T$. Then $\langle f^s, F^s \restriction T \rangle \leq^* r, s$, thus $\langle f^s, F^s \restriction T \rangle \leq^* p, q$
That is, the order $\leq^*$ does not separate p from q. □

Corollary 3.20. *Forcing with $\langle \mathbb{P}_E, \leq^* \rangle$ is the same as forcing with $\mathbb{P}^*_E$.*

The following lemma implies that a κ^+-closed forcing is κ^+-proper (see definition 3.28).

Lemma 3.21. *Let χ be large enough so that $\mathcal{P}(\mathcal{P}(\mathbb{P}^*_E)) \in H_\chi$. Let $N \prec H_\chi$, $f \in \mathbb{P}^*_E$ be such that $f \in N$, $|N| = \kappa$, and $N \supseteq N^{<\kappa}$. Then there is $f^* \leq^* f$ such for each $D \in N$ a dense open subset of $\mathbb{P}^*_E$, there is $g \geq^* f^*$ such that $g \in D \cap N$.*

Proof. Let $\langle D_\xi \mid \xi < \kappa \rangle$ be an enumeration of all dense open subsets of $\mathbb{P}^*_E$ appearing in N. Since $\mathbb{P}^*_E$ is κ^+-closed, and for each $\zeta < \kappa$, $\langle D_\xi \mid \xi < \zeta \rangle \in N$, it is possible to construct a $\leq^*$-decreasing sequence $\langle f^\xi \mid \xi \leq \kappa \rangle$ so that $f^0 \leq^* f$, and for each $\xi < \kappa$, $f^\xi \in D_\xi \cap N$. The lemma is proved by taking $f^* = f^\kappa$. □

The following definition is used only in the lemma following it. It defines a '1-point' extension of a condition in $\mathbb{P}^*_E$.

Definition 3.22. Assume $f \in \mathbb{P}^*_E$, $a \subseteq \operatorname{dom} E$, $\beta \geq_E \operatorname{mc}(a)$, and $\nu < \kappa$. We define $f_{\langle \nu, \beta, a \rangle} \in \mathbb{P}^*_E$ to be the function g defined as follows:

(1) $\operatorname{dom} g = \operatorname{dom} f$.
(2) $g(\alpha) = \begin{cases} f(\alpha)^\frown \langle \pi_{\beta,\alpha}(\nu) \rangle & \alpha \in a \cap \operatorname{dom} f, \; \pi_{\beta,\alpha}(\nu) > \max f(\alpha). \\ f(\alpha) & \text{Otherwise.} \end{cases}$

For $k > 0$ we define $f_{\langle \nu_0, \beta_0, a_0 \rangle, \ldots, \langle \nu_k, \beta_k, a_k \rangle}$ recursively, setting

$$f_{\langle \nu_0, \beta_0, a_0 \rangle, \ldots, \langle \nu_k, \beta_k, a_k \rangle} = (f_{\langle \nu_0, \beta_0, a_0 \rangle, \ldots, \langle \nu_{k-1}, \beta_{k-1}, a_{k-1} \rangle})_{\langle \nu_k, \beta_k, a_k \rangle}.$$

Lemma 3.23. *Assume $p \in \mathbb{P}_E$, $n < \omega$, and for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in \operatorname{dom} p$, the set $D(\nu_0, \dots, \nu_{n-1})$ is a dense open subset of $\langle \mathbb{P}_E, \leq^* \rangle$ below $p_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$. Then there is $p^* \leq^* p$ such that for each $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} p^*$*

$$p^*_{\langle \mu_0, \dots, \mu_{n-1} \rangle} \in D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})).$$

Proof. Assume $p \in \mathbb{P}_E$, $n < \omega$, and for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in \operatorname{dom} p$, the set $D(\nu_0, \dots, \nu_{n-1})$ is a dense open subset of $\langle \mathbb{P}_E, \leq^* \rangle$ below $p_{\langle \nu_0, \dots, \nu_{n-1} \rangle}$.

Let χ be large enough so that $\mathcal{P}(\mathcal{P}(\mathbb{P}_E)) \in H_\chi$. Let $N \prec H_\chi$ be such that $p, \mathbb{P}_E \in N$, $\{D(\nu_0, \dots, \nu_{n-1}) \mid \langle \nu_0, \dots, \nu_{n-1} \rangle \in \operatorname{dom} p\} \in N$, $|N| = \kappa$, and $N \supseteq N^{<\kappa}$. Construct $f^* \leq^* f^p$ by applying 3.21 to N. By 3.13 there is $p^* \leq^* p$ such that $f^{p^*} = f^*$. For each $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} F^{p^*}$ we set

$$\begin{aligned} D^*(\mu_0, \dots, \mu_{n-1}) = \{ f \leq^* f^p \mid{} & \operatorname{dom} f \supseteq F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N, \\ & \exists H \ \langle f_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle}, H \rangle \in \\ & \qquad D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})) \}. \end{aligned}$$

(Note that $D^*(\mu_0, \dots, \mu_{n-1}) \in N$.) We show that for each $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} F^{p^*}$, the set $D^*(\mu_0, \dots, \mu_{n-1})$ is dense open below f^p. So, let us fix some $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} F^{p^*}$, and pick $g \leq^* f^p$.

First we enlarge g so as to ensure $\operatorname{dom} g \supseteq F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N$. By definition,

$$\begin{aligned} & g_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle} \leq^* \\ & \qquad f^p_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \rangle} = \\ & \qquad \qquad f^{p_{\langle \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1}) \rangle}}. \end{aligned}$$

Hence, by 3.13, there is $q \leq^* p_{\langle \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1}) \rangle}$ such that

$$f^q = g_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle}.$$

Thus there is $q^* \leq^* q$ such that $q^* \in D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1}))$. We set $g^* = g \cup (f^{q^*} \restriction (\operatorname{supp} q^* \setminus \operatorname{supp} q))$. Then

$$g^* \leq^* g,$$

and

$$g^*_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle} = f^{q^*}.$$

Hence $g^* \in D^*(\mu_0, \dots, \mu_{n-1})$, by which density was proved.

Since f^{p^*} was constructed by 3.21, we have that for each $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} F^{p^*}$, there is $g \geq^* f^{p^*}$ such that $g \in D^*(\mu_0, \dots, \mu_{n-1}) \cap N$. Thus for each $\langle \mu_0, \dots, \mu_{n-1} \rangle \in \operatorname{dom} F^{p^*}$ there are $g^{\mu_0, \dots, \mu_{n-1}} \in N$ and $H(\mu_0, \dots, \mu_{n-1}) \in N$ such that $g^{\mu_0, \dots, \mu_{n-1}} \geq^* f^{p^*}$ and

$$\begin{aligned} & \langle g^{\mu_0, \dots, \mu_{n-1}}_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle}, H(\mu_0, \dots, \mu_{n-1}) \rangle \in \\ & \qquad D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0), \dots, \pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})) \cap N. \end{aligned}$$

We note that since $g^{\mu_0, \dots, \mu_{n-1}} \in N$ we have

$$\begin{aligned} & g^{\mu_0, \dots, \mu_{n-1}}_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \cap N \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \cap N \rangle} = \\ & \qquad g^{\mu_0, \dots, \mu_{n-1}}_{\langle \mu_0, \mathrm{mc}(p^*), F^{p^*}(\mu_0) \rangle, \dots, \langle \mu_{n-1}, \mathrm{mc}(p^*), F^{p^*}(\mu_0, \dots, \mu_{n-1}) \rangle}, \end{aligned}$$

thus

$$\langle g^{\mu_0,\dots,\mu_{n-1}}_{\langle\mu_0,\mathrm{mc}(p^*),F^{p^*}(\mu_0)\rangle,\dots,\langle\mu_{n-1},\mathrm{mc}(p^*),F^{p^*}(\mu_0,\dots,\mu_{n-1})\rangle}, H(\mu_0,\dots,\mu_{n-1})\rangle \in D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1}))\cap N.$$

We shrink $\operatorname{dom} p^*$ in order to get that for each $\langle\mu_0,\dots,\mu_{n-1}\rangle\in\operatorname{dom} F^{p^*}$

$$\langle f^{p^*}_{\langle\mu_0,\mathrm{mc}(p^*),F^{p^*}(\mu_0)\rangle,\dots,\langle\mu_{n-1},\mathrm{mc}(p^*),F^{p^*}(\mu_0,\dots,\mu_{n-1})\rangle}, F^{p^*}_{\langle\mu_0,\dots,\mu_{n-1}\rangle}\rangle \leq^* \langle g^{\mu_0,\dots,\mu_{n-1}}_{\langle\mu_0,\mathrm{mc}(p^*),F^{p^*}(\mu_0)\rangle,\dots,\langle\mu_{n-1},\mathrm{mc}(p^*),F^{p^*}(\mu_0,\dots,\mu_{n-1})\rangle}, H(\mu_0,\dots,\mu_{n-1})\rangle.$$

By the openness of $D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1}))$ and the fact

$$p^*_{\langle\mu_0,\dots,\mu_{n-1}\rangle} = \langle f^{p^*}_{\langle\mu_0,\mathrm{mc}(p^*),F^{p^*}(\mu_0)\rangle,\dots,\langle\mu_{n-1},\mathrm{mc}(p^*),F^{p^*}(\mu_0,\dots,\mu_{n-1})\rangle}, F^{p^*}_{\langle\mu_0,\dots,\mu_{n-1}\rangle}\rangle,$$

we get that for each $\langle\mu_0,\dots,\mu_{n-1}\rangle\in\operatorname{dom} F^{p^*}$

$$p^*_{\langle\mu_0,\dots,\mu_{n-1}\rangle}\in D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})).$$

□

Lemma 3.24. *Let $D\subseteq\mathbb{P}_E$ be dense open, $p\in\mathbb{P}_E$ and $n<\omega$. Then there is $p^*\leq^* p$ such that either*

$$\forall\langle\nu_0,\dots,\nu_{n-1}\rangle\in\operatorname{dom} p^*\ p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\in D$$

or

$$\forall\langle\nu_0,\dots,\nu_{n-1}\rangle\in\operatorname{dom} p^*\ \forall q\leq^* p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\ q\notin D.$$

Proof. Assume D is a dense open subset of $\mathbb{P}_E$, $p\in\mathbb{P}_E$ and $n<\omega$.

For each $\langle\nu_0,\dots,\nu_{n-1}\rangle\in\operatorname{dom} p$ set

$$D^{\in}(\nu_0,\dots,\nu_{n-1}) = \{q\leq^* p_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\mid q\in D\},$$
$$D^{\perp}(\nu_0,\dots,\nu_{n-1}) = \{r\leq^* p_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\mid \forall q\in D^{\in}(\nu_0,\dots,\nu_{n-1})\ r\perp^* q\},$$
$$D(\nu_0,\dots,\nu_{n-1}) = D^{\in}(\nu_0,\dots,\nu_{n-1})\cup D^{\perp}(\nu_0,\dots,\nu_{n-1}).$$

The openness of D guarantees the $\leq^*$-openness of $D^{\in}(\nu_0,\dots,\nu_{n-1})$, and by its definition, $D^{\perp}(\nu_0,\dots,\nu_{n-1})$ is $\leq^*$-open. Hence $D(\nu_0,\dots,\nu_{n-1})$, as a union of two open sets, is open, and in fact it is also $\leq^*$-dense below $p_{\langle\nu_0,\dots,\nu_{n-1}\rangle}$.

By 3.23, there is $p^*\leq^* p$ such that

$$\forall\langle\mu_0,\dots,\mu_{n-1}\rangle\in\operatorname{dom} p^*\quad p^*_{\langle\mu_0,\dots,\mu_{n-1}\rangle}\in D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})).$$

In order not to carry the projections $\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}$ all over the proof, we define

$$E(\mu_0,\dots,\mu_{n-1}) = D(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})),$$
$$E^{\in}(\mu_0,\dots,\mu_{n-1}) = D^{\in}(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})),$$

and

$$E^{\perp}(\mu_0,\dots,\mu_{n-1}) = D^{\perp}(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\mu_{n-1})).$$

Since each of the $E(\mu_0,\dots,\mu_{n-1})$ is the disjoint union of $E^{\in}(\mu_0,\dots,\mu_{n-1})$, and $E^{\perp}(\mu_0,\dots,\mu_{n-1})$, we can shrink $\operatorname{dom} p^*$ so as to get either

$$\forall\langle\nu_0,\dots,\nu_{n-1}\rangle\in\operatorname{dom} p^*\ p^*_{\langle\nu_0,\dots,\nu_{n-1}\rangle}\in E^{\in}(\nu_0,\dots,\nu_{n-1})$$

or

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in E^{\perp}(\nu_0,\ldots,\nu_{n-1}).$$

Looking at the definition of $E^{\perp}$ and $E^{\in}$ (and implicitly the definitions of $D^{\perp}$ and $D^{\in}$), we see that we have either

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in D$$

or

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ \forall q \leq^* p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}\ q \notin D.$$

□

Theorem 3.25. *Let $D \subseteq \mathbb{P}_E$ be dense open and $p \in \mathbb{P}_E$. Then there are $p^* \leq^* p$ and $n < \omega$ such that $\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in D$.*

Proof. Assume D is a dense open subset of $\mathbb{P}_E$ and $p \in \mathbb{P}_E$. For each $n < \omega$ we set

$$\begin{aligned} D^*_n = \{p^* \leq^* p \mid (&\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in D) \text{ or} \\ &(\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ \forall q \leq^* p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}\ q \notin D)\}. \end{aligned}$$

By 3.24, D^*_n is a dense open subset of $\langle\mathbb{P}_E, \leq^*\rangle$ below p. Since $\langle\mathbb{P}_E, \leq^*\rangle$ is κ-closed, the set $D^* = \bigcap_{n<\omega} D^*_n$ is a dense open subset of $\langle\mathbb{P}_E, \leq^*\rangle$ below p. We pick $p^* \in D^*$. Then for each $n < \omega$, either

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in D,$$

or

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ \forall q \leq^* p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}\ q \notin D.$$

Towards a contradiction, let us assume that for each $n < \omega$

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ \forall q \leq^* p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle}\ q \notin D.$$

This is just a cumbersome way to write $\forall q \leq p\ q \notin D$, contradicting the density of D. Thus, there is $n < \omega$ such that

$$\forall\langle\nu_0,\ldots,\nu_{n-1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle\nu_0,\ldots,\nu_{n-1}\rangle} \in D.$$

□

Claim 3.26. *Let σ be a statement in the $\mathbb{P}_E$-forcing language and $p \in \mathbb{P}_E$. Then there is $p^* \leq^* p$ such that $p^* \parallel \sigma$.*

Proof. Let $D = \{q \in \mathbb{P}_E \mid q \parallel \sigma\}$. Then D is a dense open subset of $\mathbb{P}_E$. By 3.25 there are $p^{*\prime} \leq p$ and $k < \omega$ such that

$$\forall\langle\nu_0,\ldots,\nu_k\rangle \in \operatorname{dom} p^{*\prime}\ p^{*\prime}_{\langle\nu_0,\ldots,\nu_k\rangle} \in D.$$

That is

$$\forall\langle\nu_0,\ldots,\nu_k\rangle \in \operatorname{dom} p^{*\prime}\ p^{*\prime}_{\langle\nu_0,\ldots,\nu_k\rangle} \parallel \sigma.$$

Let

$$\begin{aligned} A_1 &= \{\langle\nu_0,\ldots,\nu_k\rangle \in \operatorname{dom} p^{*\prime} \mid p^{*\prime}_{\langle\nu_0,\ldots,\nu_k\rangle} \Vdash \sigma\}, \\ A_2 &= \{\langle\nu_0,\ldots,\nu_k\rangle \in \operatorname{dom} p^{*\prime} \mid p^{*\prime}_{\langle\nu_0,\ldots,\nu_k\rangle} \Vdash \neg\sigma\}. \end{aligned}$$

Obviously, $A_1 \cap A_2 = \emptyset$. Let $i \in \{1,2\}$ be so that $A_i \in E^k(\mathrm{mc}(p^{*\prime}))$. Let $p^* = \langle f^{p^{*\prime}}, F^{p^{*\prime}} \upharpoonright A_i \rangle$. Since $\{p^*_{\langle \nu_0,\dots,\nu_k\rangle} \mid \langle \nu_0,\dots,\nu_k\rangle \in \operatorname{dom} p^*\}$ is a maximal anti-chain below p^*, we get $p^* \parallel \sigma$. $\square$

So now we know also that in a $\mathbb{P}_E$-generic extension:

(1) There are no new bounded subsets of κ.
(2) (Hence) No cardinal below κ is collapsed.
(3) (Hence) κ is not collapsed.

One more cardinal is preserved, κ^+. We will prove the κ^+-properness of $\mathbb{P}_E$ in order to show this.

The notions $\langle N, P\rangle$-generic and properness, as defined in [11], were used for countable elementary submodels of H_χ. We need these notions for submodels of size κ.

Definition 3.27. Let χ be large enough so that $\mathcal{P}(\mathcal{P}(P)) \in H_\chi$, where P is some forcing notion. Let $N \prec H_\chi$ be such that $|N| = \kappa$, $N \supseteq N^{<\kappa}$ and $P \in N$. Then a condition $p \in P$ is called $\langle N, P\rangle$-generic if

$$p \Vdash \ulcorner \forall D \in \check{N}\ D \text{ is dense open in } \check{P} \implies D \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset \urcorner,$$

where $\underset{\sim}{G}$ is the name of a P-generic filter.

Definition 3.28. Let χ be large enough so that $\mathcal{P}(\mathcal{P}(P)) \in H_\chi$, where P is some forcing notion. The forcing notion P is called κ^+-proper if for each $N \prec H_\chi$ and each $q \in P \cap N$ such that $|N| = \kappa$, $N \supseteq N^{<\kappa}$ and $P \in N$, there is $p \leq q$ which is $\langle N, P\rangle$-generic.

Claim 3.29. *Let χ be large enough so that $\mathcal{P}(\mathcal{P}(\mathbb{P}_E)) \in H_\chi$. Assume $p \in \mathbb{P}_E$ and $N \prec H_\chi$ is such that $|N| = \kappa$, $N \supseteq N^{<\kappa}$ and $p, \mathbb{P}_E \in N$. Then there is $p^* \leq^* p$ such that p^* is $\langle N, \mathbb{P}_E\rangle$-generic.*

Proof. Assume $p \in \mathbb{P}_E$ and $N \prec H_\chi$ is such that $|N| = \kappa$, $N \supseteq N^{<\kappa}$ and $p, \mathbb{P}_E \in N$.

Construct $f \leq^* f^p$ applying 3.21 to N. By 3.13, there is $p^* \leq^* p$ such that $f^{p^*} = f$. We show that p^* is $\langle N, \mathbb{P}_E\rangle$-generic. So let $D \in N$ be a dense open subset of $\mathbb{P}_E$ and $q \leq p^*$.

Then there are $n < \omega$ and $\langle \nu_0,\dots,\nu_{n-1}\rangle \in \operatorname{dom} p$ such that $q \leq^* p_{\langle \nu_0,\dots,\nu_{n-1}\rangle}$. We set

$$D^* = \{f^p \cup (f^r \upharpoonright (\operatorname{supp} r \setminus \operatorname{supp} p)) \mid r \leq^* p_{\langle \nu_0,\dots,\nu_{n-1}\rangle},\\ \exists l < \omega\ \forall \langle \mu_0,\dots,\mu_{l-1}\rangle \in \operatorname{dom} r\ r_{\langle \mu_0,\dots,\mu_{l-1}\rangle} \in D\}.$$

We note that D^* is $\leq^*$-dense open below f^p. Since $D \in N$, we have $D^* \in N$. By the way we chose f we see that there is $g \geq^* f$ such that $g \in D^* \cap N$. Hence there are $r \in N$ and $l < \omega$ such that $r \leq^* p_{\langle \nu_0,\dots,\nu_{n-1}\rangle}$, $f^p \cup (f^r \upharpoonright (\operatorname{supp} r \setminus \operatorname{supp} p)) = g$ and

$$\forall \langle \mu_0,\dots,\mu_{l-1}\rangle \in \operatorname{dom} r\ r_{\langle \mu_0,\dots,\mu_{l-1}\rangle} \in D \cap N.$$

In fact $q \parallel^* r$. That is there is $q^* \leq^* q$ (a shrinkage of $\operatorname{dom} q$ is enough, actually) such that $q^* \leq^* r$. Since for each $\langle \mu_0,\dots,\mu_{l-1}\rangle \in \operatorname{dom} q^*$

$$q^*_{\langle \mu_0,\dots,\mu_{l-1}\rangle} \Vdash \ulcorner r_{\langle \pi_{\mathrm{mc}(q^*),\mathrm{mc}(r)}(\mu_0),\dots,\pi_{\mathrm{mc}(q^*),\mathrm{mc}(r)}(\mu_{l-1})\rangle} \in \underset{\sim}{G} \urcorner,$$

we get $q^* \Vdash_{\mathbb{P}_E} \ulcorner \check{D} \cap \check{N} \cap \underset{\sim}{G} \neq \emptyset \urcorner$. $\square$

Corollary 3.30. *$\mathbb{P}_E$ is κ^+-proper.*

Corollary 3.31. *In a $\mathbb{P}_E$-generic extension, κ^+ is preserved.*

All in all we get the Gitik-Magidor theorem for any extender:

Theorem 3.32. *Assume $j : V \to M \supset M^\kappa$, $\operatorname{crit}(j) = \kappa$ and $\mathfrak{g}(j) \subset j(\kappa)$. Let E be the extender derived from j and let G be $\mathbb{P}_E$-generic. Then in $V[G]$:*

1. *All the cardinals are preserved.*
2. $\operatorname{cf} \kappa = \omega$.
3. $2^\kappa = |j(\kappa)|$.
4. *No new bounded subsets are added to κ.*

4. Application to pcf theory.

The assumptions we use in this section are: The GCH and the existence of an elementary embedding $j : V \to M \supset M^\kappa$ such that $\operatorname{crit}(j) = \kappa$, $\mathfrak{g}(j) \subset j(\kappa)$ and E is the extender derived from j. Throughout this section D will be the cofinite filter over ω.

The forcing $\mathbb{P}_E$ is the one defined in the previous section and we let G be a $\mathbb{P}_E$-generic filter over V. The basic observation used throughout this section is that $\operatorname{tcf}_{V[G]} \prod G^\tau/D$ can be computed from $\operatorname{tcf}_V \prod_{n<\omega} j_n(\tau)/D$.

Lemma 4.1. *If $\tau \in \operatorname{dom} E$ and $p \in \mathbb{P}_E$, then there is $p^* \leq^* p$ such that*

$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*), \dots, j_n(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\tau)| + n) = j_n(\tau)\urcorner.$$

Proof. Assume $\tau \in \operatorname{dom} E$ and $p \in \mathbb{P}_E$. By 3.14 there is $p^* \leq^* p$ such that $\tau \in \operatorname{supp} p^*$. We shrink $\operatorname{dom} p^*$ so that $\forall \langle \nu \rangle \in \operatorname{dom} p^* \ \tau \in F^{p^*}(\nu)$. Hence, from the definition of $\mathbb{P}_E$, we get

$$\forall n < \omega \ \forall \langle \nu_0, \dots, \nu_n \rangle \in \operatorname{dom} p^*$$
$$p^*_{\langle \nu_0, \dots, \nu_n \rangle} \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^\tau(|f^{p^*}(\tau)| + n) = \pi_{\operatorname{mc}(p^*), \tau}(\nu_n)\urcorner.$$

Łoś theorem yields

$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*), \dots, j_n(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\tau)| + n) = j_n(\tau)\urcorner.$$

□

We would have liked to have $\rho < \tau \implies G^\rho/D < G^\tau/D$. However, the Cohen initial-segments of G^ρ, G^τ ruin this. We can get a good approximation to this monotonicity using shifts of G^ρ, hence the following definition. By $\mathbb{Z}$ we mean the set of integers $\{0, 1, -1, 2, -2, \dots\}$.

Definition 4.2. (In $V[G]$) Assume $\tau \in \operatorname{dom} E$ and $k \in \mathbb{Z}$. Then $G^{\tau,k} : \omega \to \kappa$ is

$$G^{\tau,k}(n) = \begin{cases} G^\tau(n-k) & k \leq n < \omega, \\ 0 & 0 \leq n < k. \end{cases}$$

As usual, $\underset{\sim}{G}^{\tau,k}$ will be the $\mathbb{P}_E$-name of this function.

Lemma 4.3. *(In $V[G]$) $\forall \tau \in \operatorname{dom} E \ \forall k_1, k_2 \in \mathbb{Z} \ \operatorname{cf} \prod G^{\tau,k_1}/D = \operatorname{cf} \prod G^{\tau,k_2}/D$.*

Proof. This is a basic pcf fact. □

Lemma 4.4. *(In $V[G]$) If $\tau \in \operatorname{dom} E$, $k_1, k_2 \in \mathbb{Z}$ and $k_1 < k_2$, then $G^{\tau,k_1}/D > G^{\tau,k_2}/D$.*

Proof. This is immediate since G^τ is a strictly increasing sequence. □

Lemma 4.5. *If $p \in \mathbb{P}_E$, $\rho, \tau \in \operatorname{dom} E$ and $\rho < \tau$, then there is $p^* \leq^* p$ such that $p^* \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^\rho/D < \underset{\sim}{G}^{\tau,|f^{p^*}(\rho)-f^{p^*}(\tau)|}/D < \underset{\sim}{G}^{\rho,-1}/D\urcorner$.*

Proof. Assume $\rho, \tau \in \operatorname{dom} E$, $\rho < \tau$ and $p \in \mathbb{P}_E$. By 3.14 and 4.1 there is $p^* \leq^* p$ such that $\rho, \tau \in \operatorname{supp} p^*$ and

$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*),\dots,j_n(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(G^\rho)(|f^{p^*}(\rho)| + n) = j_n(\rho)\urcorner,$$
$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*),\dots,j_n(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(G^\tau)(|f^{p^*}(\tau)| + n) = j_n(\tau)\urcorner.$$

We shrink $\operatorname{dom} p^*$ in order to have $\forall \langle \nu \rangle \in \operatorname{dom} p^* \ \rho, \tau \in F^{p^*}(\nu)$. (In fact the condition generated by 4.1 satisfies this). We set $k = |f^{p^*}(\rho)| - |f^{p^*}(\tau)|$. Thus

$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*),\dots,j_{n+1}(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^\rho)(|f^{p^*}(\rho)| + n) =$$
$$j_n(\rho) < j_n(\tau) = j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\tau)| + n) =$$
$$j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\rho)| + n - (|f^{p^*}(\rho)| - |f^{p^*}(\tau)|)) =$$
$$j_\omega(\underset{\sim}{G}^{\tau,k})(|f^{p^*}(\rho)| + n)\urcorner,$$

and

$$\forall n < \omega \ j_\omega(p^*)_{\langle \operatorname{mc}(p^*),\dots,j_{n+1}(\operatorname{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau,k})(|f^{p^*}(\rho)| + n) =$$
$$j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\rho)| + n - (|f^{p^*}(\rho)| - |f^{p^*}(\tau)|)) =$$
$$j_\omega(\underset{\sim}{G}^\tau)(|f^{p^*}(\tau)| + n) = j_n(\tau) < j_{n+1}(\rho) =$$
$$j_\omega(\underset{\sim}{G}^\rho)(|f^{p^*}(\rho)| + n + 1) = j_\omega(\underset{\sim}{G}^{\rho,-1})(|f^{p^*}(\rho)| + n)\urcorner.$$

Łoś theorem and shrinking $\operatorname{dom} p^*$ a bit yields $\forall n < \omega \ \forall \langle \nu_0, \dots, \nu_{n+1}\rangle \in \operatorname{dom} p^*$

$$p^*_{\langle \nu_0,\dots,\nu_{n+1}\rangle} \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^\rho(|f^{p^*}(\rho)| + n) = \pi_{\operatorname{mc}(p^*),\rho}(\nu_n) <$$
$$\pi_{\operatorname{mc}(p^*),\tau}(\nu_n) = \underset{\sim}{G}^{\tau,k}(|f^{p^*}(\rho)| + n)\urcorner$$

and

$$p^*_{\langle \nu_0,\dots,\nu_{n+1}\rangle} \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^{\tau,k}(|f^{p^*}(\rho)| + n) = \pi_{\operatorname{mc}(p^*),\tau}(\nu_n) <$$
$$\pi_{\operatorname{mc}(p^*),\rho}(\nu_{n+1}) = \underset{\sim}{G}^{\rho,-1}(|f^{p^*}(\rho)| + n)\urcorner.$$

Which means $p^* \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^\rho/D < \underset{\sim}{G}^{\tau,k}/D < \underset{\sim}{G}^{\rho,-1}/D\urcorner$. □

Definition 4.6. (In $V[G]$) For each $\tau \in \operatorname{dom} E$ we set $G^{\tau *} = G^{\tau,k}$ where $k \in \mathbb{Z}$ is chosen so that $G^\kappa/D \leq G^{\tau *}/D < G^{\kappa,-1}$. For each $\tau \in \operatorname{dom} E$, $\underset{\sim}{G}^{\tau *}$ is the $\mathbb{P}_E$-name of $G^{\tau *}$.

An immediate corollary of this definition and 4.1 is

Corollary 4.7. *Assume $p \in \mathbb{P}_E$ and $\tau \in \operatorname{dom} E$. Then there is $p^* \leq^* p$ such that for each $n < \omega$*

$$j_\omega(p^*)_{\langle \mathrm{mc}(p),\dots,j_n(\mathrm{mc}(p))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau *})(|f^p(\kappa)|+n) = j_n(\tau)\urcorner.$$

Corollary 4.8. *(In $V[G]$) $\langle G^{\tau *}/D \mid \tau \in \operatorname{dom} E\rangle$ is a strictly increasing sequence in $\prod G^{\kappa,-1}/D$.*

Proof. Let $p \in \mathbb{P}_E$, $\rho, \tau \in \operatorname{dom} E$ and $\rho < \tau$. By 4.7 there is $p^* \leq^* p$ such that for each $n < \omega$

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(G^{\rho *})(|f^p(\kappa)|+n) = j_n(\rho)\urcorner,$$
$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(G^{\tau *})(|f^p(\kappa)|+n) = j_n(\tau)\urcorner,$$

and

$$j_\omega(p^*)_{\langle \mathrm{mc}(p),\dots,j_n(\mathrm{mc}(p)),j_{n+1}(\mathrm{mc}(p))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\kappa,-1})(|f^p(\kappa)|+n) = j_{n+1}(\kappa)\urcorner.$$

Since $\rho < \tau < j(\kappa)$ we have $\forall n < \omega\ j_n(\rho) < j_n(\tau) < j_{n+1}(\kappa)$, hence for each $n < \omega$

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*)),j_{n+1}(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)}$$
$$\ulcorner j_\omega(G^{\rho *})(|f^p(\kappa)|+n) < j_\omega(G^{\tau *})(|f^p(\kappa)|+n) < j_\omega(G^{\kappa,-1})(|f^p(\kappa)|+n)\urcorner.$$

Łoś theorem and shrinking p^* a bit yields

$$\forall n < \omega\ \forall \langle \nu_0,\dots,\nu_n,\nu_{n+1}\rangle \in \operatorname{dom} p^*\ p^*_{\langle \nu_0,\dots,\nu_n,\nu_{n+1}\rangle}$$
$$\Vdash_{\mathbb{P}_E} \ulcorner G^{\rho *}(|f^p(\kappa)|+n) < G^{\tau *}(|f^p(\kappa)|+n) < G^{\kappa,-1}(|f^p(\kappa)|+n)\urcorner.$$

Hence $p^* \Vdash_{\mathbb{P}_E} \ulcorner \underset{\sim}{G}^{\rho *}/D < \underset{\sim}{G}^{\tau *}/D < \underset{\sim}{G}^{\kappa,-1}\urcorner$. ☐

Lemma 4.9. *If $\tau \in \operatorname{dom} E$ and $p \Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *}\urcorner$, then there are $p^* \leq^* p$ and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$ such that*

$$\forall n < \omega\ j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+n) = \alpha_n\urcorner.$$

Proof. Let $\tau \in \operatorname{dom} E$ and $p \Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *}\urcorner$. By 4.7 there is $q \leq^* p$ such that

$$\forall n < \omega\ j_\omega(q)_{\langle \mathrm{mc}(q),\dots,j_n(\mathrm{mc}(q))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau *})(|f^p(\kappa)|+n) = j_n(\tau)\urcorner. \tag{1}$$

We construct by induction a $\leq^*$-decreasing sequence $\langle r^n \mid n < \omega\rangle$ and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$ as follows:

- $n = 0$: $r^0 = q$.
- $n = m+1$: Let $D_m = \{r \in \mathbb{P}_E \mid \exists \zeta < \kappa\ r \Vdash_{\mathbb{P}_E} \ulcorner \dot{f}(|f^{p^*}(\kappa)|+m) = \zeta\urcorner\}$. By 3.25 there are $r^{m+1} \leq^* r^m$, k, and $f : \operatorname{dom} r^{m+1} \cap [\kappa]^{k+1} \to \kappa$ such that $m \leq k < \omega$ and

$$\forall \langle \nu_0,\dots,\nu_k\rangle \in \operatorname{dom} r^{m+1}\ r^{m+1}_{\langle \nu_0,\dots,\nu_k\rangle} \Vdash_{\mathbb{P}_E} \ulcorner \dot{f}(|f^{p^*}(\kappa)|+m) = f(\nu_0,\dots,\nu_k)\urcorner.$$

  By Łoś theorem

$$j_\omega(r^{m+1})_{\langle \mathrm{mc}(r^{m+1}),\dots,j_k(\mathrm{mc}(r^{m+1}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+m) = j_\omega(f)(\mathrm{mc}(r^{m+1}),\dots,j_k(\mathrm{mc}(r^{m+1})))\urcorner. \tag{2}$$

From (1) we infer

$$j_\omega(r^{m+1})_{\langle \mathrm{mc}(r^{m+1}),\ldots,j_k(\mathrm{mc}(r^{m+1}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(f)(\mathrm{mc}(r^{m+1}),\ldots,j_k(\mathrm{mc}(r^{m+1}))) = j_\omega(\dot{f})(|f^{p^*}(\kappa)|+m) < j_\omega(\underset{\sim}{G}^{\tau *})(|f^{p^*}(\kappa)|+m) = j_m(\tau)\urcorner.$$

This means

$$j_\omega(f)(\mathrm{mc}(r^{m+1}),\ldots,j_k(\mathrm{mc}(r^{m+1}))) < j_m(\tau).$$

We set $\alpha_m = j_\omega(f)(\mathrm{mc}(r^{m+1}),\ldots,j_k(\mathrm{mc}(r^{m+1})))$. Since $\alpha_m < j_m(\tau)$, there is $g : \mathrm{dom}\, r^{m+1} \cap [\kappa]^{m+1} \to \kappa$ such that

$$\alpha_m = j_\omega(g)(\mathrm{mc}(r^{m+1}),\ldots,j_m(\mathrm{mc}(r^{m+1}))).$$

So in (2) we can substitute g for f and m for k and get

$$j_\omega(r^{m+1})_{\langle \mathrm{mc}(r^{m+1}),\ldots,j_m(\mathrm{mc}(r^{m+1}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+m) = \alpha_m\urcorner.$$

Using 3.18 we find $p^* \in \mathbb{P}_E$ such that $\forall n < \omega\ p^* \leq r^n$. □

Lemma 4.10. *Assume* $\tau \in \mathrm{dom}\, E$, $\mathrm{cf}\,\tau > \omega$, $\mathrm{cf}\,\tau \neq \kappa$ *and* $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$. *Then there is* $\rho < \tau$ *such that* $\langle \alpha_n \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$.

Proof. We split the proof according to the relation between $\mathrm{cf}\,\tau$ and κ:

- $\mathrm{cf}\,\tau > \kappa$: We note that for each $n < \omega$ there are $f_n : [\kappa]^n \to \tau$ and $\beta_n \in \mathrm{dom}\, E$ such that $j_n(f_n)(\beta_n,\ldots,j_{n-1}(\beta_n)) = \alpha_n$. Since $\mathrm{cf}\,\tau > \kappa$, there is $\rho < \tau$ such that $\forall n < \omega\ \forall \langle \nu_0,\ldots,\nu_{n-1}\rangle \in [\kappa]^n\ \rho > f_n(\nu_0,\ldots,\nu_{n-1})$. Hence $\forall n < \omega\ j_n(\rho) > \alpha_n$.
- $\mathrm{cf}\,\tau < \kappa$: Let $A = \langle \tau_\xi \mid \xi < \mathrm{cf}\,\tau\rangle$ be cofinal in τ. So for each $n < \omega$ we get $j_n(A)$ is cofinal in $j_n(\tau)$. Since $\mathrm{cf}\,\tau < \kappa$ we have that $j_n(A) = j_n'' A$. This means that for each $n < \omega$ there is $\xi_n < \mathrm{cf}\,\tau$ such that $\alpha_n < j_n(\tau_{\xi_n}) < j_n(\tau)$. Since $\mathrm{cf}\,\tau > \omega$ there is $\xi < \mathrm{cf}\,\tau$ such that for each $n < \omega$, $\xi > \xi_n$. Let $\rho = \tau_\xi$. Then for each $n < \omega$, $\rho > \tau_{\xi_n}$ and $j_n(\rho) > j_n(\tau_{\xi_n}) > \alpha_n$. Hence $\langle \alpha_n \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$.

□

Corollary 4.11. *If* $\tau \in \mathrm{dom}\, E$, $\mathrm{cf}\,\tau > \omega$ *and* $\mathrm{cf}\,\tau \neq \kappa$, *then* $\Vdash_{\mathbb{P}_E} \ulcorner \mathrm{tcf} \prod \underset{\sim}{G}^{\tau *}/D = \mathrm{cf}\,\tau\urcorner$.

Proof. Let $\tau \in \mathrm{dom}\, E$, $\mathrm{cf}\,\tau > \omega$ and $\mathrm{cf}\,\tau \neq \kappa$. By 4.8, $\langle G^{\rho *}/D \mid \kappa \leq \rho < \tau\rangle$ is a strictly increasing sequence below $G^{\tau *}/D$. We will get the conclusion of the lemma if we prove

$$\Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *} \implies \exists \rho < \tau\ \dot{f}/D < \underset{\sim}{G}^{\rho *}/D\urcorner.$$

So, let $p \Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *}\urcorner$.

By 4.9, there are $p^* \leq^* p$ and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$ such that for each $n < \omega$

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\ldots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+n) = \alpha_n\urcorner$$

and

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau *})(|f^{p^*}(\kappa)|+n) = j_n(\tau)\urcorner.$$

By 4.10, there is $\rho < \tau$ such that $\langle \alpha_n \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$. By 4.7 there is $p^{**} \leq^* p^*$ such that for each $n < \omega$

$$j_\omega(p^{**})_{\langle \mathrm{mc}(p^{**}),\dots,j_n(\mathrm{mc}(p^{**}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\rho *})(|f^{p^*}(\kappa)|+n) = j_n(\rho)\urcorner.$$

Łoś theorem and shrinking $\operatorname{dom} p^{**}$ yields $p^{**} \Vdash_{\mathbb{P}_E} \ulcorner \dot{f}/D < \underset{\sim}{G}^{\rho *}/D\urcorner$. □

Lemma 4.12. *Assume $\tau \in \operatorname{dom} E$, $\operatorname{cf} \tau = \kappa$, and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$. Then there is $\rho < j(\tau)$ such that $\langle \alpha_{n+1} \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$.*

Proof. Assume $\tau \in \operatorname{dom} E$, $\operatorname{cf} \tau = \kappa$ and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$. Then for each $0 < n < \omega$ there are $f_n : [\kappa]^n \to \tau$ and $\beta_n \in \operatorname{dom} E$ such that

$$j_n(f_n)(\beta_n, \dots, j_{n-1}(\beta_n)) = \alpha_n < j_n(\tau),$$

where $\forall \langle \nu_0, \dots, \nu_{n-2}\rangle \in [\kappa]^{n-1}\ j(f_n)(\nu_0, \dots, \nu_{n-2}, \beta_n) < j(\tau)$. Since $\operatorname{cf}_M j(\tau) = j(\kappa) > \kappa$ and $M \supset M^\kappa$, there is $\rho < j(\tau)$ such that

$$\forall 0 < n < \omega\ \forall \langle \nu_0, \dots, \nu_{n-2}\rangle \in [\kappa]^{n-1}\ j(f_n)(\nu_0, \dots, \nu_{n-2}, \beta_n) < \rho.$$

Hence $\forall\, 0 < n < \omega\ \alpha_n = j_n(f_n)(\beta_n, \dots, j_{n-1}(\beta_n)) < j_{n-1}(\rho) < j_n(\tau)$. That is $\langle \alpha_{n+1} \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$. □

Corollary 4.13. *If $\tau \in \operatorname{dom} E$ and $\operatorname{cf} \tau = \kappa$, then $\Vdash_{\mathbb{P}_E} \ulcorner \operatorname{tcf} \prod \underset{\sim}{G}^{\tau *}/D = \operatorname{cf} j(\kappa)\urcorner$.*

Proof. Assume $\tau \in \operatorname{dom} E$ and $\operatorname{cf} \tau = \kappa$. For each $\rho < j(\tau)$ there are $\tau_\rho \in \operatorname{dom} E$ and $\bar{h}_\rho : \kappa \to \tau$ such that $\rho = j(\bar{h}_\rho)(\tau_\rho)$. In the generic extension we set $h_\rho = \bar{h}''_\rho G^{\tau_\rho *,1}$. We note that $\langle h_\rho/D \mid \rho < j(\tau)\rangle$ is an increasing sequence in $\prod G^{\tau *}$, and that $\operatorname{cf}_{V[G]} j(\tau) = \operatorname{cf}_{V[G]} j(\kappa)$ since $\operatorname{cf}_M(j(\tau)) = j(\kappa) > \kappa$, $M \supset M^\kappa$ and $\mathbb{P}_E$ preserves cardinals above κ. Thus we will get the conclusion of the lemma if we prove $\Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *} \implies \exists \rho < j(\tau)\ \dot{f}/D < h_\rho/D\urcorner$. So let $p \Vdash_{\mathbb{P}_E} \ulcorner \dot{f} \in \prod \underset{\sim}{G}^{\tau *}\urcorner$.

By 4.9, there are $p^* \leq^* p$ and $\langle \alpha_n \mid n < \omega\rangle \in \prod_{n<\omega} j_n(\tau)$ such that

$$\forall n < \omega\ j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+n) = \alpha_n\urcorner$$

and

$$\forall n < \omega\ j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_n(\mathrm{mc}(p^*))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau *})(|f^{p^*}(\kappa)|+n) = j_n(\tau)\urcorner.$$

By 4.12, there is $\rho < j(\tau)$ such that $\langle \alpha_{n+1} \mid n < \omega\rangle < \langle j_n(\rho) \mid n < \omega\rangle$. By 4.7 there is $p^{**} \leq^* p^*$ such that

$$\forall n < \omega\ j_\omega(p^{**})_{\langle \mathrm{mc}(p^{**}),\dots,j_n(\mathrm{mc}(p^{**}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\underset{\sim}{G}^{\tau_\rho *})(|f^{p^*}(\kappa)|+n) = j_n(\tau_\rho)\urcorner.$$

Hence for each $n < \omega$

$$\begin{aligned} j_\omega(p^{**})_{\langle \mathrm{mc}(p^{**}),\dots,j_n(\mathrm{mc}(p^{**})),j_{n+1}(\mathrm{mc}(p^{**}))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \ulcorner j_\omega(\dot{f})(|f^{p^*}(\kappa)|+n+1) = \\ \alpha_{n+1} < j_n(\rho) = j_\omega(\bar{h}_\rho)(j_n(\tau_\rho)) = \\ j_\omega(\bar{h}_\rho)(j_\omega(\underset{\sim}{G}^{\tau_\rho *})(|f^{p^*}(\kappa)|+n)) = j_\omega(h_\rho)(|f^{p^*}(\kappa)|+n+1)\urcorner. \end{aligned}$$

Łoś theorem and shrinking $\operatorname{dom} p^{**}$ yields $p^{**} \Vdash_{\mathbb{P}_E} \ulcorner \dot{f}/D < h_\rho/D\urcorner$. □

For the specific case of $\prod G^{\kappa,-1}/D$, reading the above proof shows that we use just $\rho \in \operatorname{dom} E$, the functions $\bar{h}_\rho$ are id, and $\tau_\rho = \rho$. Thus the cofinal sequence of length $j(\kappa)$ in $\prod G^{\kappa,-1}$ constructed in the proof is $\langle G^{\rho *} \mid \rho < j(\kappa)\rangle$. Thus

Corollary 4.14. *$\langle G^{\tau *}/D \mid \kappa \leq \tau < j(\kappa)\rangle$ is cofinal in $\prod G^{\kappa,-1}$.*

The following theorem summarizes the facts proved previously for the E is a superstrong extender case.

Theorem 4.15. *Assume GCH, $j : V \to M \supset M^\kappa$, $M \supset V_{j(\kappa)}$, $\operatorname{crit} j = \kappa$ and $\mathfrak{g}(j) \subset j(\kappa)$. Let E be the extender derived from j and let G be $\mathbb{P}_E$-generic. Then $V[G]$ is a cardinal preserving generic extension in which $\operatorname{cf} \kappa = \omega$, $\kappa^\omega = j(\kappa)$, and there is a Prikry sequence G^κ in κ and a scale $\langle G^{\lambda *}/D \mid \lambda \in [\kappa, j(\kappa))\rangle$ of length $j(\kappa)$ in $\prod G^{\kappa,-1}/D$ such that $\operatorname{tcf} \prod G^{\lambda *}/D = \lambda$ for each regular cardinal $\lambda < j(\kappa)$.*

The only thing we were able to say when $\operatorname{cf}_V \tau = \omega$ is $\operatorname{cf}_{V[G]} \prod G^{\tau *}/D \leq 2^{\aleph_0}$.

5. Generic by Iteration

What we would have liked to have is:

Aim. *Assume GCH, $j : V \to M \supset M^\kappa$, $\operatorname{crit}(j) = \kappa$ and $\mathfrak{g}(j) \subset j(\kappa)$. Let E be the extender derived from j. Then there is $G \in V$ which is $j_\omega(\mathbb{P}_E)$-generic over M_ω.*

Alas, we were not able to achieve this aim. The referee has pointed out that the aim is not all that reasonable since the forcing $\mathbb{P}_E$ incorporates the forcing $\mathbb{P}^*_E$, which is essentially Cohen forcing and not like Prikry forcing at all.

In this section we use three approaches to obtain approximations to this aim. First, in theorem 5.1, we obtain a generic filter for an elementary substructure of the iterated ultrapower instead of the full iterated ultrapower. Second, in theorem 5.2, we force over V to explicitly add the Cohen component of $\mathbb{P}_E$ to obtain a filter generic over the full iterated ultrapower. Finally, in theorem 5.3, we assume the existence in V of an extender stronger than E to obtain a filter in V which is generic over the full iterated ultrapower.

We begin by showing how to construct a generic filter over an elementary submodel in M_ω.

Theorem 5.1. *Assume GCH, $j : V \to M \supset M^\kappa$, $\operatorname{crit}(j) = \kappa$, and $\mathfrak{g}(j) \subset j(\kappa)$. Let E be the extender derived from j, and $\kappa_\omega = j_\omega(\kappa)$. Let $N \in M_\omega$ be such that in M_ω we have: $j_\omega(\mathbb{P}_E) \in N$, $N \prec H^{M_\omega}_\chi$ for a large enough χ, $|N| = \kappa_\omega$ and $N \supseteq N^{<\kappa_\omega}$. Then there is $G \in V$ which is $j_\omega(\mathbb{P}_E)$-generic over N.*

Proof. By 3.29 invoked in M_ω there is $p \in j_\omega(\mathbb{P}_E)$ which forces genericity over N. Let $k < \omega$ be such that $p = j_{k,\omega}(p^k)$ and $N = j_{k,\omega}(N^k)$. We set for each $k < n < \omega$, $p^n = j_{k,n}(p^k)_{\langle \operatorname{mc}(p^k), \dots, j_{k,n-1}(\operatorname{mc}(p^k))\rangle}$ and $N^n = j_{k,n}(N^k)$.

We set for each $k \leq n < \omega$

$$G^n = \{q \in j_n(\mathbb{P}_E) \mid q \geq p^n\}$$

and then

$$G = \bigcup_{k \leq n < \omega} j''_{n,\omega} G^n.$$

Since for each $k \le n < \omega$, $p^{n+1} \le^* j_{n,n+1}(p^n)_{\langle \mathrm{mc}(p^n)\rangle}$, we get that G is a filter. We show that it intersects each dense open subset of $j_\omega(\mathbb{P}_E)$ appearing in N. So let $D \in N$ be a dense open subset of $j_\omega(\mathbb{P}_E)$.

Let D^n and $n < \omega$ be such that $k \le n < \omega$ and $j_{n,\omega}(D^n) = D$. Then p^n forces genericity over N^n. That means (by the proof of 3.29) there are $q \ge^* p^n$ and $l < \omega$ such that $q \in N^n$ and $\forall \langle \nu_0, \dots, \nu_{l-1}\rangle \in \operatorname{dom} q\ q_{\langle \nu_0,\dots,\nu_{l-1}\rangle} \in D^n \cap N^n$. Hence

$$j_{n,n+l}(q)_{\langle \mathrm{mc}(q),\dots,j_{n,n+l-1}(\mathrm{mc}(q))\rangle} \in j_{n,n+l}(D^n \cap N^n),$$

thus

$$j_{n,\omega}(q)_{\langle \mathrm{mc}(q),\dots,j_{n,n+l-1}(\mathrm{mc}(q))\rangle} \in D \cap N.$$

Noting that

$$j_{n,n+l}(q)_{\langle \mathrm{mc}(q),\dots,j_{n,n+l-1}(\mathrm{mc}(q))\rangle} \ge^* j_{n,n+l}(p^n)_{\langle \mathrm{mc}(p^n),\dots,j_{n,n+l-1}(\mathrm{mc}(p^n))\rangle} \ge^* p^{n+l},$$

we get

$$j_{n,n+l}(q)_{\langle \mathrm{mc}(q),\dots,j_{n,n+l-1}(\mathrm{mc}(q))\rangle} \in G^{n+l},$$

thus

$$j_{n,\omega}(q)_{\langle \mathrm{mc}(q),\dots,j_{n,n+l-1}(\mathrm{mc}(q))\rangle} \in G.$$

□

We continue by showing the existence of a generic filter over M_ω in a $\mathbb{P}^*_E$-generic extension of V.

Theorem 5.2. *Assume GCH, $j : V \to M \supset M^\kappa$, $\operatorname{crit}(j) = \kappa$ and $\mathfrak{g}(j) \subset j(\kappa)$. Let E be the extender derived from j. Let G^* be $\mathbb{P}^*_E$-generic over V. Then in $V[G^*]$ there is G which is $j_\omega(\mathbb{P}_E)$-generic over M_ω.*

Proof. Let G^* be $\mathbb{P}^*_E$-generic. In $V[G^*]$ we set

$$G' = \{ j_\omega(p)_{\langle \mathrm{mc}(p), j(mc(p)) \dots, j_{n-1}(\mathrm{mc}(p))\rangle} \mid f^p \in G^*,\ n < \omega\}.$$

and

$$G = \{ q \in j_\omega(\mathbb{P}_E) \mid \exists p \in G'\ q \ge p\}.$$

We show that G is $j_\omega(\mathbb{P}_E)$-generic over M_ω. So, Let $D_\omega \in M_\omega$ be a dense open subset of $j_\omega(\mathbb{P}_E)$.

Let $n < \omega$ be minimal such that there are $\alpha \in \operatorname{dom} E$ and $D : [\kappa]^n \to V$ satisfying $j_\omega(D)(\alpha, j(\alpha), \dots, j_{n-1}(\alpha)) = D_\omega$. We set

$$D^* = \{ f^p \in \mathbb{P}^*_E \mid p \in \mathbb{P}_E,\ \exists l < \omega\ j_\omega(p)_{\langle \mathrm{mc}(p), j(\mathrm{mc}(p)),\dots,j_{n+l-1}(\mathrm{mc}(p))\rangle}) \in D_\omega\}.$$

We prove that D^* is a dense open subset of $\mathbb{P}^*_E$. So, let $f \in \mathbb{P}^*_E$.

Pick $p \in \mathbb{P}_E$ such that $f^p \le^* f$ and $\alpha \in \operatorname{supp} p$. For each $\langle \nu_0, \dots, \nu_{n-1}\rangle \in \operatorname{dom} p$, the set

$$\begin{aligned} D^*(\nu_0, \dots, \nu_{n-1}) = \{ q \le^* p_{\langle \nu_0,\dots,\nu_{n-1}\rangle} \mid \exists l < \omega\ \forall \langle \mu_0, \dots, \mu_{l-1}\rangle \in \operatorname{dom} q \\ q_{\langle \mu_0,\dots,\mu_{l-1}\rangle} \in D(\pi_{\mathrm{mc}(q),\alpha}(\nu_0), \dots, \pi_{\mathrm{mc}(q),\alpha}(\nu_{n-1}))\} \end{aligned}$$

is dense open in $\langle \mathbb{P}_E, \leq^* \rangle$ below $p_{\langle \nu_0,\dots,\nu_{n-1}\rangle}$. Thus by 3.23 there is $p^* \leq^* p$ such that

$$\forall \langle \nu_0,\dots,\nu_{n-1}\rangle \in \operatorname{dom} p^* \ p^*_{\langle \nu_0,\dots,\nu_{n-1}\rangle} \in D^*(\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\nu_0),\dots,\pi_{\mathrm{mc}(p^*),\mathrm{mc}(p)}(\nu_{n-1})).$$

Thus some shrinkage of $\operatorname{dom} p^*$ yield that there is $l < \omega$ such that

$$\forall \langle \nu_0,\dots,\nu_{n-1},\mu_0,\dots,\mu_{l-1}\rangle \in \operatorname{dom} p^* \ p^*_{\langle \nu_0,\dots,\nu_{n-1},\mu_0,\dots,\mu_{l-1}\rangle} \in D(\pi_{\mathrm{mc}(p^*),\alpha}(\nu_0),\dots,\pi_{\mathrm{mc}(p^*),\alpha}(\nu_{n-1})).$$

That is

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),\dots,j_{n+l-1}(\mathrm{mc}(p^*))\rangle} \in j_\omega(D)(\alpha,\dots,j_{n-1}(\alpha)) = D_\omega.$$

Thus $f^{p^*} \in D^*$, hence the density of $D^* \in V$ is proved.

So there are $f \in G^* \cap D^*$, $l < \omega$, and $p \in \mathbb{P}_E$ such that $f^p = f$ and

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),j(\mathrm{mc}(p^*)),\dots,j_{n+l-1}\,\mathrm{mc}(p^*))\rangle} \in D_\omega.$$

By the definition of G, we get

$$j_\omega(p^*)_{\langle \mathrm{mc}(p^*),j(\mathrm{mc}(p^*)),\dots,j_{n+l-1}(\mathrm{mc}(p^*))\rangle} \in G.$$

□

In the following theorem we assume the existence of an extender on κ, but use only part of the extender as a base for the forcing notion.

Theorem 5.3. *Assume GCH,* $i\colon V \to N \supset N^\zeta$, $\mathrm{crit}(i) = \kappa$, $\mathfrak{g}(i) \subseteq i(\zeta)$. *Assume* $\kappa < \mu < i(\kappa)$ *satisfies* $\mu^{(\kappa^+)} \leq \zeta$. *Let* F *be the extender derived from* i *and* $E = F \restriction \mu$. *Then there is* $G \in V$ *which is* $i_\omega(\mathbb{P}_E)$*-generic over* N_ω.

Proof. Let $\mathfrak{A}_n = \{A \in N_n \mid A \text{ is a maximal anti-chain in } i_n(\mathbb{P}_E)\}$. For each maximal anti-chain $A \subset \mathbb{P}_E$ we set $D(A) = \{p \in \mathbb{P}_E \mid \exists a \in A\ p \leq a\}$ (thus $D(A)$ is a dense open subset of $\mathbb{P}_E$). For each $n < \omega$ we have $j_n(|\mathfrak{A}_0|) \leq j_n(|\mathbb{P}_E|^{(\kappa^+)}) \leq j_n(\zeta)$. Thus $i''_{n,n+1}\mathfrak{A}_n \in N_{n+1}$. Moreover, for $q \in i_{n+1}(\mathbb{P}_E)$ we can invoke 3.25 in N_{n+1} for $i_n(\mu^{(\kappa^+)})$-many times to get $p \leq^* q$ such that for each $A \in \mathfrak{A}_n$ there is $l < \omega$ such that $\forall \langle \nu_0,\dots,\nu_{l-1}\rangle \in \operatorname{dom} p\ p_{\langle \nu_0,\dots,\nu_{l-1}\rangle} \in i_{n,n+1}(D(A))$.

Using this fact we construct a sequence $\langle p^n \mid n < \omega\rangle$ such that $p^0 \in \mathbb{P}_E$ is arbitrary, $p^{n+1} \leq^* i_{n,n+1}(p^n)_{\langle \mathrm{mc}(p^n)\rangle}$, and for each $A \in \mathfrak{A}_n$ there is $l < \omega$ such that $\forall \langle \nu_0,\dots,\nu_{l-1}\rangle \in \operatorname{dom} p^{n+1}\ p^{n+1}_{\langle \nu_0,\dots,\nu_{l-1}\rangle} \in i_{n,n+1}(D(A))$.

For each $n < \omega$ we set

$$G_n = \{q \in i_n(\mathbb{P}_E) \mid q \geq p^n\},$$

and then

$$G = \bigcup_{n<\omega} i''_{n,\omega} G_n.$$

We show G is $i_\omega(\mathbb{P}_E)$-generic over N_ω. Let $A \in N_\omega$ be a maximal anti-chain in $i_\omega(\mathbb{P}_E)$.

We take $n < \omega$ and A_n such that $i_{n,\omega}(A_n) = A$. Then there is $l < \omega$ such that $\forall \langle \nu_0,\dots,\nu_{l-1}\rangle \in \operatorname{dom} p^{n+1}\ p^{n+1}_{\langle \nu_0,\dots,\nu_{l-1}\rangle} \in i_{n,n+1}(D(A_n))$. Hence

$$i_{n+1,n+l+1}(p^{n+1})_{\langle \mathrm{mc}(p^{n+1}),\dots,i_{n+1,n+l}(\mathrm{mc}(p^{n+1}))\rangle} \in i_{n,n+l+1}(D(A_n)).$$

Since by the construction of $\langle p^m \mid m < \omega\rangle$ we have

$$p^{n+l+1} \leq^* i_{n+1,n+l+1}(p^{n+1})_{\langle \mathrm{mc}(p^{n+1}),\dots,i_{n+1,n+l}(\mathrm{mc}(p^{n+1}))\rangle},$$

we get $i_{n+1,\omega}(p^{n+1})_{\langle \mathrm{mc}(p^{n+1}),\dots,i_{n+1,n+l}(\mathrm{mc}(p^{n+1}))\rangle} \in G \cap D(A)$. Since G is upward closed we get $G \cap A \neq \emptyset$. □

References

[1] Lev Bukovsky. Changing cofinality of a measurable cardinal. *Commentationes Mathematicae Universitatis Carolinae*, 14:689–697, 1973.

[2] James Cummings and Hugh Woodin. A book on Radin forcing. In preparation.

[3] Patrick Dehornoy. Iterated ultrapowers and Prikry forcing. *Annals of Math. Logik*, 15:109–160, 1978.

[4] Moti Gitik. Prikry type forcings. In M. Foreman and A. Kanamori, editors, *Handbook Set Theory into the 21st Century*. Springer, New York, 2006.

[5] Moti Gitik and Menachem Magidor. The singular cardinal hypothesis revisited. In Haim Judah, Winfried Just, and Hugh Woodin, editors, *Set Theory of the Continuum*, pages 243–278. Springer, 1992.

[6] Thomas Jech. On the cofinality of countable products of cardinal numbers. In Alan Baker, Béla Bollobás, and András Hajnal, editors, *A Tribute to Paul Erdös*, pages 289–305. Cambridge University Press, 1990.

[7] Peter Koepke. An Iteration Model Violating the Singular Cardinals Hypothesis. In S. Barry Cooper and John K. Truss, editors, *Sets and Proofs*, volume 258 of *London Mathematical Society Lecture Note Series*, pages 95–102. Cambridge University Press, 1999.

[8] Menachem Magidor. On the singular cardinal problem I. *Israel Journal of Mathematics*, 28:1–31, 1977.

[9] Karel Prikry. Changing measurable into accessible cardinal. *Dissertations Math.*, 68:5–52, 1970.

[10] Assaf Sharon. Generators of pcf. Master's thesis, Tel-Aviv Univeristy, 2000.

[11] Saharon Shelah. *Proper Forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer, Berlin, 1982.

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