SUPERCOMPACT EXTENDER BASED MAGIDOR-RADIN FORCING

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Abstract. The extender based Magidor-Radin forcing is being generalized to supercompact type extenders.

1. Introduction

This work[1] continues the project of generalizing the extender based Prikry forcing [3] to larger and larger cardinals. In [8, 9] the methods introduced in [3] (which generalized Prikry forcing [11] from using a measure to using an extender), were used to generalize the Magidor [7] and Radin [12] forcing notions to use a sequence of extenders. In a different direction [10] used the methods of [3] to define the extender based Prikry forcing over extenders which have higher directedness properties than their critical point. Such extenders give rise to supercompact type embeddings. Generalization of Prikry forcing to fine ultrafilters yielding supercompact type embeddings appeared in [6]. Extending this forcing notion to Magidor-Radin type forcing notions were done in [1] and [5]. In the current paper we use extenders with higher directedness properties to define the extender based Magidor-Radin forcing notion. All of the forcing notions mentioned above are of course of Prikry type. For more information on Prikry type forcing notions one should consult [2].

Before stating the theorem of this paper we need to make some notions precise. Assume E is an extender. We let $j_E : V \to M \simeq \operatorname{Ult}(V, E)$ be the natural embedding of V into the transitive collpase of the ultrapower $\operatorname{Ult}(V, E)$. We denote by $\operatorname{crit} E$ the critical point of the embedding j_E. We let $\lambda(E)$ be the least λ such that $M \supseteq {}^{<\lambda}M$. We say the extender E is $\lambda(E)$-directed.

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[1] Most of this work was done somewhat after [10] was completed. Lacking an application, which admittedly was lacking also in [10], it was mainly distributed among interested parties. Gitik, observing the utility of this forcing to some HOD constructions (see [4]), has urged us to bring the work into publishable state.

A sequence of extenders $\vec{E} = \langle E_\xi \mid \xi < \mathrm{o}(\vec{E})\rangle$, all with the same critical point $\operatorname{crit} E_\xi$ and the same directedness size $\lambda(E_\xi)$, is said be Mitchell increasing if for each $\xi < \mathrm{o}(\vec{E})$ we have $\langle E_{\xi'} \mid \xi' < \xi\rangle \in M_\xi \simeq \mathrm{Ult}(V, E_\xi)$. We will denote by $\operatorname{crit}(\vec{E})$ and $\lambda(\vec{E})$ the common values of $\operatorname{crit} E_\xi$ and $\lambda(E_\xi)$, respectively.

If $\vec{E} = \langle E_\xi \mid \xi < \mathrm{o}(\vec{E})\rangle$ is a Mithcell increasing sequence of extenders and $\alpha \in [\operatorname{crit} \vec{E}, j_{E_0}(\kappa))$ then $\bar{E} = \langle \alpha, \vec{E}\rangle$ is said to be an extender sequence. Hence an extender sequence is an ordered pair with the first coordinate being an ordinal and the second coordinate being a Mitchell increasing sequence of extenders. Note that an empty sequence of extenders is legal in an extender sequence, e.g., $\langle \alpha, \langle\rangle\rangle$ is an extender sequence. Let ES be the collection of extender sequences. If $\bar{E}$ is an extender sequence then we denote the projections to the first and second coordinates by $\mathring{\bar{E}}$ and $\overset{|}{\bar{E}}$, respectively. The ordinals at the first coordinate of an extender sequence induce an order $<$ on ES by setting $\bar{\nu} < \bar{\mu}$ if $\mathring{\bar{\nu}} < \mathring{\bar{\mu}}$. We lift the functions defined on the Mitchell increasing sequence of extenders to extender sequences in the obvious way, i.e., $\mathrm{o}(\bar{E}) = \mathrm{o}(\overset{|}{\bar{E}})$ and $\lambda(\bar{E}) = \lambda(\overset{|}{\bar{E}})$. We will also abuse notation by writing $\bar{E}_\xi$ for the extender E_ξ. There are two restrictions we have on $\lambda(\vec{E})$. The first one seems a bit technical. We demand $\lambda(\bar{E})^{<\operatorname{crit} \bar{E}} = \lambda(\bar{E})$ due to limitations we encountered in claim 3.12. The second one is more substantial. We demand $\lambda(\bar{E}) \leq j_{E_0}(\operatorname{crit}(E_0))$. (It seems this last demand can be removed for the special case $\mathrm{o}(\vec{E}) = 1$.) With all these preliminaries at hand we can write the theorem proved in this paper.

Theorem. *Assume the GCH. Let $\vec{E}$ be a Mitchell increasing sequence such that $\lambda(\vec{E}) \leq j_{E_0}(\operatorname{crit}(\vec{E}))$ and $\mu^{<\operatorname{crit} \vec{E}} < \lambda(\vec{E})$ for each $\mu < \lambda(\vec{E})$. Furthermore, assume $\epsilon \leq j_{E_0}(\kappa)$. Then there is a forcing notion $\mathbb{P}(\vec{E}, \epsilon)$ such that the following hold in $V[G]$, where $G \subseteq \mathbb{P}(\vec{E}, \epsilon)$ is generic. There is a set $G^\kappa \subseteq \mathrm{ES}$ such that $G^\kappa \cup \{\langle \operatorname{crit} \vec{E}, \vec{E}\rangle\}$ is increasing and for each $\bar{\nu} \in G^\kappa \cup \{\langle \operatorname{crit} \vec{E}, \vec{E}\rangle\}$ such that $\mathrm{o}(\bar{\nu}) > 0$ the following hold:*

1. $\{\operatorname{crit} \bar{\mu} \mid \bar{\mu} \in G^\kappa, \bar{\mu} < \bar{\nu}\} \subseteq \mathring{\bar{\nu}}$ *is a club.*
2. $\operatorname{crit} \bar{\nu}$ *and* $\lambda(\bar{\nu})$ *are preserved in* $V[G]$*, and* $(\operatorname{crit} \bar{\nu}^+ = \lambda(\bar{\nu}))^{V[G]}$.
3. *If* $\mathrm{o}(\bar{\nu}) < \operatorname{crit} \bar{\nu}$ *is V-regular then* $\operatorname{cf} \operatorname{crit} \bar{\nu} = \operatorname{cf} \mathrm{o}(\bar{\nu})$ *in* $V[G]$.
4. *(Gitik) If* $\mathrm{o}(\bar{\nu}) \in [\operatorname{crit} \bar{\nu}, \lambda(\bar{\nu}))$ *and* $\operatorname{cf}(\mathrm{o}(\bar{\nu})) \geq \operatorname{crit}(\bar{\nu})$ *then* $\operatorname{cf} \operatorname{crit} \bar{\nu} = \omega$ *in* $V[G]$.
5. *If* $\mathrm{o}(\bar{\nu}) \in [\operatorname{crit} \bar{\nu}, \lambda(\bar{\nu}))$ *and* $\operatorname{cf}(\mathrm{o}(\bar{\nu})) < \operatorname{crit}(\bar{\nu})$ *then* $\operatorname{cf} \operatorname{crit} \bar{\nu} = \operatorname{cf} \mathrm{o}(\bar{\nu})$ *in* $V[G]$.

(6) *If* $\mathrm{o}(\bar{\nu}) = \mathrm{crit}(\bar{\nu})$ *then* $\mathrm{cf}\,\mathrm{crit}\,\bar{\nu} = \omega$ *in* $V[G]$.
(7) *If* $\mathrm{o}(\bar{\nu}) = \lambda(\bar{\nu})$ *then* $\mathrm{crit}\,\bar{\nu}$ *is regular in* $V[G]$.
(8) *If* $\mathrm{o}(\bar{\nu}) = \lambda(\bar{\nu})^{++}$ *then* $\mathrm{crit}\,\bar{\nu}$ *is measurable in* $V[G]$.
(9) $2^{\mathrm{crit}\,\bar{\nu}} = \max\{\lambda(\bar{\nu}), |\epsilon|\}$.

The extenders the above theorem can handle include supercompact extenders and almost huge ones. Huge extenders are not covered by the above theorem.

Thus for example, if we assume $\langle E_\xi \mid \xi < \omega_1 \rangle$ is a Mitchell increasing sequence of extenders on κ giving rise to a $< \kappa^{++}$-closed elementary embeddings (and no more), then in the generic extension κ will change its cofinality to ω_1, and κ^+ would be collapsed. Moreover, there is a club of ordertype ω_1 cofinal in κ, and for each limit point τ in this club τ^+ of the ground model is collapsed. The GCH would be preserved, and no other cardinals are collapsed.

As another example, assume $\langle E_\xi \mid \xi < \omega_1 \rangle$ is a Mitchell increasing sequence of extenders on κ giving rise to a $< \kappa^{++}$-closed elementary embeddings which are also $\kappa^{+3} - \mathit{strong}$ (and no more), then in the generic extension κ will change its cofinality to ω_1, and κ^+ would be collapsed. Moreover, there is a club of ordertype ω_1 cofinal in κ, and for each limit point τ in this club τ^+ of the ground model is collapsed. In this case we get $2^\kappa = \kappa^{++}$ and $2^\tau = \tau^{++}$ for the limit points of the club. In fact we have $2^\kappa = (\kappa^{+3})_V$ and $2^\tau = (\tau^{+3})_V$, and we see only gap-2 in the generic extension since κ^+ of the ground mode gets collapsed as do all the τ^+ of the ground model. No other cardinal get collapsed.

The structure of the work is as follows. In section 2 a formulation of extenders useful for λ-directed extenders is presented, and an appropriate diagonal intersection operation is introduced. In section 3 the forcing notion is defined and the properties of it which do not rely on understanding the dense subsets of the forcing are presented. In section 4 claims regarding the dense subsets of the forcing notion are presented. This section is highly combinatorial in nature. In section 5 the influence of $\mathrm{o}(\vec{E})$ on the properties of κ in the generic extension is shown. The claims here rely on the structure of the dense subsets as analyzed in section 4.

This work is self contained assuming large cardinals and forcing are known.

2. λ-Directed Extenders and Normality

Assume the GCH. Let $\vec{E} = \langle E_\xi \mid \xi < \mathrm{o}(\vec{E}) \rangle$ be a Mitchell increasing sequence of λ-directed extenders such that $\lambda \leq j_{E_0}(\kappa)$ is regular and

$\lambda^{<\kappa} = \lambda$, where $\kappa = \operatorname{crit} \vec{E}$. For each $\xi < \operatorname{o}(\vec{E})$ let $j_{E_\xi} : V \to M_\xi \simeq \operatorname{Ult}(V, E_\xi)$ be the natural embedding. Let $\epsilon \leq j_{E_0}(\kappa)$ be an ordinal. (Typically ϵ will be the largest V-cardinal below $j_{E_0}(\kappa)$.) Assume $d \in [\epsilon]^{<\lambda}$ and $|d|+1 \subseteq d$. We let $\operatorname{OB}(d)$ be the set of functions $\nu : \operatorname{dom} \nu \to \operatorname{ES}$ such that $\kappa \in \operatorname{dom} \nu \subseteq d$, and if $\alpha, \beta \in \operatorname{dom} \nu$ and $\alpha < \beta$ then $\mathring{\nu}(\alpha) < \mathring{\nu}(\beta)$. Define an order on $\operatorname{OB}(d)$ by saying for each pair $\nu, \mu \in \operatorname{OB}(d)$ that $\nu < \mu$ if $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$, $|\nu| < \mathring{\mu}(\kappa)$, and for each $\alpha \in \operatorname{dom} \nu$, $\nu(\alpha) < \mathring{\mu}(\kappa)$.

For $\xi < \operatorname{o}(\vec{E})$ and a set $d \in [\epsilon]^{<\lambda}$ let

$$\operatorname{mc}_\xi(d) = \{\langle j_{E_\xi}(\alpha), \langle \alpha, \langle E_{\xi'} \mid \xi' < \xi \rangle\rangle\rangle \mid \alpha \in d\}.$$

Then define the measure $E_\xi(d)$ on $\operatorname{OB}(d)$ by setting for each $X \subseteq \operatorname{OB}(d)$, $X \in E_\xi(d) \iff \operatorname{mc}_\xi(d) \in j_{E_\xi}(X)$. For a set $d \in [\epsilon]^{<\lambda}$ let $\vec{E}(d) = \bigcap\{E_\xi(d) \mid \xi < \operatorname{o}(\bar{E})\}$. It is clear $E_\xi(d)$ is a κ-complete ultrafilter over $\operatorname{OB}(d)$ and $\vec{E}(d)$ is a κ-complete filter over $\operatorname{OB}(d)$. In addition to this, the filter $\vec{E}(d)$ has a useful normality property with a matching diagonal intersection soon to be introduced.

Claim 2.1. *If $S \subseteq \operatorname{OB}(d)$, $\nu^* \in j_{E_\xi}(S)$, and $\nu^* < \operatorname{mc}_\xi(d)$, then there is $\nu \in S$ such that $\nu^* = j_{E_\xi}(\nu)$.*

Proof. We are given $\nu^* < \operatorname{mc}_\xi(d)$. From the definition of the order we deduce that $\operatorname{dom} \nu^* \subseteq j''_{E_\xi} d$, $|\nu^*| < \operatorname{mc}_\xi(d)(j_{E_\xi}(\kappa)) = \kappa$, and for each $\alpha \in \operatorname{dom} \nu^*$, $\nu^*(\alpha) < \operatorname{mc}_\xi(d)(j_{E_\xi}(\kappa)) = \kappa$. Thus there is $e \subseteq d$ such that $|e| < \kappa$ and $\operatorname{dom} \nu^* = j''_{E_\xi} e = j_{E_\xi}(e)$. Since for each $\alpha \in e$, $\nu^*(j_{E_\xi}(\alpha)) < \kappa$ we get $j_{E_\xi}(\nu^*(j_{E_\xi}(\alpha))) = \nu^*(j_{E_\xi}(\alpha))$.

Set $\nu = \{\langle \alpha, \nu^*(j_{E_\xi}(\alpha))\rangle \mid \alpha \in e\}$. Then $\nu^* = j_{E_\xi}(\nu)$. We get $\nu \in S$ by elementarity from $j_{E_\xi}(\nu) = \nu^* \in j_{E_\xi}(S)$. □

Assume $S \subseteq \operatorname{OB}(d)$ and for each $\nu \in S$ there is a set $X(\nu) \subseteq \operatorname{OB}(d)$. Define the diagonal intersection of the family $\{X(\nu) \mid \nu \in S\}$ as follows:

$$\mathop{\triangle}_{\nu \in S} X(\nu) = \{\nu \in \operatorname{OB}(d) \mid \forall \mu \in S\ (\mu < \nu \implies \nu \in X(\mu))\}.$$

Lemma 2.2. *Assume $S \subseteq \operatorname{OB}(d)$, and for each $\nu \in S$, $X(\nu) \in \vec{E}(d)$. Then $X^* = \triangle_{\nu \in S} X(\nu) \in \vec{E}(d)$.*

Proof. We need to show for each $\xi < \operatorname{o}(\vec{E})$, $\operatorname{mc}_\xi(d) \in j_{E_\xi}(X^*)$. I.e., we need to show $\operatorname{mc}_\xi(d) \in j_{E_\xi}(X)(\nu^*)$ for each $\nu^* \in j_{E_\xi}(S)$ such that $\nu^* < \operatorname{mc}_\xi(d)$. Fix $\nu^* \in j_{E_\xi}(S)$ such that $\nu^* < \operatorname{mc}_\xi(d)$. There is $\nu \in S$ such that $\nu^* = j_{E_\xi}(\nu)$. Hence $j_{E_\xi}(X)(\nu^*) = j_{E_\xi}(X(\nu))$. Since $X(\nu) \in E_\xi(d)$ we get $\operatorname{mc}_\zeta(d) \in j_{E_\zeta}(X(\nu))$, by which we are done. □

The diagonal intersection above can be generalized to work with more than one measure in the following way. A set $T \subseteq {}^{n}\mathrm{OB}(d)$, where $n < \omega$, is said to be a tree if the following hold:

(1) Each $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in T$ is increasing.
(2) For each $k < n$ and $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in T$ we have $\langle \nu_0, \ldots, \nu_k \rangle \in T$.

Assume $T \subseteq {}^{n}\mathrm{OB}(d)$ is a tree and $\langle \nu \rangle \in T$. Set $T_{\langle \nu \rangle} = \{\langle \mu_0, \ldots, \mu_{n-2} \rangle \mid \langle \nu, \mu_0, \ldots, \mu_{n-2} \rangle \in T\}$. Denote the k-level of the tree T by $\mathrm{Lev}_k(T)$, i.e., $\mathrm{Lev}_k(T) = T \cap {}^{k+1}\mathrm{OB}(d)$. We will use $\vec{\nu}$ as a shorthand for $\langle \nu_0, \ldots, \nu_{n-1} \rangle$. For each $\vec{\nu} \in T$ we define the successor level of $\vec{\nu}$ in T by setting $\mathrm{Suc}_T(\vec{\nu}) = \{\mu \mid \vec{\nu} \frown \mu \in T\}$.

A tree $S \subseteq {}^{n}\mathrm{OB}(d)$, with all maximal branches having the same finite height $n < \omega$, is said to be an $\vec{E}(d)$-tree if the following hold:

(1) There is $\xi < \mathrm{o}(\vec{E})$ such that $\mathrm{Lev}_0(S) \in E_\xi(d)$.
(2) For each $\vec{\nu} \frown \mu \in S$ there is $\xi < \mathrm{o}(\vec{E})$ such that $\mathrm{Suc}_S(\vec{\nu}) \in E_\xi(d)$.

If S is a tree of finite height $n < \omega$ then we write $\mathrm{Lev}_{\max} S$ for $\mathrm{Lev}_{n-1} S$.

Assume S is an $\vec{E}(d)$-tree, and for each $\vec{\nu} \in \mathrm{Lev}_{\max}(S)$ there is a set $X(\vec{\nu}) \subseteq \mathrm{OB}(d)$. By recursion define the diagonal intersection of the family $\{X(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{\max} S\}$ by setting $\triangle\{X(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{\max} S\} = \triangle\{X^*(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{\max} S^*\}$, where $S^* = S \cap {}^{n-1}\mathrm{OB}(d)$ and $X^*(\vec{\mu}) = \triangle\{X(\vec{\mu} \frown \langle \nu \rangle) \mid \nu \in S_{\langle \vec{\mu} \rangle}\}$. The following is immediate.

Corollary 2.3. *Assume S is an $\vec{E}(d)$-tree, and for each $\vec{\nu} \in \mathrm{Lev}_{\max}(S)$ there is a set $X(\vec{\nu}) \in \vec{E}(d)$. Then $\triangle\{X(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{\max} S\} \in \vec{E}(d)$.*

3. The Forcing Notion

A finite sequence $\langle \bar{\nu}_0, \ldots, \bar{\nu}_k \rangle \in {}^{<\omega}\mathrm{ES}$ is said to be o-decreasing if it is increasing and $\langle \mathrm{o}(\bar{\nu}_0), \ldots, \mathrm{o}(\bar{\nu}_k) \rangle$ is non-increasing.

Definition 3.1. A condition f is in the forcing notion $\mathbb{P}^*_f(\vec{E}, \epsilon)$ if f is a function $f : d \to {}^{<\omega}\mathrm{ES}$ such that:

(1) $d \in [\epsilon]^{<\lambda}$.
(2) $d \supseteq (|d| + 1)$.
(3) For each $\alpha \in d$, $f(\alpha)$ is o-decreasing.

Assume $f, g \in \mathbb{P}^*_f(\vec{E}, \epsilon)$ are conditions. We say f is an extension of g ($f \leq^*_{\mathbb{P}^*_f(\vec{E},\epsilon)} g$) if $f \supseteq g$.

For a condition $f \in \mathbb{P}^*_f(\vec{E})$ we will write $E_\xi(f)$ and $\vec{E}(f)$ instead of $E_\xi(\mathrm{dom}\, f)$ and $\vec{E}(\mathrm{dom}\, f)$, respectively. If $T \subseteq \mathrm{OB}(e)$ and $d \subseteq e$ then $T \restriction d = \{\nu \restriction d \mid \nu \in T\}$.

Definition 3.2. A condition p is in the forcing notion $\mathbb{P}^*(\vec{E},\epsilon)$ if p is of the form $\langle f^p, T^p\rangle$, where $f^p \in \mathbb{P}^*_f(\vec{E},\epsilon)$, $T^p \in \vec{E}(f^p)$, and for each $\nu \in T^p$ and each $\alpha \in \operatorname{dom}\nu$, $\max \mathring{f}^p(\alpha) < \mathring{\nu}(\kappa)$.

Assume $p, q \in \mathbb{P}^*(\vec{E},\epsilon)$ are conditions. We say p is a direct extension of q ($p \leq^*_{\mathbb{P}^*(\vec{E},\epsilon)} q$) if $f^p \supseteq f^q$ and $T^p \restriction \operatorname{dom} f^q \subseteq T^q$. We say p is a strong direct extension of q ($p \leq^{**}_{\mathbb{P}^*(\vec{E},\epsilon)} q$) if p is a direct extension of q and $f^p = f^q$.

Since ϵ and the sequence $\vec{E}$ are fixed througout this work we designate $\mathbb{P}^*(\vec{E},\epsilon)$ by $\mathbb{P}^*$.

Definition 3.3. A condition p is in the forcing $\bar{\mathbb{P}}$ if $p = \langle p_0, \dots, p_{n^p-1}\rangle$, where $n^p < \omega$, there is a sequence $\langle \vec{E}^p_i \mid i < n^p\rangle$ such that each $\vec{E}^p_i$ is a Mitchell increasing sequence of extenders, $\langle \operatorname{crit}(\vec{E}^p_0), \dots, \operatorname{crit}(\vec{E}^p_{n^p-1})\rangle$ is strictly increasing, $\sup\{j_{E^p_{i,\xi}}(\operatorname{crit}\vec{E}^p_i) \mid \xi < \operatorname{o}(\vec{E}^p_i)\} < \operatorname{crit}\vec{E}^p_{i+1}$, $\lambda(\vec{E}^p_i) < \operatorname{crit}(\vec{E}^p_{i+1})$, and for each $i < n^p$, $p_i \in \mathbb{P}^*(\vec{E}^p_i, \epsilon^p_i)$.

Assume $p, q \in \bar{\mathbb{P}}$ are conditions. We say p is a direct extension of q ($p \leq^*_{\bar{\mathbb{P}}} q$) if $n^p = n^q$ and for each $i < n^p$, $p^i \leq^* q^i$. We say p is a strong direct extension of q ($p \leq^{**}_{\bar{\mathbb{P}}} q$) if $n^p = n^q$ and for each $i < n^p$, $p^i \leq^{**} q^i$.

The following sequence of definitions leads to the definition of the order $\leq_{\bar{\mathbb{P}}}$ (which is somewhat involved, hence the breakup to several steps). If $\nu \in \operatorname{OB}(d)$ we let $\operatorname{o}(\nu) = \operatorname{o}(\nu(\kappa))$.

Definition 3.4. Assume $f : d \to {}^{<\omega}\mathrm{ES}$ is a function, $\nu \in \operatorname{OB}(d)$, and for each $\alpha \in \operatorname{dom}\nu$, $\max \mathring{f}(\alpha) < \mathring{\nu}(\kappa)$. Define $f_{\langle\nu\rangle\downarrow}$ and $f_{\langle\nu\rangle\uparrow}$ as follows.

1. If $\operatorname{o}(\nu) = 0$ then $f_{\langle\nu\rangle\downarrow} = \emptyset$. If $\operatorname{o}(\nu) > 0$ then $f_{\langle\nu\rangle\downarrow}$ is the function g, where:
   (a) $\operatorname{dom} g = \operatorname{ran}\mathring{\nu}$.
   (b) For each $\alpha \in \operatorname{dom}\nu$, $g(\mathring{\nu}(\alpha)) = \langle \bar{\tau}_{k+1}, \dots, \bar{\tau}_{n-1}\rangle$, where $f(\alpha) = \langle \bar{\tau}_0, \dots, \bar{\tau}_{n-1}\rangle$ and $k < n$ is maximal such that $\operatorname{o}(\bar{\tau}_k) \geq \operatorname{o}(\nu(\alpha))$. Set $k = -1$ if there is no $k < n$ such that $\operatorname{o}(\bar{\tau}_k) \geq \operatorname{o}(\nu(\alpha))$.
2. Define $f_{\langle\nu\rangle\uparrow}$ to be the function g where:
   (a) $\operatorname{dom} g = \operatorname{dom} f$.
   (b) For each $\alpha \in \operatorname{dom}\nu$, $g(\alpha) = \langle \bar{\tau}_0, \dots, \bar{\tau}_k\rangle$, where $f(\alpha) = \langle \bar{\tau}_0, \dots, \bar{\tau}_{n-1}\rangle$ and $k < n$ is maximal such that $\operatorname{o}(\bar{\tau}_k) \geq \operatorname{o}(\nu(\alpha))$. Set $k = -1$ if there is no $k < n$ such that $\operatorname{o}(\bar{\tau}_k) \geq \operatorname{o}(\nu(\alpha))$.

By recursion define $f_{\langle\nu_0,\dots,\nu_k\rangle\uparrow} = (f_{\langle\nu_0,\dots,\nu_{k-1}\rangle\uparrow})_{\langle\nu_k\rangle\uparrow}$.

Let us give several examples in order to clarify the above definition. Since the definition works on each $\alpha \in \operatorname{dom} f$ independently of the other we can concentrate on one such α. Thus assume

$$\begin{aligned} f(\alpha) = \langle & \langle \alpha_0, \langle e_{00}, e_{01}, e_{02} \rangle \rangle, \\ & \langle \alpha_1, \langle e_{10}, e_{11} \rangle \rangle, \\ & \langle \alpha_2, \langle e_{20} \rangle \rangle \rangle. \end{aligned}$$

Let us add $\langle \beta, \langle e \rangle \rangle$ above $f(\alpha)$, then:

$$\begin{aligned} f_{\langle \beta, \langle e \rangle \rangle \uparrow}(\alpha) = \langle & \langle \alpha_0, \langle e_{00}, e_{01}, e_{02} \rangle \rangle, \\ & \langle \alpha_1, \langle e_{10}, e_{11} \rangle \rangle, \\ & \langle \alpha_2, \langle e_{20} \rangle \rangle, \\ & \langle \beta, \langle e \rangle \rangle \rangle \end{aligned}$$

and

$$f_{\langle \beta, \langle e \rangle \rangle \downarrow}(\beta) = \langle \rangle.$$

Let us add $\langle \beta, \langle e_0, e_1 \rangle \rangle$ above $f(\alpha)$, then:

$$\begin{aligned} f_{\langle \beta, \langle e_0, e_1 \rangle \rangle \uparrow}(\alpha) = \langle & \langle \alpha_0, \langle e_{00}, e_{01}, e_{02} \rangle \rangle, \\ & \langle \alpha_1, \langle e_{10}, e_{11} \rangle \rangle, \\ & \langle \beta, \langle e_0, e_1 \rangle \rangle \rangle \end{aligned}$$

and

$$f_{\langle \beta, \langle e_0, e_1 \rangle \rangle \downarrow}(\beta) = \langle \langle \alpha_2, \langle e_{20} \rangle \rangle \rangle$$

Let us add $\langle \beta, \langle e_0, e_1, e_2 \rangle \rangle$ above $f(\alpha)$, then:

$$\begin{aligned} f_{\langle \beta, \langle e_0, e_1, e_2 \rangle \rangle \uparrow}(\alpha) = \langle & \langle \alpha_0, \langle e_{00}, e_{01}, e_{02} \rangle \rangle, \\ & \langle \beta, \langle e_0, e_1, e_2 \rangle \rangle \rangle \end{aligned}$$

and

$$\begin{aligned} f_{\langle \beta, \langle e_0, e_1, e_2 \rangle \rangle \downarrow}(\beta) = \langle & \langle \alpha_1, \langle e_{10}, e_{11} \rangle \rangle, \\ & \langle \alpha_2, \langle e_{20} \rangle \rangle \rangle \end{aligned}$$

Finally let us Let us add $\langle \beta, \langle e_0, e_1, e_2, e_3 \rangle \rangle$ above $f(\alpha)$, then:

$$f_{\langle \beta, \langle e_0, e_1, e_2, e_3 \rangle \rangle \uparrow}(\alpha) = \langle \langle \beta, \langle e_0, e_1, e_2, e_3 \rangle \rangle \rangle$$

and

$$\begin{aligned} f_{\langle \beta, \langle e_0, e_1, e_2, e_3 \rangle \rangle \downarrow}(\beta) = \langle & \langle \alpha_0, \langle e_{00}, e_{01}, e_{02} \rangle \rangle, \\ & \langle \alpha_1, \langle e_{10}, e_{11} \rangle \rangle, \\ & \langle \alpha_2, \langle e_{20} \rangle \rangle \rangle \end{aligned}$$

Definition 3.5. The following definitions show how to reflect down a function $\mu \in \mathrm{OB}(d)$ using a larger function $\nu \in \mathrm{OB}(d)$.

(1) Assume $\mu, \nu \in \mathrm{OB}(d)$, $\mu < \nu$, and $\mathrm{o}(\mu) < \min(\mathrm{o}(\nu), \mathring{\nu}(\kappa))$. Define the function $\tau = \mu \downarrow \nu \in \mathrm{OB}(\operatorname{ran} \mathring{\nu})$ by:
  (a) $\operatorname{dom} \tau = \{\mathring{\nu}(\alpha) \mid \alpha \in \operatorname{dom} \mu\}$.
  (b) For each $\xi \in \operatorname{dom} \tau$, $\tau(\xi) = \mu(\alpha)$, were $\xi = \mathring{\nu}(\alpha)$.

(2) Assume $T \subseteq \mathrm{OB}(d)$ and $\nu \in \mathrm{OB}(d)$. If $\mathrm{o}(\nu) = 0$ then set $T_{\langle \nu \rangle \downarrow} = \emptyset$. If $\mathrm{o}(\nu) > 0$ then $T_{\langle \nu \rangle \downarrow} = \{\mu \downarrow \nu \mid \mu \in T,\ \mu < \nu,\ \mathrm{o}(\mu) < \min(\mathrm{o}(\nu), \mathring{\nu}(\kappa))\}$.

Definition 3.6. Assume $p \in \mathbb{P}^*(\vec{E})$ and $\nu \in T^p$. We define $p_{\langle \nu \rangle \downarrow}$ as follows. If $\mathrm{o}(\nu) = 0$ then $p_{\langle \nu \rangle \downarrow} = \emptyset$. If $\mathrm{o}(\nu) > 0$ then $p_{\langle \nu \rangle \downarrow}$ is the condition $q \in \mathbb{P}^*(\overset{\downarrow}{\nu})$ defined by setting $f^q = f^p_{\langle \nu \rangle \downarrow}$ and $T^q = T^p_{\langle \nu \rangle \downarrow}$. Define $p_{\langle \nu \rangle \uparrow}$ to be the condition $q \in \mathbb{P}^*(\vec{E})$, where $f^q = f^p_{\langle \nu \rangle \uparrow}$ and $T^q = T^p_{\langle \nu \rangle}$. Finally set $p_{\langle \nu \rangle} = \langle p_{\langle \nu \rangle \downarrow}, p_{\langle \nu \rangle \uparrow} \rangle$.

Generalizing the above notation, if $p = p_0 {}^\frown \cdots {}^\frown \cdots p_n \in \mathbb{P}(\vec{E})$ and $\nu \in T^{p_n}$ then we let $p_{\langle \nu \rangle} = p_0 {}^\frown \cdots {}^\frown p_{n-1} {}^\frown p_{n \langle \nu \rangle}$. By recursion we let $p_{\langle \nu_0, \dots, \nu_k \rangle} = (p_{\langle \nu_0, \dots, \nu_{k-1} \rangle})_{\langle \nu_k \rangle}$.

Of course for the above definition to make sense $T^p_{\langle \nu \rangle \downarrow} \in \overset{\downarrow}{\nu}(\operatorname{ran} \nu)$ should hold, which we prove in claim 3.9.

If $T \subseteq \mathrm{OB}(d)$ hen we let ${}^{<\omega}T = \{\langle \nu_0, \dots, \nu_n \rangle \mid n < \omega,\ \nu_0, \dots, \nu_n \in T,\ \nu_0 < \cdots < \nu_n\}$.

Definition 3.7. Assume $p, q \in \bar{\mathbb{P}}$. We say p is an extension of q ($p \leq_{\bar{\mathbb{P}}} q$) if the following hold:

(1) $n^p \geq n^q$.
(2) $\{\vec{E}^q_j \mid j < n^q\} \subseteq \{\vec{E}^p_i \mid i < n^p\}$ and $\vec{E}^q_{n^q-1} = \vec{E}^p_{n^p-1}$.
(3) For each $i < n^q$ there is $\langle \nu_0, \dots, \nu_{k-1} \rangle \in {}^{<\omega}T^{q_i}$ such that $\langle p_{j_0+1}, \dots, p_{j_1} \rangle \leq^* q_{i \langle \nu_0, \dots, \nu_{k-1} \rangle}$, where i, j_0 and j_1, are being set as follows. Let $j_1 < n^p$ satisfy $\vec{E}^p_{j_1} = \vec{E}^q_i$. If $i = 0$ then set $j_0 = -1$. If $i > 0$ then let $j_0 < j_1$ satisfy $\vec{E}^p_{j_0} = \vec{E}^q_{i-1}$.

Finally we give the definition of the forcing notion we are going to work with:

Definition 3.8. $\mathbb{P}(\vec{E}, \epsilon) = \{q \leq_{\bar{\mathbb{P}}} p \mid p \in \mathbb{P}^*(\vec{E}, \epsilon)\}$. The partial orders $\leq_{\mathbb{P}(\vec{E},\epsilon)}$ and $\leq^*_{\mathbb{P}(\vec{E},\epsilon)}$ are inherited from $\leq_{\bar{\mathbb{P}}}$ and $\leq^*_{\bar{\mathbb{P}}}$.

Since ϵ and the sequence $\vec{E}$ are fixed throughout this work we will write $\mathbb{P}$ instead of $\mathbb{P}(\vec{E}, \epsilon)$ throughout this paper.

Claim 3.9 is needed in order to show the forcing notion defined above makes sense.

Claim 3.9. *If* $T \in \vec{E}(d)$ *then* $X = \{\nu \in T \mid T_{\langle\nu\rangle\downarrow} \in \overset{\downarrow}{\nu}(\operatorname{ran} \mathring{\nu})\} \in \vec{E}(d)$.

Proof. We need to show $X \in \vec{E}(d)$. I.e., we need to show for each $\xi < \mathrm{o}(\vec{E})$, $X \in E_\xi(d)$. Fix $\xi < \mathrm{o}(\vec{E})$. We need to show $\mathrm{mc}_\xi(d_\xi) \in j_{E_\xi}(X)$. Hence it is enough showing $\mathrm{mc}_\xi(d) \in j_{E_\xi}(T)$ and $j_{E_\xi}(T)_{\langle \mathrm{mc}_\xi(d)\rangle\downarrow} \in \vec{E} \restriction \xi(d)$. Since $T \in \vec{E}(d)$ we have $\mathrm{mc}_\xi(d) \in j_{E_\xi}(T)$. So we are left with showing $j_{E_\xi}(T)_{\langle \mathrm{mc}_\xi(d)\rangle\downarrow} \in \vec{E} \restriction \xi(d)$. From the definition of the operation $\downarrow$ we get

$$j_{E_\xi}(T)_{\langle \mathrm{mc}_\xi(d)\rangle\downarrow} = \{\mu \downarrow \mathrm{mc}_\xi(d) \mid \mu \in j_{E_\xi}(T),\ \mu < \mathrm{mc}_\xi(d),\ \mathrm{o}(\mu) < \min(\kappa, \xi)\}.$$

Consider $\mu \in j_{E_\xi}(T)$ such that $\mu < \mathrm{mc}_\xi(d)$. There is $\mu^* \in T$ such that $\mu = j_{E_\xi}(\mu^*)$. Since for each $\mu^* \in T$ such that $\mathrm{o}(\mu^*) < \xi$ we have $j_{E_\xi}(\mu^*) \downarrow \mathrm{mc}_\xi(d) = \mu^*$, we get $j_{E_\xi}(T)_{\langle \mathrm{mc}_\xi(d)\rangle\downarrow} = \{\mu \in T \mid \mathrm{o}(\mu) < \xi\} \in \vec{E} \restriction \xi(d)$. □

For each condition $p \in \mathbb{P}$ let $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$. It is immediate from the definitions above that for each $0 < i < n^p - 1$ the forcing notion $\mathbb{P}/p$ factors to $P_0 \times P_1$, where $P_0 = \{q^0 \leq p^0 \mid q^0 {}^\frown p^1 \in \mathbb{P}\}$, $P_1 = \{q^1 \leq p^1 \mid p^0 {}^\frown q^1 \in \mathbb{P}\}$, $p^0 = \langle p_0, \dots, p_{i-1}\rangle$, and $p^1 = \langle p_i, \dots, p_{n^p-1}\rangle$. Together with the Prikry property (claim 5.1) and the closure of the direct order, one can analyze the cardinal structure in $V^{\mathbb{P}}$ straightforwardly.

If $e \supseteq d$ we define $\pi^{-1}_{e,d}$ to be the inverse of the operation $\restriction d$, i.e., for each $X \subseteq \mathrm{OB}(d)$ we let $\pi^{-1}_{e,d}(X) = \{\nu \in \mathrm{OB}(e) \mid \nu \restriction d \in X\}$. If $f, g \in \mathbb{P}^*_f$ are conditions then we write $\pi^{-1}_{f,g}$ for $\pi^{-1}_{\operatorname{dom} f, \operatorname{dom} g}$.

We end this section with the analysis of the cardinal structure above κ in the generic extension: The cardinals between κ and λ are collapsed, and λ and the cardinals above it are preserved. The properties of cardinals up to κ will be dealt with in later sections.

Claim 3.10. $\mathbb{P}$ *satisfies the* λ^+*-cc.*

Proof. Begin with a family of conditions $\langle p^\xi \mid \xi < \lambda^+\rangle$. Without loss of generality we can assume $n^{p^{\xi_0}} = n^{p^{\xi_1}}$ for each $\xi_0, \xi_1 < \lambda^+$. Without loss of generality we can assume $\langle p^{\xi_0}_0, \dots, p^{\xi_0}_{n^{p^{\xi_0}}-2}\rangle = \langle p^{\xi_1}_0, \dots, p^{\xi_1}_{n^{p^{\xi_1}}-2}\rangle$ for each $\xi_0, \xi_1 < \lambda^+$. Thus, without loss of generality, we can assume $n^{p^\xi} = 1$ for each $\xi < \lambda^+$. By the Δ-system lemma we can assume $\{\operatorname{dom} f^{p^\xi} \mid \xi < \lambda^+\}$ is a Δ-system with kernel d. Since $|d| < \lambda$ we can assume that for each $\xi_0, \xi_1 < \lambda^+$ and $\alpha \in d$, $f^{p^{\xi_0}}(\alpha) = f^{p^{\xi_1}}(\alpha)$. Fix $\xi_0 < \xi_1 < \lambda^+$. Set $f = f^{p^{\xi_0}} \cup f^{p^{\xi_1}}$, $T = \pi^{-1}_{f, f^{p^{\xi_0}}} T^{p^{\xi_0}} \cap \pi^{-1}_{f, f^{p^{\xi_1}}} T^{p^{\xi_1}}$, and let $p = \langle f, T\rangle$. Then $p \leq p^{\xi_0}, p^{\xi_1}$. □

Claim 3.11. $\Vdash$ *"There are no cardinals between κ and λ".*

Proof. Fix a V-regular cardinal $\tau \in (\kappa, \lambda)$. Fix a condition $p \in \mathbb{P}$ such that $\operatorname{dom} f^{p_{n^p-1}} \supseteq \tau \setminus \kappa$ will hold. Let $G \subseteq \mathbb{P}$ be generic such that $p \in G$. Set $C = \{\vec{\nu} \in {}^{<\omega}T^{p_{n^p-1}} \mid p_{\langle \vec{\nu} \rangle} \in G\}$. Then $\sup\{\sup(\tau \cap \bigcup \operatorname{dom} \vec{\nu}) \mid \vec{\nu} \in C\} = \tau$. Since $\Vdash$ "$|C| \leq \kappa$" we get $p \Vdash$ "$\operatorname{cf} \tau \leq \kappa$". $\square$

Preservation of λ will be proved by a properness type argument (claim 3.14) for which we need some preparation.

We say the elementary substructure $N \prec H_\chi$, where χ is large enough, is κ-internally approachable if there is an increasing continuous sequence of elementary substructures $\langle N_\xi \mid \xi < \kappa \rangle$ such that $N = \bigcup\{N_\xi \mid \xi < \kappa\}$, for each $\xi < \kappa$, $N_\xi \prec H_\chi$, $|N_\xi| < \lambda$, $N_\xi \cap \lambda \in \mathrm{On}$, $\mathbb{P}^*_f \in N_\xi$, $N_{\xi+1} \supseteq {}^{<\kappa}N_{\xi+1}$, and $\langle N_{\xi'} \mid \xi' < \xi \rangle \in N_{\xi+1}$.

We say the pair $\langle N, f \rangle$ is a good pair if $N \prec H_\chi$ is a κ-internally approachable elementary substructure and there is a sequence $\langle \langle N_\xi, f_\xi \rangle \mid \xi < \kappa \rangle$ such that $\langle N_\xi \mid \xi < \kappa \rangle$ witnesses the κ-internal approachablity of N, $f = \bigcup\{f_\xi \mid \xi < \kappa\}$, $\langle f_\xi \mid \xi < \kappa \rangle$ is a $\leq^*$-decreasing continuous sequence in $\mathbb{P}^*_f$, and for each $\xi < \kappa$, $f_\xi \in \bigcap\{D \in N_\xi \mid D \text{ is a dense open subset of } \mathbb{P}^*_f\}$, $f_\xi \subseteq N_{\xi+1}$, and $f_\xi \in N_{\xi+1}$.

Assume $N \prec H_\chi$ is an elementary substructure such that $|N| < \lambda$, $N \supseteq {}^{<\kappa}N$, $\mathbb{P}^*_f \in N$, $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in N$, and $D \in N$ is a dense open subset of $\mathbb{P}^*_f$. Then the set $E = \{h \in \mathbb{P}^*_f \mid \operatorname{dom} h \supseteq \bigcup_{i \leq k} \operatorname{dom} \nu_i,\ h_{\langle \nu_0, \ldots, \nu_k \rangle \uparrow} \in D\}$, which is evidently in N, is also a dense open subset of $\mathbb{P}^*_f$.

Hence, if $N \prec H_\chi$ is an elementary substructure such that $|N| < \lambda$, $N \supseteq {}^{<\kappa}N$, $\mathbb{P}^*_f \in N$, $f \in \bigcap\{D \in N \mid D \text{ is a dense open subset of } \mathbb{P}^*_f\}$, $f \subseteq N$, and $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in N \cap \mathrm{OB}(\operatorname{dom} f)$, then $f_{\langle \nu_0, \ldots, \nu_{k-1} \rangle \uparrow} \in \bigcap\{D \in N \mid D \text{ is a dense open subset of } \mathbb{P}^*_f\}$.

Hence if $\langle N, f \rangle$ is a good pair and $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in N \cap \mathrm{OB}(\operatorname{dom} f)$, then $\langle N, f_{\langle \nu_0, \ldots, \nu_{k-1} \rangle \uparrow} \rangle$ is a good pair also.

The following is immediate.

Claim 3.12. *For each set X and $f \in \mathbb{P}^*_f$ there is a good pair$\langle N, f^* \rangle$ such that $f^* \leq^* f$ and $X, f \in N$.*

Assume χ is large enough and $N \prec H_\chi$ is an elementary substructure such that $\mathbb{P} \in N$. We say the condition $p \in N$ is N-generic if for each dense open subset $D \in N$ of $\mathbb{P}$ we have $p \Vdash$ "$\check{\mathbb{P}} \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset$".

We say the forcing notion $\mathbb{P}$ is λ-proper if for an unbounded set of structures $N \prec H_\chi$ such that $\mathbb{P} \in N$ and $|N| < \lambda$, and for each condition $p \in \mathbb{P} \cap N$ there is a stronger N-generic condition.

The followig lemma shows a property stronger than properness.

Lemma 3.13. *Let $N \prec H_\chi$ be a κ-internally approachable structure, $\mathbb{P} \in N$, and $p \in N \cap \mathbb{P}$ a condition. Then there is a direct extension $p^* \leq^* p$ such that for each dense open subset $D \in N$ of $\mathbb{P}$ the set $\{s \frown p^*_{\langle \nu_0, \ldots, \nu_{n-1} \rangle \uparrow} \in D \mid \langle \nu_0, \ldots, \nu_{n-1} \rangle \in {}^{<\omega}T^{p^*},\ s \leq^* p^*_{\langle \vec{\nu} \rangle \downarrow}\}$ is predense below p^*. Moreover, if $s \leq^* p^*_{\langle \vec{\nu} \rangle \downarrow}$ and $s \frown p^*_{\langle \nu_0, \ldots, \nu_{n-1} \rangle \uparrow} \in D$ then there is a weaker condition $q \geq^* p^*_{\langle \nu_0, \ldots, \nu_{n-1} \rangle \uparrow}$ such that $s \frown q \in D \cap N$.*

Proof. Let $\langle N, f^* \rangle$ be a good pair such that $f^* \leq^* f^{p_{n^p-1}}$. Choose a set $T \in \vec{E}(f^*)$ such that $\langle f^*, T \rangle \leq^* p_{n^p-1}$. Let $\langle D_\alpha \mid \alpha < |N| \rangle$ be an enumeration of the dense open subsets of $\mathbb{P}$ appearing in N. Let $\langle \langle N_\iota, f_\iota \rangle \mid \iota < \kappa \rangle$ be a sequence witnessing $\langle N, f^* \rangle$ is a good pair. For each $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in {}^{<\omega}T$ construct the set $T^{\langle \nu_0, \ldots, \nu_{k-1} \rangle}$ as follows.

Fix $\vec{\nu} = \langle \nu_0, \ldots, \nu_{k-1} \rangle \in {}^{<\omega}T$. Let $\mathcal{D} = \{D_\alpha \mid \alpha \in \operatorname{dom} \nu_{k-1}\}$. Note $\mathcal{D} \in N$ since $|\nu_{k-1}| < \kappa$ and $N \supseteq {}^{<\kappa}N$. For each $s \in \mathbb{P}(\overset{\shortmid}{\nu}_{k-1})$ and $D \in \mathcal{D}$ define the sets $D^{\in}_{\vec{\nu},s,D}$, $D^{\perp}_{\vec{\nu},s,D}$, and $D^{*}_{\vec{\nu},s,D}$, as follows: Let $g \in D^{\in}_{\vec{\nu},s,D}$ if $g \leq f^{p_{n^p-1}}$, $\operatorname{dom} g \supseteq \operatorname{dom} \nu_{k-1}$, and $s \frown \langle g_{\langle \vec{\nu} \rangle}, T' \rangle \in D$ for some $T' \in \vec{E}(g)$. Let $h \in D^{\perp}_{\vec{\nu},s,D}$ if $h \perp g$ for each $g \in D^{\in}_{\langle \vec{\nu} \rangle,s,D}$. Set $D^*_{\vec{\nu},s,D} = D^{\in}_{\vec{\nu},s,D} \cup D^{\perp}_{\vec{\nu},s,D}$. It is immediate $D^{\in}_{\vec{\nu},s,D}$ and $D^{\perp}_{\vec{\nu},s,D}$ are open subsets of $\mathbb{P}^*_f$ below$f^{p_{n^p-1}}$. Thus $D^*_{\vec{\nu},s,D}$ is a dense open subset of $\mathbb{P}^*_f$ below $f^{p_{n^p-1}}$. Set $D^*_{\vec{\nu}} = \bigcap \{D^*_{\vec{\nu},s,D} \mid s \in \mathbb{P}(\overset{\shortmid}{\nu}_{k-1}),\ D \in \mathcal{D}\}$. Note $D^*_{\vec{\nu}} \in N$ is a dense open subset of $\mathbb{P}^*_f$ below $f^{p_{n^p-1}}$. Let $\iota < \kappa$ be minimal such that $\vec{\nu}, \mathcal{D}, D^*_{\vec{\nu}} \in N_\iota$. Then $f_\iota \in D^*_{\vec{\nu}} \cap N_{\iota+1}$. Thus for each $s \in \mathbb{P}(\overset{\shortmid}{\nu}_{k-1})$ and $D \in \mathcal{D}$ either there is a set $T^{\vec{\nu},s,D} \in \vec{E}(f_\iota) \cap N_{\iota+1}$ such that $s \frown \langle f_{\iota \langle \vec{\nu} \rangle}, T^{\vec{\nu},s,D} \rangle \in D$ or $s \frown \langle h, T'' \rangle \notin D$ for each $h \leq^* f_{\iota \langle \vec{\nu} \rangle}$ and $T'' \in \vec{E}(h)$. Set $T^{\nu} = \bigcap \{T^{\vec{\nu},s,D} \mid s \in \mathbb{P}(\overset{\shortmid}{\nu}_{k-1}),\ D \in \mathcal{D},\ s \frown \langle f_{\iota \langle \vec{\nu} \rangle}, T^{\nu,s,D} \rangle \in D\}$.

Set $T^* = \triangle \{\pi^{-1}_{f^*, f_{\iota(\vec{\nu})}} T^{\vec{\nu}} \mid \vec{\nu} \in {}^{<\omega}T\}$. Set $p^* = p \restriction n^p - 1 \frown \langle f^*, T^* \rangle$. We claim p^* satisfies the lemma. To show this fix a dense open subset $D \in N$ and a condition $q \leq p^*$.

Let $\alpha < |N|$ be such that $D = D_\alpha$. Without loss of generality assume $q \in D$, $q_{n^q-1} \leq p^*_{\langle \nu_0, \ldots, \nu_{k-1} \rangle \uparrow}$, $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in {}^{<\omega}T^*$, and $\alpha \in \operatorname{dom} \nu_{k-1}$. Set $s = q \restriction n^q - 1$. Thus $q = s \frown \langle f^{q_{n^q-1}}, T^{q_{n^q-1}} \rangle \in D$. Let $\iota < \kappa$ be minimal such that $\langle \nu_0, \ldots, \nu_{k-1} \rangle, \{D_\alpha \mid \alpha \in \operatorname{dom} \nu_{k-1}\} \in N_\iota$. Since $f^{q_{n^q-1}} \leq^* f_{\iota \langle \nu_0, \ldots, \nu_{k-1} \rangle}$ we must have $s \frown \langle f_{\iota \langle \vec{\nu} \rangle}, T^{\vec{\nu},s,D} \rangle \in D$, hence $s \frown p^*_{\langle \vec{\nu} \rangle \uparrow} \in D$. It is clear q and $s \frown p^*_{\langle \vec{\nu} \rangle \uparrow}$ are compatible. In addition $s \frown \langle f_{\iota \langle \vec{\nu} \rangle}, T^{\vec{\nu},s,D} \rangle \in N$, thus we are done. $\square$

Corollary 3.14. $\mathbb{P}$ *is λ-proper.*

Corollary 3.15. $\Vdash$ *"λ is a cardinal".*

4. Dense open sets and measure one sets

In order to reduce clutter later on, given a condition $p \in \mathbb{P}^*$, we will say a tree is a p-tree instead of saying it is an $\vec{E}(f^p)$-tree. If S is a p-tree and r is a function with domain S then we define the function $\vec{r}$ by setting for each $\vec{\nu} = \langle \nu_0, \dots, \nu_n \rangle \in S$, $\vec{r}(\vec{\nu}) = r(\nu_0) {}^\frown \cdots {}^\frown r(\nu_0, \dots, \nu_n)$. A function r is said to be a $\langle p, S \rangle$-function if S is a p-tree, for each $\vec{\nu} \in \operatorname{Lev}_{<\max} S$, $\vec{r}(\vec{\nu}) \leq^{**} p_{\langle \vec{\nu} \rangle \downarrow}$, and for each $\vec{\nu} \in \operatorname{Lev}_{\max} S$, $\vec{r}(\vec{\nu}) \leq^{**} p_{\langle \vec{\nu} \rangle}$.

4.1. **One of the measures suffices.**

The aim of this subsection is to prove claim 4.3, which together with corollary 4.12 will allow the investigation of the cardinal structure below κ. Note the proof of corollary 4.12 depends on claim 4.3. The following lemma, which is quite technical, takes its core argument from the proof of the Prikry property for Radin forcing.

Lemma 4.1. *Assume $p \in \mathbb{P}^*$ is a condition, S is a p-tree of height one, and r is a $\langle p, S \rangle$-function. Then there is a strong direct extension $p^* \leq^{**} p$ such that $\{r(\nu) \mid \langle \nu \rangle \in S\}$ is predense below p^*.*

Proof. We derive two functions r_0 and r_1 from the function r as follows. By the definition of a $\langle p, S \rangle$-function we know that $r(\nu) \leq^* p_{\langle \nu \rangle}$ for each $\langle \nu \rangle \in S$. For each $\langle \nu \rangle \in S$ work as follows. By the definition of $p_{\langle \nu \rangle}$ we have two conditions $p_0(\nu)$ and $p_1(\nu)$ such that $p_{\langle \nu \rangle} = p_0(\nu) {}^\frown p_1(\nu)$. Thus $r(\nu) \leq^* p_0(\nu) {}^\frown p_1(\nu)$. By the definition of the direct order there are two conditions $r_0(\nu)$ and $r_1(\nu)$ such that $r(\nu) = r_0(\nu) {}^\frown r_1(\nu)$, $r_0(\nu) \leq^* p_0(\nu)$ and $r_1(\nu) \leq^* p_1(\nu)$. Thus we have the definition of the functions r_0 and r_1.

Fix $\xi < \operatorname{o}(\vec{E})$ so that $S \in E_\xi(f^p)$ will hold. We collect the information from the sets $T^{r_0(\nu)}$ and $T^{r_1(\nu)}$ into the sets T^* and R, respectively, as follows.. The information from the sets $T^{r_1(\nu)}$'s is collected by setting $R = \triangle_{\langle \nu \rangle \in S} T^{r_1(\nu)}$. By lemma 2.2 $R \in \vec{E}(f^p)$.

The information from the sets $T^{(r_0(\nu))}$'s is collected into the set T^* as follows. The set T^* will be the union of the three sets T^0, T^1, and T^2, which we construct now.

(Note that if we were dealing with classical Radin forcing then the sets T^0, T^1, and T^2, would have been as follows:: T^0 would be a measure one set in the sense of the measures with indices below ξ, T^1 would have been a measure one set in the sense if the ξ-th measure, and T^2 would have been a measure one set in the sense of the measures with indices above ξ.)

The construction of T^0 is easy. Set $T^0 = T^{j_{E_\xi}(r_0)(\operatorname{mc}_\xi(f^p))}$. It is obvious $T^0 \in \vec{E} \restriction \xi(f^p)$.

The constructin of T^1 is slightly more involved than the construction of T^0. Set $T^{1\prime} = \{\langle\nu\rangle \in S \mid T^0_{\langle\nu\rangle\downarrow} = T^{r_0(\nu)}\}$. From the construction of T^0 it is clear $T^{1\prime} \in E_\xi(f^p)$. For each $\mu \in T^0$ set $X(\mu) = \{\langle\nu\rangle \in S \mid \mu < \nu,\ \mu \downarrow \nu \in T^{r_0(\nu)}\}$. From the construction of T^0 we get $X(\mu) \in E_\xi(f^p)$. Set $T^1 = \{\nu \in T^{1\prime} \mid \forall\mu \in T^0\ (\mu < \nu \implies \nu \in X(\mu))\}$. We show $T^1 \in E_\xi(f^p)$. Thus we need to show $\mathrm{mc}_\xi(f^p) \in j_{E_\xi}(T^1)$. Since $\mathrm{mc}_\xi(f^p) \in j_{E_\xi}(T^{1\prime})$ it is enough to show that if $\mu \in j_{E_\xi}(T^0)$ and $\mu < \mathrm{mc}_\xi(f^p)$ then $\mathrm{mc}_\xi(f^p) \in j_{E_\xi}(X)(\mu)$. So fix $\mu \in j_{E_\xi}(T^0)$ such that $\mu < \mathrm{mc}_\xi(f^p)$. Then $|\mu| < \kappa$, $\operatorname{dom}\mu \subseteq j''_{E_\xi}(\operatorname{dom} f^p)$, and $\sup\operatorname{ran}\mathring{\mu} < \kappa$. Necessarily there is $\mu^* \in T^0$ such that $\mu = j_{E_\xi}(\mu^*)$. Hence $j_{E_\xi}(X)(\mu) = j_{E_\xi}(X(\mu^*)) \ni \mathrm{mc}_\xi(f^p)$, by which we are done.

We construct now the set T^2. For each $\mu \in R$ set $Y(\mu) = \{\nu \downarrow \mu \in R_{\langle\mu\rangle\downarrow} \mid \nu \in T^1,\ R_{\langle\mu\rangle\downarrow\langle\nu\downarrow\mu\rangle\downarrow} \in \overset{\downarrow}{\nu}(\operatorname{dom}\nu)\}$. Now let $T^2 = \{\mu \in R \mid \exists\tau < \mathrm{o}(\mu)\ Y(\mu) \in \overset{\downarrow}{\mu}_\tau(\operatorname{dom}\mu)\}$. We show $T^2 \in E_\zeta(f^p)$ for each $\zeta > \xi$. We need to show for each $\zeta > \xi$, $\mathrm{mc}_\zeta(f^p) \in j_{E_\zeta}(T^2)$. Fix $\zeta > \xi$. We show $\mathrm{mc}_\zeta(f^p) \in j_{E_\zeta}(T^2)$. It is enough to show there is $\tau < \zeta$ such that $j_{E_\zeta}(Y)(\mathrm{mc}_\zeta(f^p)) \in E_\tau(f^p)$. We claim ξ can serve as the needed $\tau < \zeta$. Thus it is enough to show $j_{E_\zeta}(Y)(\mathrm{mc}_\zeta(f^p)) \in E_\xi(f^p)$. Hence we need to show

$$\{\nu \downarrow \mathrm{mc}_\zeta(f^p) \in j_{E_\zeta}(R)_{\langle\mathrm{mc}_\zeta(f^p)\rangle\downarrow} \mid \nu \in j_{E_\zeta}(T^1),\ j_{E_\zeta}(R)_{\langle\mathrm{mc}_\zeta(f^p)\rangle\downarrow\langle\nu\downarrow\mathrm{mc}_\zeta(f^p)\rangle\downarrow} \in \overset{\downarrow}{\nu}(\operatorname{dom}\nu)\} \in E_\xi(f^p).$$

Note $R^* = j_{E_\zeta}(R)_{\langle\mathrm{mc}_\zeta(f^p)\rangle\downarrow} \in \vec{E} \restriction \zeta(f^p)$, and if $\nu \in j_{E_\zeta}(T^1)$ and $\nu < \mathrm{mc}_\zeta(f^p)$, then there is $\nu^* \in T^1$ such that $\nu = j_{E_\zeta}(\nu^*)$. Moreover, $\nu^* = \nu \downarrow \mathrm{mc}_\zeta(f^p)$. Hence it is enough to show

$$\{\nu^* \in R^* \mid \nu^* \in T^1,\ R^*_{\langle\nu^*\rangle\downarrow} \in \overset{\downarrow}{\nu}{}^*(\operatorname{dom}\nu^*)\} \in E_\xi(f^p).$$

We are done since the last formula holds.

Having constructed T^0, T^1, and T^2 we set $p^* = \langle f^p, T^* \cap R\rangle$. We will be done by showing $\{r(\nu) \mid \nu \in S\}$ is predense below p^*. Assume $q \leq p^*$. We need to exhibit $\nu \in S$ so that $q \parallel r(\nu)$. We work as follows. Fix $\langle\mu_0, \dots, \mu_{n-1}\rangle \in {}^{<\omega}T^{p^*}$ such that $q \leq^* p^*_{\langle\mu_0,\dots,\mu_{n-1}\rangle}$. There are three cases to handle:

(1) Assume there is $i < n$ such that $\langle\mu_0, \dots, \mu_{i-1}\rangle \in {}^{<\omega}T^0$ and $\mu_i \in T^1$. The construction of T^1 yields $\langle\mu_0, \dots, \mu_{i-1}\rangle \in T^{r_0(\mu_i)}$ and the construction of R yields $\langle\mu_{i+1}, \dots, \mu_{n-1}\rangle \in {}^{<\omega}T^{r_1(\mu_i)}$. Hence $r_0(\mu_i)_{\langle\mu_0,\dots,\mu_{i-1}\rangle} {}^\frown r_1(\mu_i)_{\langle\mu_{i+1},\dots,\mu_{n-1}\rangle}$ and q are $\leq^*$-compatible, by which this case is done.

(2) Assume $\langle \mu_0, \dots, \mu_{n-1} \rangle \in {}^{<\omega}T^0$. By the construction of T^1 the set $X = \{\nu \in T^1 \mid \langle \mu_0, \dots, \mu_{n-1} \rangle \downarrow \nu \in {}^{<\omega}T^0_{\langle \nu \rangle \downarrow}\} \in E_\xi(f^p)$. Choose $\nu^* \in T^{q_{n^{n^q}-1}}$ such that $\nu = \nu^* \restriction f^p \in X$. Then $q_{\langle \nu^* \rangle} \leq^* p_{\langle \mu_0, \dots, \mu_{n-1}, \nu \rangle}$. Now we can procced as in the first case above.

(3) The last case is when there is $i < n$ such that $\langle \mu_0, \dots, \mu_{i-1} \rangle \in {}^{<\omega}T^0$ and $\mu_i \notin T^0 \cup T^1$. By the construction of T^2 there is $\tau < \mathrm{o}(\mu_i)$ such that $Y = Y(\mu_i) \in \mu_{i\tau}(\operatorname{dom} \mu_i)$. Hence there are $\mu_{i\tau}(\operatorname{dom} \mu_i)$-many $\nu \downarrow \mu_i$ such that $\nu \in T^1$, $\nu \downarrow \mu_i \in T^*_{\langle \mu_i \rangle \downarrow}$ and $T^*_{\langle \mu_i \rangle \downarrow \langle \nu \rangle \downarrow \mu_i} \in \nu(\operatorname{dom} \nu)$.

Thus there is $\sigma^* \in T^{q_i}$ such that $\sigma = \sigma^* \restriction \operatorname{dom} f^{p_i} \in Y$, where $\sigma = \nu \downarrow \mu_i$ and $\nu \in T^1$. Thus $q_{\langle \sigma^* \rangle} \leq^* p_{\langle \mu_0, \dots, \mu_{i-1}, \nu, \mu_i, \dots, \mu_{n-1} \rangle}$ and we can proceed as in the first case above.

□

Corollary 4.2. *Assume $p \in \mathbb{P}$ is a condition, S is a p_{n^p-1}-tree of height one, and r is a $\langle p_{n^p-1}, S \rangle$-function. Then there is a strong direct extension $p^* \leq^{**} p$ such that $p^* \restriction n^p - 1 = p \restriction n^p - 1$ and $\{p \restriction n^p - 1 {}^\frown r(\nu) \mid \langle \nu \rangle \in S\}$ is predense open below p^*.*

Generalize the notions of p-tree and $\langle p, S \rangle$-function to arbitrary condition $p \in \mathbb{P}$ as follows. By recursion we say the tree S is a p-tree if there is $n < \omega$ for which following hold:

(1) $\mathrm{Lev}_{<n}(S)$ is a $p \restriction n^p - 1$-tree.
(2) For each $\vec{\nu} \in \mathrm{Lev}_{n-1}(S)$, $S_{\langle \vec{\nu} \rangle}$ is a p_{n^p-1}-tree.

Let $p \in \mathbb{P}$ be an arbitrary condition. By recursion we say the function r is a $\langle p, S \rangle$-function if there is $n < \omega$ such that:

(1) S is a p-tree.
(2) $\mathrm{Lev}_{<n}(S)$ is a $p \restriction n^p - 1$-tree.
(3) $r \restriction \mathrm{Lev}_{<n} S$ is a $\langle \mathrm{Lev}_{<n} S, p \restriction n^p - 1 \rangle$-function.
(4) For each $\vec{\nu} \in \mathrm{Lev}_{n-1}(S)$ the function s with domain $S_{\langle \vec{\nu} \rangle}$, define by setting $s(\vec{\mu}) = r(\vec{\nu} {}^\frown \vec{\mu})$, is a $\langle p_{n^p-1}, S_{\langle \vec{\nu} \rangle} \rangle$-function.

Claim 4.3. *Assume $p \in \mathbb{P}$ is a condition, S is a p-tree, and r is a $\langle p, S \rangle$-function. Then there is a strong direct extension $p^* \leq^{**} p$ such that $\{\vec{r}(\vec{\nu}) \mid \vec{\nu} \in S\}$ is predense below p^*.*

Proof. If S is a p_{n^p-1}-tree then we are done by corollary 4.2. Thus assume there is $n < \omega$ such that $\mathrm{Lev}_{<n} S$ is a $p \restriction n^p - 1$-tree. Construct the strong direct extension $q(\vec{\nu}) \leq^{**} p_{\langle \vec{\nu} \rangle \uparrow}$ and the $\langle p_{\langle \vec{\nu} \rangle \uparrow}, S_{\langle \vec{\nu} \rangle} \rangle$-function $s_{\vec{\nu}}$ for each $\vec{\nu} \in \mathrm{Lev}_{n-1} S$ as follows. For each $\vec{\nu} \in \mathrm{Lev}_{n-1} S$ let $s_{\vec{\nu}}$ be the function with domain $S_{\langle \vec{\nu} \rangle}$ defined by setting $s_{\vec{\nu}}(\vec{\mu}) = r(\vec{\nu} {}^\frown \vec{\mu})$ for each $\vec{\mu} \in S_{\langle \vec{\nu} \rangle}$. By corollary 4.2 there is a strong direct extension $q(\vec{\nu}) \leq^{**} p_{\langle \vec{\nu} \rangle \uparrow}$ such that $\{s_{\vec{\nu}}(\vec{\mu}) \mid \vec{\mu} \in \mathrm{Lev}_{\max}(S(\vec{\nu}))\}$ is predense below $q(\vec{\nu})$. Let

$q \leq^{**} p_{n^p-1}$ be a strong direct extension satisfying $q \leq^{**} q(\vec{\nu})$ for each $\vec{\nu} \in \mathrm{Lev}_{n-1}(S)$. Hence $\{s_{\vec{\nu}}(\vec{\mu}) \mid \vec{\mu} \in \mathrm{Lev}_{\max}(S(\vec{\nu}))\}$ is predense below q for each $\vec{\nu} \in \mathrm{Lev}_{n-1} S$. Hence $\{\vec{r}(\vec{\nu}) ^\frown s_{\vec{\nu}}(\vec{\mu}) \mid \vec{\mu} \in \mathrm{Lev}_{\max} S(\vec{\nu})\}$ is predense below $\vec{r}(\vec{\nu}) ^\frown q$. Let $p^* \leq^{**} p$ be a strong direct extension such that $p^*_{n^p-1} = q$ and $p^* \restriction n^p - 1 \leq^{**} p \restriction n^p - 1$ is a strong direct extension constructed by recursion so as to satisfy $\{\vec{r}(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{n-1} S\}$ is predense below $p^* \restriction n^p - 1$. Necessarily $\{\vec{r}(\vec{\nu}) \mid \vec{\nu} \in \mathrm{Lev}_{\max} S\}$ is predense below p^* □

4.2. **Dense open sets and direct extensions.** In this subsection we prove corollary 4.12, which is the basic tool to be used in the next section to analyse the properties of the cardinal κ and the cardinal structure below it.

An essential obstacle in the extender based Radin forcing in comparison to the plain Radin forcing is that while in the later forcig notion if we have two direct extensions $q, r \leq^* p$ then q and r are compatible, in the former forcing notion this does not hold. This usually entails some inductions, taking place inside elementary substructures, which construct long increasing seqeunce of conditions from $\mathbb{P}^*_f$, which at the end will be combined into one conditions. This method breaks if the elementary substructures in question are not closed enough (which is our case if we want to handle λ successor of singular). The point of lemma 4.7 is to show how we can construction a condition p such that if a direct extension $q \leq^* p$ has some favorable circumstances then the condition p will suffice for this circumstances. This will enable us to work more like in a plain Radin forcing.

So as we just pointed out, we aim to prove lemma 4.7. This lemma is proved by recursion with the non-recursive case being lemma 4.4. Since the notation in lemma 4.7 is kind of hairy we present the cases $k = 1$ and $k = 2$ in lemma 4.5 and lemma 4.6, respectively.

Lemma 4.4. *Assume $\langle N, f^* \rangle$ is a good pair and $D \in N$ is a dense open set. Let $p \in \mathbb{P}$ be a condition such that $f^{p_{n^p-1}} = f^*$. If there is an extension $s \leq p \restriction n^p - 1$ and a direct extension $q \leq^* p_{n^p-1}$ such that $s ^\frown q \in D$ then there is a set $T^* \in \vec{E}(f^*)$ such that $\langle f^*, T^* \rangle \leq^{**} p_{n^p-1}$ and $s ^\frown \langle f^*, T^* \rangle \in D$.*

Proof. Assume $s \leq p \restriction n^p - 1$, $q \leq^* p_{n^p-1}$, and $s ^\frown q \in D$. Set $D^\in = \{g \mid \exists T \in \vec{E}(g) \;\; s ^\frown \langle g, T \rangle \in D\}$ and $D^\perp = \{g \mid \forall h \in D^\in \; g \perp h\}$. Then $D^\perp \in N$ is open by its definiton and $D^\in \in N$ is open since D is open. The set $D^* = D^\in \cup D^\perp \in N$ is dense open, hence $f^* \in D^*$. Since $f^* \geq f^q \in D^\in$ we get $f^* \notin D^\perp$, thus $f^* \in D^\in$. □

Lemma 4.5. *Assume $\langle N, f^*\rangle$ is a good pair, $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition such that $f^{p_{n^p-1}} = f^*$. If there is an extension $s \leq p \restriction n^p - 1$ and $\xi < \mathrm{o}(\vec{E})$ such that $\{\nu \in T^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle\nu\rangle}\ s \frown q \in D\} \in E_\xi(f^*)$, then there is a p_{n^p-1}-tree S of height one, and a $\langle p_{n^p-1}, S\rangle$-function r, such that for each $\langle\nu\rangle \in S$, $s \frown r(\nu) \in D$.*

Proof. Assume $X = \{\nu \in T^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle\nu\rangle}\ s \frown q \in D\} \in E_\xi(f^*)$. Set

$$D^\in = \{g \mid \exists T \in \vec{E}(g), \text{ there is a } \langle g, T\rangle\text{-tree } S \text{ of height one and } \\ \text{a } \langle\langle g, T\rangle, S\rangle\text{-function } r \text{ such that } \forall\langle\nu\rangle \in S\ s \frown r(\nu) \in D\}$$

and $D^\perp = \{g \mid \forall h \in D^\in\ g \perp h\}$. Then $D^\perp \in N$ is open by its definiton and $D^\in \in N$ is open since D is open. The set $D^* = D^\in \cup D^\perp \in N$ is dense open, hence $f^* \in D^*$. For each $\nu \in X$ fix a direct extension $t(\nu) \leq^* p_{n^p-1\langle\nu\rangle\downarrow}$, and a direct extension $q(\nu) \leq^* p_{n^p-1\langle\nu\rangle\uparrow}$ such that $s \frown t(\nu) \frown q(\nu) \in D$. Since $\langle N, f^*_{\langle\nu\rangle\uparrow}\rangle$ is a good pair we get by the previous lemma a set $T(\nu) \in \vec{E}(f^*_{\langle\nu\rangle\uparrow})$ satisfying $\langle f^*_{\langle\nu\rangle\uparrow}, T(\nu)\rangle \leq^{**} p_{n^p-1\langle\nu\rangle\uparrow}$ and $s \frown t(\nu) \frown \langle f^*_{\langle\nu\rangle\uparrow}, T(\nu)\rangle \in D$.

Set $g = f^* \cup f^{j_{E_\xi}(t)(\mathrm{mc}_\xi(f^*))}$. Set $X^* = \pi^{-1}_{g,f^*}(X)$. By removing a measure zero set from X^* we can assume for each $\nu \in X^*$, $g_{\langle\nu\rangle\downarrow} = f^{t(\nu\restriction\mathrm{dom}\, f^*)}$. Choose a set $T \in \vec{E}(g)$ such that $\langle g, T\rangle \leq^* p_{n^p-1}$. Define the funcrion r with domain X^* by setting for each $\nu \in X^*$, $r(\nu) = \langle g_{\langle\nu\rangle\downarrow}, T^{(t(\nu\restriction\mathrm{dom}\, f^*))} \cap T_{\langle\nu\rangle\downarrow}\rangle \frown \langle g_{\langle\nu\rangle\uparrow}, \pi^{-1}_{g,f^*} T(\nu) \cap T\rangle$. Note $r(\nu) \leq^{**} \langle g, T\rangle_{\langle\nu\rangle}$, thus r is a $\langle g, X^*\rangle$-function. Since D is open we get for each $\nu \in X^*$, $s \frown r(\nu) \in D$. Thus $g \in D^\in$. Since $g \leq f^* \in D^*$ we get $f^* \in D^\in$. □

Lemma 4.6. *Assume $\langle N, f^*\rangle$ is a good pair, $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition such that $f^{p_{n^p-1}} = f^*$. If there is $s \leq p \restriction n^p - 1$ such that $\{\langle\nu_0, \nu_1\rangle \in {}^2T^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle\nu_0,\nu_1\rangle}\ s \frown q \in D\}$ is an $\vec{E}(f^*)$-tree, then there is a p_{n-1}-tree S of height two, and a $\langle p_{n^p-1}, S\rangle$-function r such that for each $\langle\nu_0, \nu_1\rangle \in S$, $s \frown \vec{r}(\nu_0, \nu_1) \in D$.*

Proof. Assume $X = \{\langle\nu_0, \nu_1\rangle \in {}^2T^{p^*_{n^p-1}} \mid \exists s \leq p \restriction n^p - 1\ \exists q \leq^* p_{n^p-1\langle\nu_0,\nu_1\rangle}\ q \in D\}$ is an $\vec{E}(f^*)$-tree. Set

$$D^\in = \{g \mid \exists T \in \vec{E}(g), \text{ there is a } \langle g, T\rangle\text{-tree } S \text{ of height two and } \\ \text{a } \langle\langle g, T\rangle, S\rangle\text{-function } r \text{ such that} \\ \forall\langle\nu_0, \nu_1\rangle \in S\ s \frown \vec{r}(\nu_0, \nu_1) \in D\}$$

and $D^\perp = \{g \mid \forall h \in D^\in\ g \perp h\}$. Then $D^\perp \in N$ is open by its definiton and $D^\in \in N$ is open since D is open. The set $D^* = D^\in \cup$

$D^{\perp} \in N$ is dense open, hence $f^* \in D^*$. For each $\langle \nu_0, \nu_1 \rangle \in X$ fix a direct extension $t(\nu_0, \nu_1) \leq^* p_{n^p-1\langle \nu_0 \rangle \downarrow}$, a direct extension $t_0(\nu_0, \nu_1) \leq^* p_{n^p-1\langle \nu_0 \rangle \uparrow \langle \nu_1 \rangle \downarrow}$, and a direct extension $q(\nu_0, \nu_1) \leq^* p_{n^p-1\langle \nu_0, \nu_1 \rangle \uparrow}$ such that $s \frown t(\nu_0, \nu_1) \frown t_0(\nu_0, \nu_1) \frown q(\nu_0, \nu_1) \in D$. For each $\nu_0 \in \operatorname{Lev}_0(X)$ we can remove a measure zero set from $\operatorname{Suc}_X(\nu_0)$ so that we can assume there is a direct extension $t(\nu_0) \leq^* p^*_{n^p-1\langle \nu_0 \rangle \downarrow}$ such that $t(\nu_0) = t(\nu_0, \nu_1)$ for each $\nu_1 \in \operatorname{Suc}_X(\nu_0)$. By the previous lemma there is a $p_{n^p-1\langle \nu_0 \rangle \uparrow}$-tree $S(\nu_0)$ of height one, and a $\langle p_{n^p-1\langle \nu_0 \rangle \uparrow}, S(\nu_0) \rangle$-function r_{ν_0} satisfying for each $\nu_1 \in S(\nu_0)$, $s \frown t(\nu_0) \frown r_{\nu_0}(\nu_1) \in D$.

Set $g = f^* \cup f^{j_{E_\xi}(t)(\operatorname{mc}_\xi(f^*))}$, where $\operatorname{Lev}_0(X) \in E_\xi(f^*)$. Set $X^* = \{\langle \nu_0, \nu_1 \rangle \mid \nu_0 \in \pi^{-1}_{g,f^*} \operatorname{Lev}_0(X),\ \nu_1 \in \pi^{-1}_{g,f^*} S(\nu_0 \restriction \operatorname{dom} f^*)\}$. By removing a measure zero set from $\operatorname{Lev}_0(X^*)$ we can assume for each $\nu_0 \in X^*$, $g_{\langle \nu_0 \rangle \downarrow} = f^{t(\nu_0 \restriction \operatorname{dom} f^*)}$. Choose a set $T \in \vec{E}(g)$ such that $\langle g, T \rangle \leq^* p^*_{n^p-1}$. For each $\nu_0 \in \operatorname{Lev}_0(X^*)$ let r'_{ν_0} be the function with domain $\operatorname{Suc}_{X^*}(\nu_0)$ defined by shrinking the trees in r_{ν_0} so that both $r'_{\nu_0}(\nu_1) \leq^{**} r_{\nu_0 \restriction \operatorname{dom} f^*}(\nu_1 \restriction \operatorname{dom} f^*)$ and $r'_{\nu_0}(\nu_1) \leq^{**} \langle g, T \rangle_{\langle \nu_0 \rangle \uparrow \langle \nu_1 \rangle}$ will hold for each $\nu_1 \in \operatorname{Suc}_{X^*}(\nu_0)$. Define the function r with domain X^* by setting for each $\langle \nu_0, \nu_1 \rangle \in X^*$, $r(\nu_0) = \langle g_{\langle \nu_0 \rangle \downarrow}, T_{\langle \nu_0 \rangle \downarrow} \cap \pi^{-1}_{g_{\langle \nu_0 \rangle \downarrow}, f^{t(\nu_0 \restriction \operatorname{dom} f^*)}} T^{(t(\nu_0 \restriction \operatorname{dom} f^*))} \rangle$ and $r(\nu_0, \nu_1) = r'_{\nu_0}(\nu_1)$.

Note $\vec{r}(\nu_0, \nu_1) \leq^{**} \langle g, T \rangle_{\langle \nu_0, \nu_1 \rangle}$, thus r is a $\langle g, X^* \rangle$-function. Since D is open we get $s \frown \vec{r}(\nu_0, \nu_1) \in D$ for each $\langle \nu_0, \nu_1 \rangle \in X^*$. Thus $g \in D^{\in}$. Since $g \leq f^* \in D^*$ we get $f^* \in D^{\in}$. □

As discussed earlier, the following lemma is the intended one, with the previous ones serving as an introduction to the technique used in the proof.

Lemma 4.7. *Assume $\langle N, f^* \rangle$ is a good pair, $k < \omega$, $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition such that $f^{p_{n^p-1}} = f^*$. If there is $s \leq p \restriction n^p - 1$ such that $\{\vec{\nu} \in {}^k T^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle \vec{\nu} \rangle}\ s \frown q \in D\}$ is an $\vec{E}(f^*)$-tree, then there is a p_{n^p-1}-tree S of height k, and a $\langle p_{n^p-1}, S \rangle$-function r such that for each $\vec{\nu} \in \operatorname{Lev}_{\max} S$, $s \frown \vec{r}(\vec{\nu}) \in D$.*

Proof. Assume $X = \{\langle \mu \rangle \frown \vec{\nu} \in {}^k T p_{n^p-1} \mid \exists q \leq^* p_{n^p-1\langle \langle \mu \rangle \frown \vec{\nu} \rangle}\ s \frown q \in D\}$ is an $\vec{E}(f^*)$-tree. For each $\langle \mu \rangle \frown \vec{\nu} \in X$ fix a direct extension $t(\mu \frown \langle \vec{\nu} \rangle) \leq^* p_{n^p-1\langle \mu \rangle \downarrow}$ and a direct extension $q(\mu \frown \vec{\nu}) \leq^* p_{n^p-1\langle \mu \rangle \uparrow \langle \vec{\nu} \rangle}$ such that $s \frown t(\langle \mu \rangle \frown \vec{\nu}) \frown q(\langle \mu \rangle \frown \vec{\nu}) \in D$. For each $\mu \in \operatorname{Lev}_0(X)$ we can remove a measure zero set from $X_{\langle \mu \rangle}$ so that we will have a direct extension $t(\mu) \leq^* p_{n^p-1\langle \mu \rangle \downarrow}$ such that $t(\mu) = t(\langle \mu \rangle \frown \vec{\nu})$ for each $\vec{\nu} \in X_{\langle \mu \rangle}$. By recursion there is a $p_{n^p-1\langle \mu \rangle \uparrow}$-tree $S(\mu)$ of height $k-1$, and a $\langle p_{n^p-1\langle \mu \rangle \uparrow}, S(\mu) \rangle$-function r_μ satisfying for each $\vec{\nu} \in S(\mu)$, $s \frown t(\mu) \frown r_\mu(\vec{\nu}) \in D$.

Set $g = f^* \cup f^{j_{E_\xi}(t)(\mathrm{mc}_\xi(f^*))}$, where $\mathrm{Lev}_0(X) \in E_\xi(f^*)$. Set $X^* = \{\langle\mu\rangle^\frown \vec{\nu} \mid \mu \in \pi^{-1}_{g,f^*} \mathrm{Lev}_0(X),\ \vec{\nu} \in \pi^{-1}_{g,f^*}(S(\mu \restriction \mathrm{dom}\, f^*))\}$. By removing a measure zero set from $\mathrm{Lev}_0(X^*)$ we can assume for each $\mu \in X^*$, $g_{\langle\mu\rangle\downarrow} = f^{t(\mu\restriction \mathrm{dom}\, f^*)}$. Choose a set $T \in \vec{E}(g)$ such that $\langle g, T\rangle \leq^* p_{n^p-1}$. For each $\mu \in \mathrm{Lev}_0(X^*)$ let r'_{ν_0} be the function with domain $X^*_{\langle\mu\rangle}$ defined by shrinking the trees in r_μ so that both $\vec{r}'_\mu(\vec{\nu}) \leq^{**} \vec{r}_{\mu\restriction \mathrm{dom}\, f^*}(\vec{\nu} \restriction X_{\langle\mu\rangle\restriction \mathrm{dom}\, f^*})$ and $r'_\mu(\vec{\nu}) \leq^{**} \langle g, T\rangle_{\langle\mu\rangle\uparrow\langle\vec{\nu}\rangle}$ will hold for each $\vec{\nu} \in X^*_{\langle\mu\rangle}$. Define the function r with domain X^* by setting for each $\langle\mu\rangle^\frown \vec{\nu} \in X^*$, $r(\mu) = \langle g_{\langle\mu\rangle\downarrow}, T_{\langle\mu\rangle\downarrow} \cap \pi^{-1}_{g_{\langle\mu\rangle\downarrow}, f^{t(\mu\restriction \mathrm{dom}\, f^*)}} T^{(t(\mu\restriction \mathrm{dom}\, f^*))}\rangle$ and $r(\langle\mu\rangle^\frown \vec{\nu}) = r'_\mu(\vec{\nu})$.

Note $\vec{r}(\langle\mu\rangle^\frown \vec{\nu}) \leq^{**} \langle g, T\rangle_{\langle\langle\mu\rangle^\frown\vec{\nu}\rangle}$, thus r is a $\langle g, X^*\rangle$-function. Since D is open we get $s^\frown \vec{r}(\langle\mu\rangle^\frown \vec{\nu}) \in D$ for each $\langle\langle\mu\rangle^\frown \vec{\nu}\rangle \in X^*$. Thus $g \in D^\in$. Since $g \leq f^* \in D^*$ we get $f^* \in D^\in$. □

Lemma 4.8. *Assume $\langle f, T\rangle \in \mathbb{P}$ is a condition, $k < \omega$, and $S \subseteq {}^kT$ is not an $\vec{E}(f)$-tree. Then there is a set $T^* \in \vec{E}(f)$ such that $\langle f, T^*\rangle \leq^* \langle f, T\rangle$ and ${}^kT^* \cap S = \emptyset$.*

Proof. By removing measure zero sets from the levels of S we can find $n < k$ so that the following will hold:

(1) For each $l < n$ and $\langle\nu_0, \dots, \nu_{l-1}\rangle \in S$, $\mathrm{Suc}_S(\nu_0, \dots, \nu_l) \in E_\xi(f)$ for some $\xi < \mathrm{o}(\vec{E})$.

(2) For each $\langle\nu_0, \dots, \nu_{n-1}\rangle \in S$, $\mathrm{Suc}_S(\nu_0, \dots, \nu_{n-1}) \notin E_\xi(f)$ for each $\xi < \mathrm{o}(\vec{E})$.

Shrink T so that $\{\langle f, T\rangle_{\langle\nu_0,\dots,\nu_{n-1}\rangle} \mid \langle\nu_0, \dots, \nu_{n-1}\rangle \in \mathrm{Lev}_n(S)\}$ is predense below $\langle f, T\rangle$. We are done by setting $A = \triangle\{T \setminus \mathrm{Suc}_{S_n}(\nu_0, \dots, \nu_{n-1}) \mid \langle\nu_0, \dots, \nu_{n-1}\rangle \in S_n\}$ and $T^* = T \cap A$. □

Corollary 4.9. *Assume $\langle N, f^*\rangle$ is a good pair, $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition such that $f^{p_{n^p-1}} = f^*$. Assume $s \leq p \restriction n^p - 1$. Then one and only one of the following holds:*

(1) There is a p_{n^p-1}-tree S, and a $\langle p_{n^p-1}, S\rangle$-function r, such that for each $\vec{\nu} \in \mathrm{Lev}_{\max} S$, $s^\frown \vec{r}(\vec{\nu}) \in D$.

(2) There is a set $T^* \in \vec{E}(f^*)$ such that $\langle f^*, T^*\rangle \leq^{**} p_{n^p-1}$ and for each $\vec{\nu} \in {}^{<\omega}T^*$ and $q \leq^* \langle f^*, T^*\rangle_{\langle\vec{\nu}\rangle}$, $s^\frown q \notin D$.

It is about time we get rid of the conditional appearing in the former statements show we have densely many times p_{n^p-1}-trees and functions.

Claim 4.10. *Assume $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition. Then there is an extension $s \leq p \restriction n^p - 1$ and $k < \omega$ so that $\{\vec{\nu} \in {}^kT^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle\vec{\nu}\rangle}\ s^\frown q \in D\}$ is an $\vec{E}(f^p)$-tree.*

Proof. Towards contradiction assume the claim fails. Then for each $s \leq p \restriction n^p - 1$ and $k < \omega$ the set $S(s,k) = \{\vec{\nu} \in {}^kT^{p_{n^p-1}} \mid \exists q \leq^* p_{n^p-1\langle\vec{\nu}\rangle}\ s \frown q \in D\}$ is not an $\vec{E}(f^p)$-tree. Thus there is a set $T(s,\kappa) \in \vec{E}(f^{p_{n^p-1}})$ satisfying ${}^kT(s,k) \cap S(s,k) = \emptyset$. Set $T^* = \bigcap\{T(s,k) \mid k < \omega,\ s \leq p \restriction n^p - 1\}$. Consider the condition $p^* = p \restriction n^p - 1 \frown \langle f^{p_{n^p-1}}, T^*\rangle$. By the density of the set D there is an extension $s \leq p \restriction n^p - 1$, $\vec{\nu} \in {}^kT^*$, and $q \leq^* p^*_{n^p-1\langle\vec{\nu}\rangle}$, such that $s \frown q \in D$. Hence $\vec{\nu} \in S(s,k)$. However ${}^kT^* \cap S(n,k) = \emptyset$, contradiction. ☐

Corollary 4.11. *Assume $\langle N, f^*\rangle$ is a good pair, $D \in N$ is a dense open set, and $p \in \mathbb{P}$ is a condition such that $f^{p_{n^p-1}} = f^*$. Then there is a maximal antichain A below $p \restriction n^p - 1$ such that for each $s \in A$ there is a p_{n^p-1}-tree S and a $\langle p_{n^p-1}, S\rangle$-function r, such that for each $\vec{\nu} \in \operatorname{Lev}_{\max} S$, $s \frown \vec{r}(\vec{\nu}) \in D$.*

Corollary 4.12. *Assume $D \in N$ is a dense open set and $p \in \mathbb{P}$ is a condition. Then there is a direct extension $p^* \leq^* p$, a p^*-tree S, and a $\langle p^*, S\rangle$-function r such that for each $\vec{\nu} \in \operatorname{Lev}_{\max}(S)$, $\vec{r}(\vec{\nu}) \in D$.*

5. κ Properties in the Genric Extension

Claim 5.1. *The forcing notion $\mathbb{P}$ is of Prikry type.*

Proof. Assume $p \in \mathbb{P}$ is a condition and σ is a formula in the $\mathbb{P}$-forcing language. We will be done by exhibiting a direct extension $p^* \leq^* p$ such that $p^* \parallel \sigma$. Set $D = \{q \leq p \mid q \parallel \sigma\}$. The set D is dense open, hence by corollary 4.12 there is a direct extension $p^* \leq^* p$, a p^*-tree S, and a $\langle p^*, S\rangle$-function r, such that for each $\vec{\nu} \in \operatorname{Lev}_{\max} S$, $\vec{r}(\vec{\nu}) \in D$. Set $X_0 = \{\vec{\nu} \in \operatorname{Lev}_{\max} S \mid \vec{r}(\vec{\nu}) \Vdash \neg\sigma\}$ and $X_1 = \{\vec{\nu} \in \operatorname{Lev}_{\max} S \mid \vec{r}(\vec{\nu}) \Vdash \sigma\}$. Since the sets X_0 and X_1 are a disjoint partition of $\operatorname{Lev}_{\max} S$, only one of them is a measure one set. Fix $i < 2$ such that X_i is a measure one set. Set $S_i = \{\langle \nu_0, \dots, \nu_k\rangle \mid \langle \nu_0, \dots \nu_n\rangle \in X_i,\ k \leq n\}$. Using claim 4.3 shrink the trees appearing in the condition p^* so that $\{\vec{r}(\vec{\nu}) \mid \vec{\nu} \in \operatorname{Lev}_{\max} S_i\}$ is predense below p^*. Thus $p^* \Vdash \sigma_i$, where $\sigma_0 =$ "$\neg\sigma$" and $\sigma_1 =$ "σ". ☐

Lemma 5.2. $\Vdash$ *"κ is a cardinal".*

Proof. If $\operatorname{o}(\vec{E}) = 1$ then there are no new bounded subset of κ in $V^{\mathbb{P}}$, hence no cardinal below κ is collapsed, hence κ is preserved. If $\operatorname{o}(\vec{E}) > 1$ then an unbounded number of cardinals below κ is preserved, hence κ is preserved. ☐

Claim 5.3. *If $\operatorname{o}(\vec{E}) < \kappa$ is regular then $\Vdash$ "$\operatorname{cf} \kappa = \operatorname{cf} \operatorname{o}(\vec{E})$".*

Proof. It is immediate $\Vdash$ "$\operatorname{cf}\kappa \leq \operatorname{cf}\operatorname{o}(\vec{E})$". Hence we need to show $\Vdash$ "$\operatorname{cf}\kappa \not< \operatorname{cf}\operatorname{o}(\vec{E})$". Assume $\sigma < \kappa$ and $p \Vdash$ "$\sigma < \operatorname{cf}\operatorname{o}(\vec{E})$ and $\underset{\sim}{f} : \sigma \to \kappa$". We will be done by exhibiting a direct extension $p^* \leq^* p$ such that $p^* \Vdash$ "$\underset{\sim}{f}$ is bounded". Let $\langle N, f^* \rangle$ be a good pair such that $p, \underset{\sim}{f}, \sigma \in N$ and $f^* \leq^* f^{p_{n^p-1}}$. Shrink $T^{p_{n^p-1}}$ so as to satisfy for each $\nu \in T^{p_{n^p-1}}$, $\mathring{\nu}(\kappa) > \sigma$.

Factor $\mathbb{P}$ as follows. Set $P_0 = \{s \leq p \restriction n^p - 1 \mid \exists q \leq p_{n^p-1}\ s ^\frown q \in \mathbb{P}\}$ and $P_1 = \{q \leq p_{n^p-1} \mid \exists s \leq p \restriction n^p - 1\ s ^\frown q \in \mathbb{P}\}$. For each $\xi < \sigma$ work as follows. Set $D_\xi = \{q \leq p_{n^p-1} \mid$ There exists a P_0-name $\underset{\sim}{\rho}$ such that $q \Vdash_{P_1}$ "$\underset{\sim}{f}(\xi) = \underset{\sim}{\rho}$"$\}$. Since $D_\xi \in N$ is a dense open subset of $\mathbb{P}$ below p_{n^p-1} there is a a direct extension $p^\xi = \langle f^*, T^\xi \rangle \leq^* p_{n^p-1}$, a p^ξ-tree S^ξ, and a $\langle p^\xi, S^\xi \rangle$-function r_ξ satisfying for each $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$, $\vec{r}_\xi(\vec{\nu}) \in D_\xi$. Thus for each $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$ there is a P_0-name $\underset{\sim}{\rho}^{\xi,\vec{\nu}}$ so that $\vec{r}_\xi(\vec{\nu}) \Vdash_{P_1}$ "$\underset{\sim}{f}(\xi) = \underset{\sim}{\rho}^{\xi,\vec{\nu}}$". Since $|P_0| < \kappa$ there is $\zeta^{\xi,\vec{\nu}} < \kappa$ such that $p \restriction n^p - 1 \Vdash_{P_0}$ "$\underset{\sim}{\rho}^{\xi,\vec{\nu}} < \zeta^{\xi,\vec{\nu}}$".

Let m_ξ be a function witnessing S^ξ is a p_{n^p-1}-tree, i.e., $m_\xi : \{\emptyset\} \cup \operatorname{Lev}_{<\max} S \to \operatorname{o}(\vec{E})$ is a function satisfying for each $\vec{\nu} \in \operatorname{dom} m_\xi$, $\operatorname{Suc}_S(\vec{\nu}) \in E_{m_\xi(\vec{\nu})}(f^{p_{n^p-1}})$. (We use the convention $\operatorname{Suc}_S(\langle\rangle) = \operatorname{Lev}_0(S)$.) Since $\operatorname{o}(\vec{E}) < \kappa$ we can remove a measure zero set from S^ξ (and $\operatorname{dom} m_\xi$) and get for each $\vec{\nu}_0, \vec{\nu}_1 \in \operatorname{dom} m^\xi$, if $|\vec{\nu}_0| = |\vec{\nu}_1|$ then $m_\xi(\vec{\nu}_0) = m_\xi(\vec{\nu}_1)$. Thus $|\operatorname{ran} m_\xi| < \omega$. Set $\tau_\xi = \sup \operatorname{ran} m_\xi$. Shrink T^ξ so that $\{\vec{r}_\xi(\vec{\nu}) \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi\}$ is predense below p^ξ. Note, if $\mu \in \operatorname{Lev}_0 T^\xi$, $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$, $\operatorname{o}(\mu) > \tau_\xi$ and $\vec{\nu} \not< \mu$, then $\vec{r}(\vec{\nu}) \perp p^\xi_{\langle\mu\rangle}$. Hence $p \restriction n^p - 1 ^\frown p^\xi_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}(\xi) < \sup\{\zeta^{\xi,\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\}$".

Set $T^* = \bigcap_{\xi<\sigma} T^\xi$ and $p^* = p \restriction n^p - 1 ^\frown \langle f^*, T^* \rangle$. We claim $p^* \Vdash$ "$\underset{\sim}{f}$ is bounded". To show this set $\tau = \sup\{\tau_\xi \mid \xi < \sigma\}$. Note $\tau < \operatorname{o}(\vec{E})$. Since $\{p^*_{\langle\mu\rangle} \mid \mu \in T^*,\ \operatorname{o}(\mu) = \tau\}$ is predense below p^* it is enough to show that $p^*_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}$ is bounded" for each $\mu \in T^*$ such that $\operatorname{o}(\mu) = \tau$. So fix $\mu \in T^*$ such that $\operatorname{o}(\mu) = \tau$. Set $\zeta = \sup\{\zeta^{\xi,\vec{\nu}} \mid \xi < \sigma, \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\}$. Note $\zeta < \kappa$. We get for each $\xi < \sigma$, $p^*_{\langle\mu\rangle} \leq^* p \restriction n^p - 1 ^\frown p^\xi_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}(\xi) < \sup\{\zeta^{\xi,\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\} < \zeta < \kappa$". $\square$

Claim 5.4 (Gitik)**.** *If* $\operatorname{o}(\vec{E}) \in [\kappa, \lambda)$ *and* $\operatorname{cf}(\operatorname{o}(\vec{E})) \geq \kappa$ *then* $\Vdash$ "$\operatorname{cf}\kappa = \omega$".

Proof. Fix a condition $p \in \mathbb{P}$ such that $\operatorname{o}(\vec{E}) + 1 \subseteq\in \operatorname{dom} f^{p_{n^p-1}}$. Partition $T^{p_{n^p-1}}$ into $\operatorname{o}(\vec{E})$ disjoint subsets $\{A_\xi \mid \xi < \operatorname{o}(\vec{E})\}$ by setting for each $\xi < \mathrm{o}(\vec{E})$,

$$A_\xi = \{\nu \in T^{p_{n^p-1}} \mid \xi \in \operatorname{dom}\nu,\ \mathrm{o}(\nu(\kappa)) = \operatorname{otp}((\operatorname{dom}\nu) \cap \xi)\}.$$

Let G be generic. Choose a condition $p \in G$. Let $\langle \nu_\xi \mid \xi < \kappa\rangle$ be the increasing enumeration of the set $\{\nu_0, \ldots, \nu_k \mid p \restriction n^p - 1 {}^\frown p_{n^p-1\langle \nu_0,\ldots,\nu_\kappa\rangle} \in G\}$. Set $\zeta_0 = 0$. For each $n < \omega$ set $\zeta_{n+1} = \min\{\xi > \zeta_n \mid \nu_\xi \in A_{sup((\operatorname{dom}\nu_{\zeta_n})\cap \mathrm{o}(\vec{E}))}\}$. We are done since $\kappa = \sup_{n<\omega} \zeta_n$. □

Using the same method as above we get the following claim.

Claim 5.5. *If* $\mathrm{o}(\vec{E}) \in [\kappa, \lambda)$ *and* $\mathrm{cf}(\mathrm{o}(\vec{E})) < \kappa$ *then* $\Vdash$ "$\mathrm{cf}\,\kappa = \mathrm{cf}(\mathrm{o}(\vec{E}))$"*.*

Claim 5.6. *If* $\mathrm{o}(\vec{E}) = \lambda$ *then* $\Vdash$ *"κ is regular".*

Proof. Assume $\sigma < \kappa$ and $p \Vdash$ "$\underset{\sim}{f} : \sigma \to \kappa$". We will be done by exhibiting a direct extension $p^* \leq^* p$ such that $p^* \Vdash$ "$\underset{\sim}{f}$ is bounded". Let $\langle N, f^*\rangle$ be a good pair such that $p, \underset{\sim}{f}, \sigma \in N$ and $f^* \leq^* f^{p_{n^p-1}}$. Shrink $T^{p_{n^p-1}}$ so as to satisfy for each $\nu \in \operatorname{Lev}_0(T^{p_{n^p-1}})$, $\mathring{\nu}(\kappa) > \sigma$.

Factor $\mathbb{P}(\vec{E})$ as follows. Set $P_0 = \{s \leq p \restriction n^p - 1 \mid \exists q \leq p_{n^p-1}\ s {}^\frown q \in \mathbb{P}(\vec{E})\}$ and $P_1 = \{q \leq p_{n^p-1} \mid \exists s \leq p\ \restriction n^p - 1 s {}^\frown q \in \mathbb{P}(\vec{E})\}$. For each $\xi < \sigma$ work as follows. Set $D_\xi = \{q \leq p_{n^p-1} \mid$ There exists a P_0-name $\underset{\sim}{\rho}$ such that $q \Vdash_{P_1}$ "$\underset{\sim}{f}(\xi) = \underset{\sim}{\rho}$"$\}$. Since $D_\xi \in N$ is a dense open subset of $\mathbb{P}$ below p_{n^p-1} there is a a direct extension $p^\xi = \langle f^*, T^\xi\rangle \leq^* p_{n^p-1}$, a p^ξ-tree S^ξ, and a $\langle p^\xi, S^\xi\rangle$-function r_ξ satisfying for each $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$, $\vec{r}_\xi(\vec{\nu}) \in D_\xi$. Thus for each $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$ there is a P_0-name $\underset{\sim}{\rho}^{\xi,\vec{\nu}}$ so that $\vec{r}_\xi(\vec{\nu}) \Vdash$ "$\underset{\sim}{f}(\xi) = \underset{\sim}{\rho}^{\xi,\vec{\nu}}$". Since $|P_0| < \kappa$ there is $\zeta^{\xi,\vec{\nu}} < \kappa$ such that $p \restriction n^p - 1 \Vdash_{P_0}$ "$\underset{\sim}{\rho}^{\xi,\vec{\nu}} < \zeta^{\xi,\vec{\nu}}$".

Let m_ξ be a function witnessing S^ξ is a p_{n^p-1}-tree, i.e., $m_\xi : \{\emptyset\} \cup \operatorname{Lev}_{<\max} S \to \mathrm{o}(\vec{E})$ is a function satisfying for each $\vec{\nu} \in \operatorname{dom} m_\xi$, $\operatorname{Suc}_S(\vec{\nu}) \in E_{m_\xi(\vec{\nu})}(f^{p_{n^p-1}})$. (We use the convention $\operatorname{Suc}_S(\langle\rangle) = \operatorname{Lev}_0(S)$.) Since $\lambda = \mathrm{o}(\vec{E})$ is regular and $|S^\xi| < \lambda$ we get $\tau_\xi = \sup\operatorname{ran} m_\xi < \lambda$. Shrink T^ξ so that $\{\vec{r}_\xi(\vec{\nu}) \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi\}$ is predense below p^ξ. Note, if $\mu \in T^\xi$, $\vec{\nu} \in \operatorname{Lev}_{\max} S^\xi$, $\mathrm{o}(\mu) > \mathring{\mu}(\tau_\xi)$ and $\vec{\nu} \not< \mu$, then $\vec{r}(\vec{\nu}) \perp p^\xi_{\langle\mu\rangle}$. Hence $p \restriction n^p - 1 {}^\frown p^\xi_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}(\xi) < \sup\{\zeta^{\xi,\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\}$".

Set $T^* = \bigcap_{\xi<\sigma} T^\xi$ and $p^* = p \restriction n^p - 1 {}^\frown \langle f^*, T^*\rangle$. We claim $p^* \Vdash$ "$\underset{\sim}{f}$ is bounded". To show this set $\tau = \sup\{\tau_\xi \mid \xi < \sigma\}$. Note $\tau < \mathrm{o}(\vec{E}) = \lambda$. Since $\{p^*_{\langle\mu\rangle} \mid \mu \in T^*,\ \mathrm{o}(\mu) = \mathring{\mu}(\tau)\}$ is predense below p^* it is enough to show that $p^*_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}$ is bounded" for each $\mu \in \operatorname{Lev}_0 T^*$ such that $\mathrm{o}(\mu) = \mathring{\mu}(\tau)$. So fix $\mu \in \operatorname{Lev}_0 T^*$ such that $\mathrm{o}(\mu) = \mathring{\mu}(\tau)$. Set $\zeta = \sup\{\zeta^{\xi,\vec{\nu}} \mid \xi < \sigma, \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\}$. Note $\zeta < \kappa$. We get for each $\xi < \sigma$, $p^*_{\langle\mu\rangle} \leq^* p \restriction n^p - 1 {}^\frown p^\xi_{\langle\mu\rangle} \Vdash$ "$\underset{\sim}{f}(\xi) < \sup\{\zeta^{\xi,\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{\max} S^\xi, \vec{\nu} < \mu\} < \zeta < \kappa$". ☐

Definition 5.7. An ordinal $\rho < \mathrm{o}(\vec{E})$ is a repeat point of $\vec{E}$ if for each $d \in [\epsilon]^{<\lambda}$, $\bigcap_{\xi<\rho} E_\xi(d) = \bigcap_{\xi<\mathrm{o}(\vec{E})} E_\xi(d)$.

Lemma 5.8. *Assume $\rho < \mathrm{o}(\vec{E})$ is a repeat point of $\vec{E}$.*

1. *If $p, q \in \mathbb{P}$ are compatible then $j_{E_\rho}(p)_{\langle \mathrm{mc}_\rho(p)\rangle}$ and $j_{E_\rho}(q)_{\langle \mathrm{mc}_\rho(q)\rangle}$ are compatible.*
2. *Assume $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{A} \subseteq \kappa$". Then for each $p \in \mathbb{P}$ there is a direct extension $p^* \leq^* p$ such that $j_{E_\rho}(p^*)_{\langle \mathrm{mc}_\rho(p^*)\rangle} \parallel$ "$\kappa \in j_{E_\rho}(\underset{\sim}{A})$".*

Proof. (1) Let $r \leq p, q$. By definition of the order there are extensions $p' \leq p$ and $q' \leq q$ such that $r \leq^* p', q'$. By elementarity $j_{E_\rho}(p')_{\langle \mathrm{mc}_\rho(p')\rangle} \leq j_{E_\rho}(p)_{\langle \mathrm{mc}_\rho(p)\rangle}$ and $j_{E_\rho}(q')_{\langle \mathrm{mc}_\rho(q')\rangle} \leq j_{E_\rho}(q)_{\langle \mathrm{mc}_\rho(q)\rangle}$. Thus we will be done by showing $j_{E_\rho}(p')_{\langle \mathrm{mc}_\rho(p')\rangle}$ and $j_{E_\rho}(q')_{\langle \mathrm{mc}_\rho(q')\rangle}$ are compatible.

So, without loss of generality assume p and q are $\leq^*$ compatible. By elementarity $j_{E_\rho}(p)$ and $j_{E_\rho}(q)$ are compatible. Note

$$p_{n^p-1} = j_{E_\rho}(p_{n^p-1})_{\langle \mathrm{mc}_\rho(p_{n^p-1})\rangle\downarrow}$$

and

$$q_{n^q-1} = j_{E_\rho}(q_{n^q-1})_{\langle \mathrm{mc}_\rho(q_{n^q-1})\rangle\downarrow}.$$

Then

$$j_{E_\rho}(p)_{\langle \mathrm{mc}_\rho(p_{n^p-1})\rangle} = p {}^\frown \langle j_{E_\rho}(p_{n^p-1})_{\langle \mathrm{mc}_\rho(p_{n^p-1})\rangle\uparrow}\rangle$$

and

$$j_{E_\rho}(q)_{\langle \mathrm{mc}_\rho(q_{n^q-1})\rangle} = q {}^\frown \langle j_{E_\rho}(q_{n^q-1})_{\langle \mathrm{mc}_\rho(q_{n^q-1})\rangle\uparrow}\rangle.$$

We are done.

(2) Let $\langle N, f^*\rangle$ be a good pair such that $f^* \leq^* f^{p_{n^p-1}}$ and $p, \underset{\sim}{A} \in N$. Set $T = \pi^{-1}_{f^*, f^{p_{n^p-1}}} T^p$. Fix $\nu \in T$ and consider the condition $p \restriction n^p - 1 {}^\frown \langle f^*, T\rangle_{\langle\nu\rangle}$. By the Prikry property there are $s \leq^* p \restriction n^p - 1$, $r \leq^* \langle f^*, T\rangle_{\langle\nu\rangle\downarrow}$, and $q \leq^* \langle f^*, T\rangle_{\langle\nu\rangle\uparrow}$ such that $s {}^\frown r {}^\frown q \parallel$ "$\nu(\kappa) \in \underset{\sim}{A}$". Since the set $\{t \leq p \mid t \parallel$ "$\nu(\kappa) \in \underset{\sim}{A}$"$\}$ is dense open below p and belongs to N, we get by lemma 3.13 that there is a set $T_1(\nu) \in \vec{E}(f^*)$ such that $s {}^\frown r {}^\frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle \parallel$ "$\nu(\kappa) \in \underset{\sim}{A}$".

Thus for each $\nu \in T$ there is $s(\nu) \leq^* p \restriction n^p - 1$, $r(\nu) \leq^* \langle f^*, T\rangle_{\langle\nu\rangle\downarrow}$, and $T_1(\nu) \in \vec{E}(f^*)$ such that $s(\nu) {}^\frown r(\nu) {}^\frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle \parallel$ "$\nu(\kappa) \in \underset{\sim}{A}$". We can find a set $T_{=\rho} \in E_\rho(f^*)$ and $s \leq^* p \restriction n^p - 1$

such that for each $\nu \in T_{=\rho}$, $s(\nu) = s$. Thus for each $\nu \in T_{=\rho}$, $s \frown r(\nu) \frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle \parallel$ "$\nu(\kappa) \in \underset{\sim}{A}$". Then by removing a measure zero set from $T_{=\rho}$ we can have either

$$\forall \nu \in T_{=\rho}\ s \frown r(\nu) \frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle \Vdash \text{“}\nu(\kappa) \in \underset{\sim}{A}\text{”}$$

or

$$\forall \nu \in T_{=\rho}\ s \frown r(\nu) \frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle \Vdash \text{“}\nu(\kappa) \notin \underset{\sim}{A}\text{”}.$$

Let $g = f^* \cup f^{j(r)(\mathrm{mc}_\rho(f^*))}$. Set $T^* = \pi^{-1}_{g,f^*} T^{j_{E_\rho(r)}(\mathrm{mc}_\rho(f^*))} \cap \pi^{-1}_{g,f^*} \triangle_{\nu \in T_{=\rho}} T_1(\nu)$. Setting $p^* = s \frown \langle g, T^*\rangle$ we get for each $\nu \in T^*_{=\rho}$, $p^*_{\langle\nu\rangle} \leq^* s \frown r(\nu) \frown \langle f^*_{\langle\nu\rangle\uparrow}, T_1(\nu)\rangle$. Thus by removing a measure zero set from $T^*_{=\rho}$ we get either

$$\forall \nu \in T^*_{=\rho}\ p^*_{\langle\nu\rangle} \Vdash \text{“}\nu(\kappa) \in \underset{\sim}{A}\text{”}$$

or

$$\forall \nu \in T^*_{=\rho}\ p^*_{\langle\nu\rangle} \Vdash \text{“}\nu(\kappa) \notin \underset{\sim}{A}\text{”}.$$

Going to the ultrapower we get $j_{E_\rho}(p^*)_{\langle \mathrm{mc}_\rho(g)\rangle} \parallel$ "$\kappa \in j_{E_\rho}(\underset{\sim}{A})$". □

Corollary 5.9. *Assume $\rho < \mathrm{o}(\vec{E})$ is a repeat point of $\vec{E}$. Then $\Vdash$ "κ is measurable".*

Proof. If $G \subseteq \mathbb{P}$ is generic then it is a simple matter to check that

$$U = \{\underset{\sim}{A}[G] \mid p \in G,\ j_{E_\rho}(p)_{\langle \mathrm{mc}_\rho(p_{n^p-1})\rangle} \Vdash \text{“}\kappa \in (p_{n^p-1}) \in j_{E_\rho}(\underset{\sim}{A})\text{”}\}$$

is the witnessing ultrafilter. □

Claim 5.10. *If $\mathrm{o}(\vec{E}) = \lambda^{++}$ then $\Vdash$ "κ is measurable".*

Proof. By the previous corollary it is enough to exhibit $\rho < \lambda^{++}$ which is a repeat point of $\vec{E}$. Fix $d \in [\epsilon]^{<\lambda}$ and consider the sequence $\langle \bigcap_{\xi' < \xi} E_{\xi'}(d) \mid \xi < \lambda^{++}\rangle$. This is a $\subseteq$-decreasing sequence of filters on $\mathrm{OB}(d)$. Since there are λ^+ filters on $\mathrm{OB}(d)$ there is $\rho_d < \lambda^{++}$ such that $\bigcap_{\xi < \rho_d} E_\xi(d) = \bigcap_{\xi < \lambda^{++}} E_\xi(d)$. Set $\rho = \sup\{\rho_d \mid d \in [\epsilon]^{<\lambda}\}$. Then ρ is a repeat point of $\vec{E}$. □

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