SUPERCOMPACT EXTENDER BASED PRIKRY FORCING

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Abstract. The extender based Prikry forcing notion is being generalized to supercompact extenders.

1. Introduction

In this work a generalization of the extender base Prikry forcing notion to supercompact extenders is presented. The extender based Prikry forcing notion was introduced in [1]. A generalization of it to arbitrary short extenders appeared in [2]. These papers contain historical exposition of the topics at hand. The theorem proved in the current paper is the following:

Theorem. *Assume the GCH, $j : V \to M$ is an elementary embedding such that M is transitive, $M \supseteq {}^{<\mu}M$, $\operatorname{crit}(j) = \kappa$, μ is regular such that $\kappa < \mu \leq j(\kappa)$, and if $\mu = \lambda^+$ then $\operatorname{cf} \lambda \geq \kappa$. Then there is a generic extension $V[G]$ of the universe V such that:*

1. *V and $V[G]$ have the same bounded subsets of κ, thus κ and all the cardinals below it are preserved.*
2. *$\operatorname{cf}^{V[G]} \kappa = \omega$, and all V-cardinals in (κ, μ) are collapsed.*
3. *All the cardinals $\geq \mu$ are preserved.*
4. *$2^\kappa = |j(\mu)|$.*

The structure of this work is as follows. In section 2 we list facts about elementary embeddings and extenders. In section 3 the Prikry forcing with supercompact extender is presented.

The notation used is standard. Knowledge of forcing, extenders, supercompact cardinals, elementary embeddings and their iterations is assumed.

2. Preliminaries

Definition 2.1. Let $j : V \to M$ be an elementary embedding.

1. For each $\alpha < j(\mu)$ let λ_α be the minimal $\lambda \leq \mu$ such that $\alpha < \lambda_\alpha$.
2. For each $\alpha < j(\mu)$ define $E(\alpha)$ by
$$\forall A \subseteq \mu \; \bigl(A \in E(\alpha) \iff \alpha \in j(A)\bigr).$$
It is well know that $E(\alpha)$ is a κ-complete ultrafilter over μ. Note that $E(\alpha)$ concentrates on λ_α. Let $i_\alpha : V \to N_\alpha \simeq \operatorname{Ult}(V, E(\alpha))$ be the natural elementary embedding from V to the ultrapower of V with $E(\alpha)$.

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(3) The extender E derived from j is the system

$$E = \langle\langle E(\alpha) \mid \alpha < j(\mu)\rangle, \langle \pi_{\beta,\alpha} \mid \alpha, \beta \in j(\mu),\ \alpha \in \operatorname{ran} i_\beta\rangle\rangle,$$

where $\pi_{\beta,\alpha} : \mu \to \mu$ is a function such that $j(\pi_{\beta,\alpha})(\beta) = \alpha$. ($\alpha \in \operatorname{ran} i_\beta$ means there are many such functions. Any one of them will do as $\pi_{\beta,\alpha}$.) We assume that it is known how to construct $i : V \to M \simeq \operatorname{Ult}(V, E)$. Moreover we assume $i = j$. E.g., j is the natural embedding $j : V \to \operatorname{Ult}(V, E)$.

We assume the theory of iterated ultrapowers is known and give only the definitions and proposition we need in order to get to the ω-iterate of V. Since j is definable from a set parameter, terms of the forms $j(j)$ have definite meaning.

Definition 2.2. Assume $j : V \to M$ is an elementary embedding. Define by induction for $n < \omega$:

$$j_{0,1} = j,\ M_0 = V,$$

and for each $n < \omega$,

$$j_{n+1,n+2} = j(j_{n,n+1}) : M_{n+1} \to M_{n+2}.$$

We 'complete' the list of j's by setting for each $n < m < \omega$,

$$\begin{aligned} j_{n,n} &= \operatorname{id}, \\ j_{n,m} &= j_{m-1,m} \circ j_{n,m-1}, \\ j_n &= j_{0,n}. \end{aligned}$$

We let M_ω be the direct limit of the system $\langle M_n, j_{n,m} \mid n \le m < \omega\rangle$ with the limit embeddings $j_{n,\omega}$ for each $n < \omega$.

Since the ultrapower is taken with measures constructed from $\alpha \ge j(\kappa)$ we cannot use, e.g., $j_{1,\omega}(\alpha) = \alpha$ for $\alpha < j(\kappa)$. To emphasis this we present the following lemmas (which are not needed in this work).

Lemma 2.3. *Assume $0 < n < \omega$ and $x \in M_n$. Then there are $\alpha_0 < j(\mu)$, …, $\alpha_{n-1} < j_n(\mu)$ and a function $f : {}^n\mu \to V$ such that*

$$x = j_n(f)(j_{1,n}(\alpha_0), j_{2,n}(\alpha_1), \dots, j_{n,n}(\alpha_{n-1})).$$

Proof. It is immediate that for each $x \in M$ there is $\alpha < j(\mu)$ and a function $f : \mu \to V$ in V such that $j(f)(j_{1,1}(\alpha)) = j(f)(\alpha) = x$.

Assume $x \in M_{n+1}$. By elementarity there is $\alpha_n < j_{n+1}(\mu)$ and a function $F : j_n(\mu) \to M_n$ in M_n such that $j_{n,n+1}(F)(\alpha_n) = x$. By recursion there are $\alpha_0 < j(\mu), \dots, \alpha_{n-1} < j_n(\mu)$ and a function $g : {}^n\mu \to V$ in V such that

$$F = j_n(g)(j_{1,n}(\alpha_0), j_{2,n}(\alpha_1), \dots, j_{n,n}(\alpha_{n-1})).$$

Hence we get

$$\begin{aligned} j_{n,n+1}(F)(\alpha_n) =& j_{n,n+1}\big(j_n(g)(j_{1,n}(\alpha_0), j_{2,n}(\alpha_1), \dots, j_{n,n}(\alpha_{n-1}))\big)(\alpha_n) = \\ =& j_{n+1}(g)\big(j_{1,n+1}(\alpha_0), j_{2,n+1}(\alpha_1), \dots, j_{n,n+1}(\alpha_{n-1})\big)(\alpha_n), \end{aligned}$$

which by defining $f(\xi_0, \dots, \xi_n) = g(\xi_0, \dots, \xi_{n-1})(\xi_n)$ yields

$$x = j_{n,n+1}(F)(\alpha_n) = j_{n+1}(f)(j_{1,n+1}(\alpha_0), j_{2,n+1}(\alpha_1), \dots, j_{n,n+1}(\alpha_{n-1}), j_{n,n}(\alpha_n)).$$

□

Corollary 2.4. *Assume $x \in M_\omega$. Then there are $n < \omega$, ordinals $\alpha_0 < j(\mu)$, ..., $\alpha_{n-1} < j_n(\mu)$ and a function $f : {}^n\mu \to V$ such that*

$$x = j_{n,\omega}(f)(j_{1,\omega}(\alpha_0), j_{2,\omega}(\alpha_1), \dots, j_{n,\omega}(j_{n,n}(\alpha_{n-1}))).$$

Proof. Since M_ω is the direct limit of the M_n-s, there is $n < \omega$ and $x_n \in M_n$ such that $j_{n,\omega}(x_n) = x$. By the previous lemma there are $\alpha_0 < j(\mu)$, ..., $\alpha_{n-1} < j_n(\mu)$ and a function $f : {}^n\mu \to V$ such that

$$x_n = j_n(f)(j_{1,n}(\alpha_0), j_{2,n}(\alpha_1), \dots, j_{n,n}(\alpha_{n-1})).$$

Applying $j_{n,\omega}$ to both sides of the last equation we get

$$x = j_\omega(f)(j_{1,\omega}(\alpha_0), j_{2,\omega}(\alpha_1), \dots, j_{n,\omega}(\alpha_{n-1})).$$

□

3. The forcing

Assume the GCH, and let $j : V \to M$ be an elementary embedding such that M is transitive, $M \supseteq {}^{<\mu}M$, $\operatorname{crit}(j) = \kappa$, μ is regular such that $\kappa < \mu \leq j(\mu)$, and if $\mu = \lambda^+$ then $\operatorname{cf}(\lambda) \geq \kappa$. Let E be the extender derived from j.

The measures used by conditions in the forcing we will define are not on μ, but on functions taking values in μ, a collection which we call $\operatorname{OB}(d)$. $\operatorname{OB}(d)$ is defined so as to satisfy $(j \restriction d)^{-1} \in j(\operatorname{OB}(d))$, a fact which is proved immediately after the definition.

Definition 3.1. Assume $d \in [j(\mu)]^{<\mu}$ and $\kappa, |d| \in d$. Then $\nu \in \operatorname{OB}(d) \iff$

(1) $\nu : \operatorname{dom} \nu \to \mu$ is a function such that $\operatorname{dom} \nu \subseteq d$;
(2) $\kappa, |d| \in \operatorname{dom} \nu$;
(3) $|\nu| \leq \nu(|d|)$;
(4) $\forall \alpha < j(\mu)\ (j(\alpha) \in \operatorname{dom} \nu \implies \nu(j(\alpha)) = \alpha)$;
(5) $\forall \alpha \in \operatorname{dom} \nu\ \nu(\alpha) < \lambda_\alpha$;
(6) $\forall \alpha, \beta \in \operatorname{dom} \nu\ \big(\alpha < \beta \implies \nu(\alpha) < \nu(\beta)\big)$.

On $\operatorname{OB}(d)$ the partial order $<$ is defined by: $\nu_0 < \nu_1 \iff$

$$\operatorname{dom} \nu_0 \subseteq \operatorname{dom} \nu_1,$$

and

$$\forall \alpha \in \operatorname{dom} \nu_0 \setminus j''\mu\ \big(\nu_0(\alpha) < \nu_1(\alpha)\big).$$

Claim 3.2. *Assume $d \in [j(\mu)]^{<\mu}$ and $\kappa, |d| \in d$. Then $(j \restriction d)^{-1} \in j(OB(d))$.*

Proof. Set $f = (j \restriction d)^{-1}$. We need to show the clauses of definition 3.1 hold for f in the sense of M.

(1) f is a function with domain $j''d \subseteq j(d)$.
(2) Since $\kappa, |d| \in d$ we get $j(\kappa), j(|d|) \in j''d$, hence $j(\kappa), j(|d|) \in \operatorname{dom} f$.
(3) $|f| = |d| = f(j(|d|))$.
(4) This clause is vacuous.
(5) Obvious.
(6) Obvious.

□

Definition 3.3. A condition f is in the forcing notion $\mathbb{P}^*_E$ if $f : d \to {}^{<\omega}\mu$ is a function such that:

(1) $d \in [j(\mu)]^{<\mu}$;
(2) $\kappa, |d| \in d$;
(3) For each $\alpha \in d \setminus j''\mu$ there is $k < \omega$ so that
$$f(\alpha) = \langle f_0(\alpha), \dots, f_{k-1}(\alpha)\rangle \subset \lambda_\alpha.$$

The forcing notion $\mathbb{P}^*_E$ is equipped with the partial order $f \leq^*_{\mathbb{P}^*_E} g \iff f \supseteq g$. (Thus $\langle \mathbb{P}^*_E, \leq^*\rangle$ is the Cohen forcing adding $|j(\mu)|$ subsets to μ).

Definition 3.4. (1) Assume $T \subseteq \mathrm{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$. Then for each $n < \xi$,
$$\mathrm{Lev}_n(T) = \{\langle \nu_0, \dots, \nu_n\rangle \in \mathrm{OB}(d)^{n+1} \mid \langle \nu_0, \dots, \nu_n\rangle \in T\},$$
and
$$\mathrm{Suc}_T(\nu_0, \dots, \nu_{n-1}) = \{\lambda \in \mathrm{OB}(d) \mid \langle \nu_0, \dots, \nu_{n-1}, \lambda\rangle \in T\}.$$
For notational convenience let $\mathrm{Suc}_T(\langle\rangle) = \mathrm{Lev}_0(T)$. Assume $\langle\nu\rangle \in T$. Define
$$T_{\langle\nu\rangle} = \{\langle \nu_0, \dots, \nu_{k-1}\rangle \mid k < \omega,\ \langle \nu, \nu_0, \dots, \nu_{k-1}\rangle \in T\},$$
and by recursion when $\langle \nu_0, \dots, \nu_n\rangle \in T$ define
$$T_{\langle \nu_0, \dots, \nu_n\rangle} = (T_{\langle \nu_0, \dots, \nu_{n-1}\rangle})_{\langle \nu_n\rangle}.$$

(2) A measure $E(d)$ is defined on $\mathrm{OB}(d)$ as follows:
$$\forall X \subseteq \mathrm{OB}(d)\ \big(X \in E(d) \iff \mathrm{mc}(d) \in j(X)\big),$$
where $\mathrm{mc}(d)$ is defined by
$$\mathrm{mc}(d) = \{\langle j(\alpha), \alpha\rangle \mid \alpha \in d\}.$$
The measure $E^{(n+1)}(d)$ $(n < \omega)$ on $\mathrm{OB}(d)^{n+1}$ is defined by recursion as follows.
$$X \in E^{(n+1)}(d) \iff$$
$$\{\langle \nu_0, \dots, \nu_{n-1}\rangle \in \mathrm{Lev}_{n-1}(X) \mid \mathrm{Suc}_X(\nu_0, \dots, \nu_{n-1}) \in E(d)\} \in E^{(n)}(d),$$
where we set $E^{(0)} = \{\langle\rangle\}$ and consider it a measure on $\mathrm{OB}(d)^0 = \{\langle\rangle\}$. Note that essentially $E^{(1)}(d) = E(d)$. The measure $E^{(\omega)}(d)$ on $\mathrm{OB}(d)^{<\omega}$ is defined by recursion as follows:
$$X \in E^{(\omega)}(d) \iff \forall n < \omega\ \ \mathrm{Lev}_n(X) \in E^{(n+1)}(d).$$

(3) A set $T \subseteq \mathrm{OB}(d)^{<\xi}$ $(1 < \xi \leq \omega)$ ordered by end-extension is called a tree if it is closed under initial segments. A tree $T \subseteq \mathrm{OB}(d)^{<\omega}$ is called an $E(d)$-tree if for each $\langle \nu_0, \dots, \nu_{n-1}\rangle \in T$ we have $\nu_{k-1} < \nu_k$ $(k < n)$, and
$$\forall \langle \nu_0, \dots, \nu_{n-1}\rangle \in T\ \ \mathrm{Suc}_T(\nu_0, \dots, \nu_{n-1}) \in E(d).$$
Note that if T is an $E(d)$-tree then $T \in E^{(\omega)}(d)$. Hence for each $n < \omega$, $\mathrm{Lev}_n(T) \in E^{(n+1)}(d)$. Note also that if T is a tree such that $T \in E^{(\omega)}(d)$, then we can find a subtree $S \subseteq T$ such that S is an $E(d)$-tree.

(4) For a function $f \in \mathbb{P}^*_E$ we write $\mathrm{OB}(f)$, $E(f)$, and $\mathrm{mc}(f)$, for $\mathrm{OB}(\operatorname{dom} f)$, $E(\operatorname{dom} f)$, and $\mathrm{mc}(\operatorname{dom} f)$, respectively.

Definition 3.5. A condition p in the forcing notion $\mathbb{P}_E$ is of the form $\langle f, A\rangle$, where:
(1) $f \in \mathbb{P}^*_E$;

(2) A is an $E(f)$-tree.

We write f^p, A^p, and $\mathrm{mc}(p)$, for f, A, and $\mathrm{mc}(f)$, respectively.

Note that if $d_1 \subseteq d_2$ then a function $\mathrm{OB}(d_2) \to \mathrm{OB}(d_1)$ defined by

$$A \mapsto \{\nu \restriction d_1 \mid \nu \in A\}$$

is a projection of $E(d_2)$ to $E(d_1)$, thus the following definition makes sense.

Definition 3.6. Let $p, q \in \mathbb{P}_E$. We say that p is a Prikry extension of q ($p \leq^*_{\mathbb{P}_E} q$) if:

(1) $f^p \leq^*_{\mathbb{P}^*_E} f^q$;
(2) $\{\langle \nu_0 \restriction \operatorname{dom} f^q, \dots, \nu_{n-1} \restriction \operatorname{dom} f^q\rangle \mid n < \omega,\ \langle \nu_0, \dots, \nu_{n-1}\rangle \in A^p\} \subseteq A^q$.

Definition 3.7. Let $f \in \mathbb{P}^*_E$ and $\nu \in \mathrm{OB}(\operatorname{dom} f)$ be such that $\nu(\kappa) > \max f(\kappa)$. Define $f_{\langle \nu \rangle} \in \mathbb{P}^*_E$ to be a function $g \in \mathbb{P}^*_E$ with domain $\operatorname{dom} f$ satisfying for each $\alpha \in \operatorname{dom} g$,

$$g(\alpha) = \begin{cases} f(\alpha)^\frown\langle \nu(\alpha)\rangle & \alpha \in \operatorname{dom} \nu \text{ and } \big(\nu(\alpha) > \max f(\alpha) \text{ or } \alpha \in j''\mu\big), \\ f(\alpha) & \text{Otherwise.} \end{cases}$$

Assume $\langle \nu_0, \dots, \nu_{n-1}\rangle \in \mathrm{OB}(f)^n$ is such that $\max f(\kappa) < \nu_0(\kappa)$, and $\nu_i < \nu_{i+1}$ for each $i < n-1$. Define $f_{\langle \nu_0, \dots, \nu_{n-1}\rangle}$ recursively as $(f_{\langle \nu_0, \dots, \nu_{n-2}\rangle})_{\langle \nu_{n-1}\rangle}$.

Let $p \in \mathbb{P}_E$ and $\langle \nu_0, \dots, \nu_{n-1}\rangle \in A^p$ be such that $\max f^p(\kappa) < \nu_0(\kappa)$, and $\nu_i < \nu_{i+1}$ for each $i < n-1$. Define $p_{\langle \nu_0, \dots, \nu_{n-1}\rangle} \in \mathbb{P}_E$ to be the condition $\langle f^p_{\langle \nu_0, \dots, \nu_{n-1}\rangle}, A^p_{\langle \nu_0, \dots, \nu_{n-1}\rangle}\rangle \in \mathbb{P}_E$.

Whenever the notation $\langle \nu_0, \dots, \nu_{n-1}\rangle$ is used, where $\nu_k \in \mathrm{OB}(d)$ ($k < n$), it is implicitly assumed that $\nu_{k-1} < \nu_k$ ($k < n$).

Definition 3.8. Assume $p, q \in \mathbb{P}_E$. We say that p is an extension of q ($p \leq_{\mathbb{P}_E} q$) if there is $\langle \nu_0, \dots, \nu_{n-1}\rangle \in A^q$ such that

$$p \leq^*_{\mathbb{P}_E} q_{\langle \nu_0, \dots, \nu_{n-1}\rangle}.$$

Note that if $p, q \in \mathbb{P}_E$ are conditions such that $f^p \parallel_{\mathbb{P}^*_E} f^q$, then $p \parallel_{\mathbb{P}_E} q$. Thus an anti-chain in $\mathbb{P}_E$ induces an anti-chain in $\mathbb{P}^*_E$. Since $\langle \mathbb{P}^*_E, \leq^*\rangle$ has the μ^+-cc, the following is immediate.

Claim 3.9. *The forcing $\mathbb{P}_E$ has the μ^+-cc.*

The following claim is immediate since $\operatorname{crit} j = \kappa$ and $M \supset {}^\kappa M$.

Claim 3.10. *$\langle \mathbb{P}_E, \leq^*\rangle$ is κ-closed.*

The following lemma, merging subtrees into a tree, is immediate. It is used in the proof of claim 3.12 as a replacement for diagonal intersection of sets.

Lemma 3.11. *Assume $p \in \mathbb{P}_E$, and for each $n < \omega$ and $\langle \nu_0, \dots, \nu_{n-1}\rangle \in A^p$ there is a tree $T(\nu_0, \dots, \nu_{n-1}) \subseteq A^p_{\langle \nu_0, \dots, \nu_{n-1}\rangle}$ such that $\langle f^p_{\langle \nu_0, \dots, \nu_{n-1}\rangle}, T(\nu_0, \dots, \nu_{n-1})\rangle \in \mathbb{P}_E$. Then there is a condition $p^* \leq^*_{\mathbb{P}_E} p$ such that $f^{p^*} = f^p$, and for each $n < \omega$ and $\langle \nu_0, \dots, \nu_{n-1}\rangle \in A^{p^*}$,*

$$p^*_{\langle \nu_0, \dots, \nu_{n-1}\rangle} \leq^*_{\mathbb{P}_E} \langle f^p_{\langle \nu_0, \dots, \nu_{n-1}\rangle}, T(\nu_0, \dots, \nu_{n-1})\rangle.$$

Claim 3.12. *Assume $p \in \mathbb{P}_E$ and $D \subseteq \mathbb{P}_E$ is dense open. Then there are a condition $p^* \leq^*_{\mathbb{P}_E} p$ and $n < \omega$ such that*

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*} \ p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \in D.$$

Proof. Let χ be large enough.

First step. The first step in the proof is the construction of an elementary submodel $N \prec H_\chi$ and a condition $f^* \leq^* f^p$ such that $|N| < \mu$, $N \cap \mu \in \mathrm{On}$, $N \supset N^{<\kappa}$, $p, \mathbb{P}_E, D \in N$, $f^* \in \bigcap \{D \in N \mid D \text{ is a dense open subset of } \mathbb{P}^*_E \text{ below } f^p\}$, and $f^* \subset N$.

Observe that for a set X such that $|X| < \mu$ there is an elementary submodel $N \prec H_\chi$ such that $|N| < \mu$, $N \supseteq X$, $N \cap \mu \in \mathrm{On}$, and $N \supset N^{<\kappa}$. (If $\mu = \lambda^+$ then we need that $\mathrm{cf}\,\lambda \geq \kappa$ in order for the previous sentence to be correct).

The construction of N and f^* is done by induction of length κ where in each step $\xi < \kappa$ of the induction a pair $\langle N_\xi, f_\xi \rangle$ is constructed so that $N_\xi \prec H_\chi$, $|N_\xi| < \mu$, $N_\xi \cap \mu \in \mathrm{On}$, $N_\xi \supset N_\xi^{<\kappa}$, $N_{\xi_1} \subseteq N_{\xi_2}$ ($\xi_1 < \xi_2 < \kappa$), $f_{\xi_2} \leq^* f_{\xi_1}$ ($\xi_1 < \xi_2 < \kappa$), $f_\xi \in N_{\xi+1}$, $f_\xi \subset N_{\xi+1}$, and $f_\xi \in \bigcap \{D \in N_\xi \mid D \text{ is a dense open subset of } \mathbb{P}^*_E \text{ below } f^p\}$. (As can be seen, each pair $\langle N_\xi, f_\xi \rangle$ satisfies almost all the demands for being $\langle N, f^* \rangle$. The one thing which might fail is $f_\xi \subset N_\xi$.) The induction is carried as follows.

- $\xi = 0$: Let $N_0 \prec H_\chi$ be an elementary submodel such that $|N_0| < \mu$, $N_0 \supset N_0^{<\kappa}$, and $p, \mathbb{P}_E, D \in N_0$. Choose a condition $f_0 \leq^* f^p$ such that $f \in \bigcap \{D \in N_0 \mid D \text{ is a dense open subset of } \mathbb{P}^*_E \text{ below } f^p\}$.
- $0 < \xi < \kappa$: Let $N_\xi \prec H_\chi$ be an elementary submodel such that $|N_\xi| < \mu$, $N_\xi \supset N_\xi^{<\kappa}$, $N_\xi \supset \bigcup_{\xi' < \xi} N_{\xi'}$, $\bigcup_{\xi'<\xi} f_{\xi'} \in N_\xi$, and $N_\xi \supset \bigcup_{\xi'<\xi} f_{\xi'}$. Pick a condition $f_\xi \in \bigcap \{D \in N_\xi \mid D \text{ is a dense open subset of } \mathbb{P}^*_E \text{ below } f^p\}$ such that $f_\xi \leq^* \bigcup_{\xi'<\xi} f_{\xi'}$.

When the induction terminates set $N = \bigcup_{\xi<\kappa} N_\xi$ and $f^* = \bigcup_{\xi<\kappa} f_\xi$. It is trivial to verify that all the demands specified in the beginning of the proof are satisfied by the pair $\langle N, f^* \rangle$. With N and f^* at our disposal we proceed to the second step of the proof.

Second step. Let A be a tree such that $\langle f^*, A \rangle \leq^* p$. Since $\big(j(N) \supset {}^{<j(\kappa)}j(N)\big)_M$ and $N \supset \{\kappa\} \cup \mathrm{dom}\, f^*$ we deduce that $\mathrm{mc}(f^*) \in j(N)$. Thus by removing a measure zero set from A we can ensure that $A \subset N$. For each $k < \omega$ and $\langle \nu_0, \dots, \nu_{k-1} \rangle \in A$ set

$$\begin{aligned} D_{\langle \nu_0, \dots, \nu_{k-1} \rangle} = \{ f \leq^* f^p \mid \exists T \ \langle f_{\langle \nu_0, \dots, \nu_{k-1} \rangle}, T \rangle \in D \text{ or} \\ \forall g \leq^* f \ \forall T \ \langle g_{\langle \nu_0, \dots, \nu_{k-1} \rangle}, T \rangle \notin D \}. \end{aligned}$$

For each $\langle \nu_0, \dots, \nu_{k-1} \rangle \in A$ the set $D_{\langle \nu_0, \dots, \nu_{k-1} \rangle}$ is a dense open subsets of $\mathbb{P}^*_E$ below f^p, and since $A \subset N$ these sets are in N. Thus $f^* \in D_{\langle \nu_0, \dots, \nu_{k-1} \rangle}$ for each $\langle \nu_0, \dots, \nu_{k-1} \rangle \in A$. That is, for each $k < \omega$ and $\langle \nu_0, \dots, \nu_{k-1} \rangle \in A$ either

$$\exists T \ \langle f^*_{\langle \nu_0, \dots, \nu_{k-1} \rangle}, T \rangle \in D,$$

or

$$\forall g \leq^* f^* \ \forall T \ \langle g_{\langle \nu_0, \dots, \nu_{k-1} \rangle}, T \rangle \notin D.$$

Third step. We claim there is $n < \omega$ and $X \in E^{(n)}(f^*)$ such that for each $\langle \nu_0, \dots, \nu_{n-1} \rangle \in X$ there is a tree T such that $\langle f^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle}, T \rangle \in D$. Towards a contradiction assume that this is not the case. Then there is $Y \in E^{(\omega)}(f^*)$ such that for each $k < \omega$ and $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in Y$,

$$\forall g \leq^* f^* \ \forall T \ \langle g_{\langle \nu_0,\ldots,\nu_{k-1}\rangle}, T\rangle \notin D.$$

Let $p^* \leq^* p$ be a condition such that $f^{p^*} = f^*$ and $A^{p^*} \subseteq Y$. Since D is dense open, there is a condition $q \in D$ such that $q \leq p^*$. By the definition of the order $\leq_{\mathbb{P}_E}$ there is $k < \omega$ and $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in A^{p^*}$ such that $q \leq^* p^*_{\langle \nu_0,\ldots,\nu_{k-1}\rangle}$. Set $g = f^* \cup \big(f^q \restriction (\operatorname{dom} f^q \setminus \operatorname{dom} f^*)\big)$. A contradiction is derived by noting that $g \leq^* f^*$ and $\langle g_{\langle \nu_0,\ldots,\nu_{k-1}\rangle}, A^q\rangle = q \in D$.

Fourth step. Fix $n < \omega$ and $X \in E^{(n)}(f^*)$ such that for each $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in X$ there is a tree $T(\nu_0, \ldots, \nu_{n-1})$ such that $\langle f^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle}, T(\nu_0, \ldots, \nu_{n-1})\rangle \in D$. By 3.11 there is a condition $p^* \leq^* \langle f^*, A\rangle$ such that for each $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in A^{p^*}$,

$$p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \leq^* \langle f^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle}, T(\nu_0, \ldots, \nu_{n-1})\rangle.$$

Since the set D is open we get that for each $\langle \nu_0, \ldots, \nu_{n-1} \rangle \in A^{p^*}$,

$$p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \in D.$$

□

Definition 3.13. Assume $G \subset \mathbb{P}_E$ is generic. For each $\alpha < j(\mu)$ set

$$G^\alpha = \bigcup \{ f^p(\alpha) \mid p \in G,\ \alpha \in \operatorname{dom} f^p \}.$$

Note that for $\alpha < \mu$, the tail of $G^{j(\alpha)}$ looks like $\langle \alpha, \alpha, \ldots\rangle$. Let $\langle G^\alpha_n \mid n < \omega\rangle$ be the increasing enumeration of G^α. Let $\underset{\sim}{G}^\alpha$ and $\underset{\sim}{G}^\alpha_n$ be the $\mathbb{P}_E$-names of G^α and G^α_n, respectively.

Claim 3.14. *Assume $\lambda \in [\kappa, \mu)$ and $\sup j''\lambda < j(\lambda)$. Then $G^{\sup j''\lambda}$ is an ω-sequence unbounded in λ.*

Proof. Fix $\lambda \in [\kappa, \lambda)$ such that $\alpha = \sup j''\lambda < j(\lambda)$. Assume $\tau < \lambda$. Then $j(\tau) < \alpha$, hence $\alpha \in (j(\tau), j(\lambda))$, i.e., $(\tau, \lambda) \in E(\alpha)$. Thus if $A \in E(f^p)$ then $\{\nu \in A \mid \nu(\alpha) > \tau\} \in E(f^p)$. Fix a condition $p \in \mathbb{P}_E$ such that $\alpha \in \operatorname{dom} f^p$. Set

$$A^* = \langle \langle \nu_0, \ldots, \nu_n\rangle \in A^p \mid n < \omega,\ \nu_0(\alpha) > \tau\rangle.$$

Then $p^* = \langle f^p, A^*\rangle$ is a Prikry extension of p, and for each $n < \omega$ and $\langle \nu_0, \ldots, \nu_n \rangle \in A^{p^*}$,

$$p^*_{\langle \nu_0,\ldots,\nu_{n-1}\rangle} \Vdash_{\mathbb{P}_E} \text{“}\underset{\sim}{G}^\alpha_{|f^p(\alpha)|} > \tau\text{”}.$$

I.e., $p^* \Vdash_{\mathbb{P}_E}$ “$\underset{\sim}{G}^\alpha_{|f^p(\alpha)|} > \tau$”. □

Since for a V-regular cardinal $\lambda \in (\kappa, \mu)$ we have $\sup j''\lambda < j(\lambda)$, we conclude that λ is collapsed. Thus we get:

Claim 3.15. *All V-cardinals in (κ, μ) are collapsed in a $\mathbb{P}_E$-generic extension.*

The following is the basic observation regarding the sequences G^α. It is immediate from the definition of the sequences G^α and Łoś theorem.

Claim 3.16. *Assume $p \in \mathbb{P}_E$ and $\alpha < j(\mu)$. Then there is a condition $p^* \leq^*_{\mathbb{P}_E} p$ such that for each $n < \omega$,*

$$j_\omega(p^*)_{\langle j_{1,\omega}(\mathrm{mc}(p^*)),\dots,j_{n+1,\omega}(j_n(\mathrm{mc}(p^*)))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \text{“} j_\omega(\underset{\sim}{G}^\alpha_{|f^{p^*}(\alpha)|+n}) = j_{n+1,\omega}(j_n(\alpha))\text{”}.$$

Proof. Set $n_0 = |f^p(\alpha)|$. Note that for each $n < n_0$, $p \Vdash_{\mathbb{P}_E}$ “$\underset{\sim}{G}^\alpha_n = f^p_n(\alpha)$”. Construct by induction a $\leq^*_{\mathbb{P}_E}$-decreasing sequence of conditions as follows. Set $p_0 = p$. Assume p_n was constructed. Construct p_{n+1} as follows. Let

$$D_n = \{q \leq_{\mathbb{P}_E} p \mid \exists \xi\ q \Vdash_{\mathbb{P}_E} \text{“}\underset{\sim}{G}^\alpha_{n_0+n} = \check{\xi}\text{”}\}.$$

By 3.12 there is a Prikry extension $p_{n+1} \leq^*_{\mathbb{P}_E} p_n$ and $k < \omega$ such that

$$\forall \langle \nu_0, \dots, \nu_{k-1}\rangle \in A^{p_{n+1}}\ p_{n+1\langle \nu_0,\dots,\nu_{k-1}\rangle} \in D.$$

Note that since D is open we can make k larger than n. Thus we get by the definition of the forcing notion

$$\forall \langle \nu_0, \dots, \nu_n\rangle \in A^{p_{n+1}}\ p_{n+1\langle \nu_0,\dots,\nu_n\rangle} \Vdash_{\mathbb{P}_E} \text{“}\underset{\sim}{G}^\alpha_{n_0+n} = \nu_n(\alpha)\text{”}.$$

Going to the $n+1$ ultrapower we get

$$j_{n+1}(p_{n+1})_{\langle j_{1,n+1}(\mathrm{mc}(p_{n+1})),\dots,j_{n+1,n+1}(j_n(\mathrm{mc}(p_{n+1})))\rangle} \Vdash_{j_{n+1}(\mathbb{P}_E)} \text{“} j_{n+1}(\underset{\sim}{G}^\alpha_{n_0+n}) = j_n(\alpha)\text{”}.$$

And sending to M_ω yields

$$j_\omega(p_{n+1})_{\langle j_{1,\omega}(\mathrm{mc}(p_{n+1})),\dots,j_{n+1,\omega}(j_n(\mathrm{mc}(p_{n+1})))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \text{“} j_\omega(\underset{\sim}{G}^\alpha_{n_0+n}) = j_{n+1,\omega}(j_n(\alpha))\text{”}.$$

When the induction terminates choose a condition p^* such that for each $n < \omega$, $p^* \leq^*_{\mathbb{P}_E} p_n$. We get for each $n < \omega$,

$$j_\omega(p^*)_{\langle j_{1,\omega}(\mathrm{mc}(p^*)),\dots,j_{n+1,\omega}(j_n(\mathrm{mc}(p^*)))\rangle} \leq^*_{\mathbb{P}_E} j_\omega(p_{n+1})_{\langle j_{1,\omega}(\mathrm{mc}(p_{n+1})),\dots,j_{n+1,\omega}(j_n(\mathrm{mc}(p_{n+1})))\rangle},$$

by which we get

$$j_\omega(p^*)_{\langle j_{1,\omega}(\mathrm{mc}(p^*)),\dots,j_{n+1,\omega}(j_n(\mathrm{mc}(p^*)))\rangle} \Vdash_{j_\omega(\mathbb{P}_E)} \text{“} j_\omega(\underset{\sim}{G}^\alpha_{n_0+n}) = j_{n+1,\omega}(j_n(\alpha))\text{”}.$$

□

Observe that the sets $\{j_{n+1,\omega}(j_n(\alpha)) \mid n < \omega\}$ and $\{j_{n+1,\omega}(j_n(\beta)) \mid n < \omega\}$ are disjoint, where $\alpha < \beta < j(\mu)$. Thus the sequences G^α and G^β are tail-disjoint. Thus in $V[G]$, $|\{G^\alpha \mid \alpha < j(\mu\}| = |j(\mu)|$.

Claim 3.17. *In $V[G]$, $2^\kappa \geq |j(\mu)|$.*

Definition 3.18. A triple $\langle \mathbb{P}, \leq, \leq^*\rangle$ is called a Prikry type forcing notion if

(1) $\langle \mathbb{P}, \leq\rangle$ and $\langle \mathbb{P}, \leq^*\rangle$ are forcing notions;
(2) $\leq^* \subseteq \leq$;
(3) For each σ a formula in the $\langle \mathbb{P}, \leq\rangle$-forcing language, and a condition $p \in \mathbb{P}$, there is a condition $p^* \leq^* p$ such that $p^* \parallel_{\langle \mathbb{P}, \leq\rangle} \sigma$.

Corollary 3.19. *$\langle \mathbb{P}_E, \leq, \leq^*\rangle$ is a Prikry type forcing notion.*

Proof. Assume σ is a formula in the $\mathbb{P}_E$-forcing language, and $p \in \mathbb{P}_E$. Set

$$D = \{q \in \mathbb{P}_E \mid q \parallel_{\mathbb{P}_E} \sigma\}.$$

Trivially, D is a dense open subset of $\mathbb{P}_E$. By 3.12 there are a condition $p' \leq^*_{\mathbb{P}_E} p$ and $n < \omega$ such that

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p'} \ p'_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \in D.$$

That is

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p'} \ p'_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \parallel_{\mathbb{P}_E} \sigma.$$

Set

$$T_0 = \{\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p'} \mid p'_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \Vdash_{\mathbb{P}_E} \neg\sigma\},$$

and

$$T_1 = \{\langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p'} \mid p'_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \Vdash_{\mathbb{P}_E} \sigma\}.$$

Since $\mathrm{Lev}_{n-1}(A^{p'}) \in E^{(n)}(\mathrm{mc}(p'))$, $T_0 \cap T_1 = \emptyset$, and $\mathrm{Lev}_{n-1}(A^{p'}) = T_0 \cup T_1$, we have either $T_0 \in E^{(n)}(\mathrm{mc}(p'))$ or $T_1 \in E^{(n)}(\mathrm{mc}(p'))$. Set $i < 2$ so that $T_i \in E^{(n)}(\mathrm{mc}(p'))$. Let

$$T = \{\langle \nu_0, \dots, \nu_{k-1} \rangle \in A^{p'} \mid k < \omega,\ \langle \nu_0, \dots, \nu_{n-1} \rangle \in T_i\}.$$

Set $p^* = \langle f^{p'}, T \rangle$. Then either

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*} \ p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \Vdash_{\mathbb{P}_E} \neg\sigma$$

or

$$\forall \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*} \ p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \Vdash_{\mathbb{P}_E} \sigma.$$

Since $\{p^*_{\langle \nu_0, \dots, \nu_{n-1} \rangle} \mid \langle \nu_0, \dots, \nu_{n-1} \rangle \in A^{p^*}\}$ is a maximal anti-chain below p^*, we have either $p^* \Vdash_{\mathbb{P}_E} \neg\sigma$ or $p^* \Vdash_{\mathbb{P}_E} \sigma$. ☐

Claim 3.20. *The cardinal μ is preserved in a $\mathbb{P}_E$-generic extension.*

Proof. Since all the cardinals in (κ, μ) are collapsed and $\mathrm{cf}^{V^{\mathbb{P}_E}}(\kappa) = \omega$, collapse of μ means that $\mathrm{cf}^{V^{\mathbb{P}_E}}(\mu) < \kappa$.

Assume $p \Vdash_{\mathbb{P}_E}$ "$\dot{f} : \check{\lambda} \to \check{\mu}$", where $\lambda < \kappa$. We will exhibit a condition $p^* \leq^*_{\mathbb{P}_E} p$ forcing that $\dot{f}$ is bounded in μ. For each $\zeta < \lambda$ set $D_\zeta = \{q \leq_{\mathbb{P}_E} p \mid \exists \xi < \mu \ q \Vdash_{\mathbb{P}_E}$ "$\dot{f}(\check{\zeta}) = \check{\xi}$"$\}$. Since the sets D_ζ are dense open subsets of $\mathbb{P}_E$ below p, we can construct by induction, using 3.12, the $\leq^*$-decreasing sequence $\langle p^\zeta \mid \zeta < \lambda \rangle$ and the sequence $\langle k_\zeta < \omega \mid \zeta < \lambda \rangle$ satisfying $p^0 = p$, and for each $\zeta < \lambda$,

$$\forall \langle \nu_0, \dots, \nu_{k_\zeta} \rangle \in A^{p^{\zeta+1}} \ \exists \xi < \mu \ p^{\zeta+1}_{\langle \nu_0, \dots, \nu_{k_\zeta} \rangle} \Vdash_{\mathbb{P}_E} \text{“}\dot{f}(\check{\zeta}) = \check{\xi}\text{”}.$$

Choose a condition p^* such that for each $\zeta < \lambda$, $p^* \leq^*_{\mathbb{P}_E} p^\zeta$. Now we have for each $\zeta < \lambda$,

$$\forall \langle \nu_0, \dots, \nu_{k_\zeta} \rangle \in A^{p^*} \ \exists \xi < \mu \ p^*_{\langle \nu_0, \dots, \nu_{k_\zeta} \rangle} \Vdash_{\mathbb{P}_E} \text{“}\dot{f}(\check{\zeta}) = \check{\xi}\text{”}.$$

For each $\zeta < \lambda$ define a function $F_\zeta : \mathrm{Lev}_{k_\zeta}(A^{p^*}) \to \mu$ so that

$$\forall \langle \nu_0, \dots, \nu_{k_\zeta} \rangle \in A^{p^*} \ p^*_{\langle \nu_0, \dots, \nu_{k_\zeta} \rangle} \Vdash_{\mathbb{P}_E} \text{“}\dot{f}(\check{\zeta}) = \check{F}_\zeta(\nu_0, \dots, \nu_{k_\zeta})\text{”}.$$

Set $\mu^* = \sup\{F_\zeta(\nu_0, \dots, \nu_{k_\zeta}) \mid \zeta < \lambda,\ \langle \nu_0, \dots, \nu_{k_\zeta} \rangle \in A^{p^*}\}$. By its definition, $p^* \Vdash_{\mathbb{P}_E}$ "$\operatorname{ran} \dot{f} \subseteq \check{\mu}^*$". Since the sup is taken over a set of size less than μ, and μ is regular we get $\mu^* < \mu$. □

Theorem 3.21. *Assume $G \subset \mathbb{P}_E$ is generic. Then in $V[G]$:*

1. *V and $V[G]$ have the same bounded subsets of κ, and thus κ and all the cardinals below it are preserved.*
2. *All cardinals in (κ, μ) are collapsed, and $\operatorname{cf}^{V[G]} \kappa = \omega$.*
3. *All the cardinals $\geq \mu$ are preserved.*
4. $2^\kappa = |j(\mu)|$.

Proof.

1. The Prikry property of $\langle \mathbb{P}_E, \leq, \leq^* \rangle$ (3.19), together with the κ-closure of $\langle \mathbb{P}_E, \leq^* \rangle$ (3.10), yield that V and $V[G]$ have the same bounded subset of κ.
2. Fix a V-regular cardinal $\lambda \in [\kappa, \mu)$. By simple density argument, $G^{\sup j''\lambda}$ is an ω-sequence unbounded in λ. Thus $\operatorname{cf}^{V[G]} \lambda = \omega$. Since all the regular cardinals in (κ, μ) are collapsed, so are the singulars in the range.
3. By the μ^+-cc of $\mathbb{P}_E$ (3.9) all the cardinals above μ are preserved, and μ is preserved by 3.20.
4. On the one hand, the μ^+-cc together with $|\mathbb{P}_E| \leq |j(\mu)|$ imply $2^\kappa \leq |j(\mu)|$. On the other hand, 3.17 gives $2^\kappa \geq |j(\mu)|$.

□

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