THE SHORT EXTENDERS GAP THREE FORCING USING A MORASS

CARMI MERIMOVICH

Abstract. We show how to construct Gitik's short extenders gap-3 forcing using a morass, and that the forcing notion is of Prikry type.

1. Introduction

In [7] a forcing notion blowing up the powerset of a cardinal κ carrying an extender together with changing κ's cofinality to ω in one step was introduced. The size of the powerset in the generic extension was set to the size of the extender. It was felt at the time that this is the optimal assumption. That is, if one begins with a model for κ of cofinality ω with large powerset, then in the core model one should find an extender on κ of the powerset size. Going this way [8] and [5] assumed $2^\kappa > \kappa^+$ and found, quite unexpectedly, *two* possibilities. One possibility was indeed that in the core model the cardinal κ carries an extender of size 2^κ. The other possibility, however, was that in the core model the cardinal κ is a singular cardinal of cofinality ω, and there is an increasing sequence of cardinals κ_n, with limit κ, each carrying a rather short extender. In the sequence of papers [3, 4, 5] it was shown that indeed this other possibility can be used to blow up the powerset of κ. In [3] and in a simpler form in [11] the following was proved.

Theorem (M. Gitik [3])**.** *Assume the GCH and let $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of cardinals such that for each $n < \omega$ there is a $\langle \kappa_n, \kappa_n^{+n+2} \rangle$ extender. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Then there is a cardinal preserving generic extension adding no new bounded subsets to κ such that $2^\kappa = \kappa^{++}$.*

The paper [12] begun as an attempt to simplify [11]. As is evident by the result, this aim was not achieved. However, it was discovered along the way that the forcing notion is of Prikry type.

Widening the gap, the following was proved in each of [4], [2], and [6], where the presentation gets simpler from paper to paper.

Theorem (M. Gitik [4])**.** *Assume the GCH and let $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of cardinals such that for each $n < \omega$ there is a $\langle \kappa_n, \kappa_n^{+n+3} \rangle$ extender. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Then there is a cardinal preserving generic extension adding no new bounded subsets to κ such that $2^\kappa = \kappa^{+3}$.*

In the current paper we are going to reprove the last theorem using a morass to define the forcing notion and to show the forcing notion is of Prikry type.

Date: November 18, 2009.

2010 *Mathematics Subject Classification.* Primary 03E35, 03E55.

Key words and phrases. Singular cardinal hypothesis, Prikry property, Extender based forcing, Simplified morass.

A straightforward generalization of the method in [3] in order to achieve the result in [4] fails due to the appearance of large antichains in the forcing construction. The solution used in [4] was to do a preparation before the main forcing. The resulting generic object G' is used in the main forcing notion in order to restrict the length of the antichains. The paper [6] notes that Assaf Sharon and Tadatoshi Miyamoto pointed out that the preparation generic G' resembles Velleman's simplified $\langle \kappa^{++}, 1\rangle$-morass with linear limits [15]. Intrigued by this note we looked at [14] and [15] where simplified gap-1 morasses with and without linear limits were introduced. It is immediate to see that the generic filter G' codes a simplified morass with linear limits. (In itself G' looks stronger than the morass with linear limits.) So an attempt to construct the forcing from some kind of a morass seemed reasonable. A point of weakness in using the morass is the chain condition. Using G' the forcing has the κ^{++}-cc. Using a simplified morass (i.e., without linear limits or piste) we got only κ^{+3}-cc. However, adding the assumption that the morass has some stationarity the forcing has a kind of properness from which one can deduce that κ^{++} is preserved. This is more in the spirit of [10].

It should be noted that the morass assumption is a reasonable one together with the large cardinals assumption since Velleman presented a κ^{++}-closed, κ^{+3}-cc forcing notion adding a morass to the universe, thus this forcing does not change the large cardinals status of cardinals below κ.

We stress that the results and techniques, except for the Prikry property proof which we took from [12] and the properness, are all due to Gitik, and we followed very closely the proofs in [6] when writing this paper.

Gitik presented his forcing notions in the TAU set theory seminar of the year 2007 from which the notes [11, 6] grew out. We thank the participants of the seminar Eilon Beillinski, Omer Ben-Naria, Assaf Ferber, Assaf Rinot, and Liad Tal. Of course we thank Moti Gitik for the organization, presentation, and for being rather patient with the enormous amount of questions he had to answer by phone, email, and in person.

We have tried to keep this work resemble [12] as far as possible. In fact some of the claims are identical in both papers. We hope this shows that the essential ingredient converting the gap-2 forcing to the gap–3 forcing is the limitation put on extending a condition by the morass.

This paper is self contained assuming one knows forcing and large cardinals theory. We present the relevant material about morasses in section 2. In section 3 we present the gap-3 forcing notion.

2. Simplified gap-1 morass

Simplified gap-1 morasses were introduced in [14] as a simplifications of Jensen's gap-1 morasses [1]. Introduction to gap-1 simplified morasses can be found in [14], [10], and [9].

Since in the next section we are going to use a neat $\langle \kappa^{++}, 1\rangle$-simplified morass, we will concentrate from now on such morass. Unless otherwise noted, in this section the definitions and facts are from [14].

In the following the notation $\mathbb{M} \restriction X$ denotes the set $\{Y \subsetneq X \mid Y \in \mathbb{M}\}$.

Definition 2.1. Assume κ is a cardinal.

- A family of subsets $\mathbb{M} \subseteq [\kappa^{+3}]^{\leq \kappa^+}$ is a simplified $\langle \kappa^{++}, 1\rangle$-morass if:

(1) $\langle \mathbb{M}, \subsetneq \rangle$ is well-founded. Define the $\mathbb{M}$-rank function $\rho_{\mathbb{M}}$ (or just ρ when $\mathbb{M}$ is clear from context) by recursion:

$$\rho_{\mathbb{M}}(Y) = \sup\{\rho_{\mathbb{M}}(X) + 1 \mid X \in \mathbb{M} \restriction Y\}.$$

The height of the morass $\mathbb{M}$, denoted by $\rho(\mathbb{M})$, is defined by

$$\rho(\mathbb{M}) = \sup\{\rho_{\mathbb{M}}(Y) + 1 \mid Y \in \mathbb{M}\}.$$

(2) $\mathbb{M}$ is locally small, i.e., for each $Y \in \mathbb{M}$, $|\mathbb{M} \restriction Y| < \kappa^{++}$.
(3) $\mathbb{M}$ is homogeneous, i.e., for each two sets $Y_0, Y_1 \in \mathbb{M}$ of the same rank there is an order preserving bijection $\pi_{Y_0,Y_1} : Y_0 \to Y_1$ such that $\mathbb{M} \restriction Y_1 = \{\pi''_{Y_0,Y_1} X \mid X \in \mathbb{M} \restriction Y_0\}$.
(4) $\mathbb{M}$ is directed, i.e., for each two sets $X, Y \in \mathbb{M}$ there is a set $Z \in \mathbb{M}$ such that $Z \supseteq X, Y$.
(5) $\mathbb{M}$ is locally almost directed, i.e., for each $Y \in \mathbb{M}$ one of the following holds:
  (5.1) $\mathbb{M} \restriction Y$ is directed.
  (5.2) There are sets $Y_0, Y_1 \in \mathbb{M}$ and ordinals $\alpha_0 \in Y_0$, $\alpha_1 \in Y_1$ such that

$$\mathbb{M} \restriction Y = \{Y_0, Y_1\} \cup (\mathbb{M} \restriction Y_0) \cup (\mathbb{M} \restriction Y_1),$$

  and

$$Y_0 \cap Y_1 = Y_0 \cap \alpha_0 = Y_1 \cap \alpha_1.$$

  The triple $\langle Y, Y_0, Y_1 \rangle$ was dubbed Δ-system like in [4], and the pair $\langle Y_0, Y_1 \rangle$ was dubbed split-end in [14], The ordinals α_0 and α_1 are called the witnessing ordinals of the Δ-system $\langle Y, Y_0, Y_1 \rangle$.
(6) $\mathbb{M}$ covers κ^{+3}, i.e., $\kappa^{+3} = \bigcup \mathbb{M}$.

- A simplified $\langle \kappa^{++}, 1 \rangle$-morass $\mathbb{M}$ is neat if for each $Y \in \mathbb{M}$ of rank > 0, $Y = \bigcup (\mathbb{M} \restriction Y)$.
- A simplified $\langle \kappa^{++}, 1 \rangle$-morass $\mathbb{M}$ is stationary if $\mathbb{M}$ is a stationary subset of $[\kappa^{+3}]^{\kappa^+}$.

We quote facts about the simplified morass.

Fact 2.2 (D. Velleman [14])**.** *Assume $\mathbb{M}$ is a simplified $\langle \kappa^{++}, 1 \rangle$-morass.*

(1) Assume $X \in \mathbb{M}$ and $\rho(X) < \tau < \rho(\mathbb{M})$. Then there is a set $Y \in \mathbb{M}$ such that $\rho(Y) = \tau$ and $Y \supsetneq X$.
(2) Assume $Z \in \mathbb{M}$, $X \in \mathbb{M} \restriction Z$, and $\rho(X) < \tau < \rho(Z)$. Then there is a set $Y \in \mathbb{M}$ such that $X \subsetneq Y \subsetneq Z$ and $\rho(Y) = \tau$.
(3) Assume the sets $Y_0, Y_1 \in \mathbb{M}$ are of the same $\mathbb{M}$-rank and $\alpha \in Y_0 \cap Y_1$. Then $Y_0 \cap (\alpha + 1) = Y_1 \cap (\alpha + 1)$.
(4) Assume the sets $Y_0, Y_1 \in \mathbb{M}$ are of the same $\mathbb{M}$-rank and $\sup(Y_0 \cap \alpha) = \sup(Y_1 \cap \alpha) = \alpha$. Then $Y_0 \cap \alpha = Y_1 \cap \alpha$.
(5) $\rho(\mathbb{M}) = \kappa^{++}$.
(6) Assume $Z \in \mathbb{M}$, $\lambda < \operatorname{cf}(\rho(Z))$, and for each $\xi < \lambda$, $X_\xi \in \mathbb{M} \restriction Z$. Then there is $Y \in \mathbb{M} \restriction Z$ such that $X_\xi \subsetneq Y$ for each $\xi < \lambda$.

A definition is needed for the last fact. Assume χ is large enough. An elementary substructure $N \prec H_\chi$ is said to be $\mathbb{M}$-admissible if $|N| = \kappa^+$, $N \cap \kappa^{++} \in On$, $\mathbb{M} \in N$, $N \cap \kappa^{+3} \in \mathbb{M}$ and $\mathbb{M} \restriction (N \cap \kappa^{+3}) = \mathbb{M} \cap N$. The appendix in [16] *proved that if $\mathbb{M}$ is a stationary simplified morass then there is a club $C \subseteq [\kappa^{+3}]^{\leq \kappa^+}$ such that $\mathbb{M} \cap C$ is a stationary coding set* [18]*. As a corollary we get:*

(7) *Assume χ is large enough and $\mathbb{M}$ is a stationary subset of $[\kappa^{+3}]^{\leq\kappa^+}$. Then $\{N \cap \kappa^{+3} \mid N \prec H_\chi$ is $\mathbb{M}$-admissble$\}$ is stationary.*

The following fact also tends to be used when dealing with morasses.

Fact 2.3 (Stanley)**.** *Assume $\mathbb{M}$ is a simplified $\langle \kappa^{++}, 1\rangle$-morass. $X_0, X_1 \in \mathbb{M}$ are of the same rank, and α is a limit point of both X_0 and X_1. Then $X_0 \cap \alpha = X_1 \cap \alpha$.*

At the behest of the referee we add several more facts and their proofs about the gap-1 morass which are derived from the above facts.

Fact 2.4. *Assume $\mathbb{M}$ is a simplified $\langle \kappa^{++}, 1\rangle$-morass.*

(1) *For each $X \in \mathbb{M}$, $\rho(X) < \kappa^{++}$.*
(2) *Assume $X_0, X_1 \in \mathbb{M}$ are different sets of the same rank. Then $\sup X_0 \neq \sup X_1$.*
(3) *Assume $X_0, X_1 \in \mathbb{M}$ are of the same rank. Then there is a Δ-system $\langle Z, Z_0, Z_1\rangle$ in $\mathbb{M}$ such that $X_0 \subseteq Z_0$, $X_1 \subseteq Z_1$, $X_0 \nsubseteq Z_1$, and $X_1 \subseteq Z_0$*
(4) *Assume $X_0, X_1 \in \mathbb{M}$ are of the same rank. Then there is a sequence of Δ-system $\langle\langle Z^i, Z^i_0, Z^i_1\rangle \mid i < n\rangle$, where $Z^i_0 \supseteq X_0$ and $Z^{i+1} \subseteq Z^i_0$, such that $\pi_{X_0,X_1} = \pi_{Z^{n-1}_0, Z^{n-1}_1} \circ \cdots \circ \pi_{Z^0_0, Z^0_1} \restriction X_0$.*

Proof. (1) This is immediate since $\mathbb{M}$ is locally small.

(2) Towards a contradiction assume $\alpha = \sup X_0 = \sup Y_0$. First assume $\max X_0$ exists. Since, by homogeneity, X_0 and X_1 have the same order type then $\max X_1$ also exists. By 2.2 (3), $X_0 \cap (\alpha+1) = X_1 \cap (\alpha+1)$, thus $X_0 = X_1$. Contradiction.

    Assume $\max X_0$ does not exist. Hence also $\max X_1$ does not exist. Then α is a limit point of both X_0 and X_1, thus, by 2.3, $X_0 \cap \alpha = X_1 \cap \alpha$, thus $X_0 = X_1$. Contradiction again and we are done.

(3) We choose Z of minimal rank such that $Z \supseteq X_0, X_1$. By local directedness either $\mathbb{M} \restriction Z$ is directed or we have a Δ-system $\langle Z, Z_0, Z_1\rangle$. If $\mathbb{M} \restriction Z$ is directed then there would have been a set $Y \in \mathbb{M} \restriction Z$ such that $Y \supseteq X_0, X_1$, in contradiction to the minimality of the rank of Z. Thus we have a Δ-system $\langle Z, Z_0, Z_1\rangle$. Recall that $\mathbb{M} \restriction Z = \mathbb{M} \restriction Z_0 \cup \mathbb{M} \restriction Z_1 \cup \{Z_0, Z_1\}$. If either $Z_0 \supseteq X_0, X_1$ or $Z_1 \supseteq X_0, X_1$ then we would have been again in contradiction with the minimality of Z. Thus by renaming Z_0 and Z_1, if necessary, we get $Z_0 \supseteq X_0$ and $Z_1 \supseteq X_1$.

(4) Let $\langle Z^0, Z^0_0, Z^0_1\rangle$ be a Δ-system in $\mathbb{M}$ such that $Z^0_0 \supseteq X_0$, $Z^0_1 \supseteq X_1$, and $Z^0_0 \cap Z^0_1 \nsupseteq X_0, X_1$. Such a Δ-system exists by the previous item. If $\pi''_{Z^0_0, Z^0_1} X_0 = X_1$ then we are done.

    Thus assume this is not the case. Set $\pi''_{Z^0_0, Z^0_1} X_0 = X'_0$. By recursion there is a sequence of Δ-systems $\langle\langle Z^i, Z^i_0, Z^i_1\rangle \mid 1 \leq i < n\rangle$, such that $\pi_{X'_0, X_1} = \pi_{Z^{n-1}_0, Z^{n-1}_1} \circ \cdots \circ \pi_{Z^1_0, Z^1_1} \restriction X'_0$. Since $\pi_{X_0,X_1} = \pi_{X'_0,X_1} \circ \pi_{X_0,X'_0}$ we get $\pi_{X_0,X_1} = \pi_{Z^{n-1}_0, Z^{n-1}_1} \circ \cdots \circ \pi_{Z^0_0, Z^0_1} \restriction X_0$.

$\square$

As for the existence of a simplified morass, [14] introduced a forcing notion adding a stationary simplified morass to the universe. In addition it showed how to derive a neat simplified morass from a simplified morass. We present a slight variation of this forcing, appearing in [17] and [10], adding directly a neat stationary simplified $\langle \kappa^{++}, 1\rangle$-morass to the universe.

Definition 2.5. A condition p in the forcing P is a subset of $[\kappa^{+3}]^{\leq\kappa^+}$ such that:

(1) $\langle p, \subsetneq\rangle$ is well-founded. Define the obvious p-rank function for $Y \in p$ by setting $\rho_p(Y) = \sup\{\rho_p(X) + 1 \mid X \in p \restriction Y\}$.
(2) p is small, i.e., $|p| < \kappa^{++}$.
(3) p has a maximal element, denoted by $\max p$, i.e., for each $Y \in p$, $Y \subseteq \max p$.
(4) p is homogeneous, i.e., for each $Y_0, Y_1 \in p$ of the same p-rank there is an order preserving bijection $\pi : Y_0 \to Y_1$ such that $p \restriction Y_1 = \{\pi'' X \mid X \in p \restriction Y_0\}$.
(5) p is directed, i.e., for each $X, Y \in p$ there is $Z \in p$ such that $Z \supseteq X, Y$.
(6) p is locally almost directed, i.e., for each $Y \in p$ one of the following holds:
  (6.1) $p \restriction Y$ is directed.
  (6.2) There are two sets $Y_0, Y_1 \in p$ and two ordinals $\alpha_0 \in Y_0$ and $\alpha_1 \in Y_1$, such that:
$$p \restriction Y = \{Y_0, Y_1\} \cup (p \restriction Y_0) \cup (p \restriction Y_1),$$
  and
$$Y_0 \cap Y_1 = Y_0 \cap \alpha_0 = Y_1 \cap \alpha_1.$$
(7) p is neat, i.e., for each $Y \in p$ satisfying $\rho_p(Y) > 0$, $Y = \bigcup(p \restriction Y)$.

The partial order on P is defined by $p \leq_P q$ iff $\max q \in p$ and $p \restriction \max q = q$.

Assuming the GCH it is obvious that P is a κ^{++}-closed, κ^{+3}-cc forcing notion, that GCH is preserved and it is not hard to see that a P-generic filter is a stationary neat simplified $\langle\kappa^{++}, 1\rangle$-morass. In fact if $\mathbb{M}$ is the generic filter, then for each cardinal $\lambda < \kappa^{++}$ the set $\{X \in \mathbb{M} \mid \mathrm{cf}(\rho_{\mathbb{M}}(X)) = \lambda\}$ is stationary. (This property is important to us since claim 3.12 depends on the set $\{X \in \mathbb{M} \mid \mathrm{cf}(\rho(X)) = \kappa^+\}$ being stationary.)

Let us use the notation $\sup^+(A)$, where A is a set of ordinals, to mean the minimal ordinal which is greater than each $\alpha \in A$, i.e.,
$$\sup{}^+(A) = \begin{cases} \sup A & \max A \text{ does not exist,} \\ (\max A) + 1 & \max A \text{ exists.} \end{cases}$$
It will be convenient to set $\sup^+ \emptyset = 0$.

Conditions in the forcing notion defined in the next section have as a component subset of $\mathbb{M}$ with certain properties. We call such subsets submorasses and define them next.

Definition 2.6. Assume $\mathbb{M}$ is a simplified $\langle\kappa^{++}, 1\rangle$-morass and $\lambda < \kappa$ is a cardinal.

- A family $M \in [\mathbb{M}]^{<\lambda}$ is called an $\mathbb{M}$-submorass (or just submorass if $\mathbb{M}$ is clear from context) if:
  (1) M has a maximum. I.e., there is a set $Z \in M$ such that $M \subseteq (\mathbb{M} \restriction Z) \cup \{Z\}$. We denote the set Z by $\max M$.
  (2) M is homogeneous, i.e., each two sets $Y_0, Y_1 \in M$ of the same $\mathbb{M}$-rank are of the same M-rank, and $M \restriction Y_1 = \{\pi''_{Y_0,Y_1} X \mid X \in M \restriction Y_0\}$.
  (3) M is locally almost directed, i.e., for each $Y \in M$ one of the following holds:
    (3.1) $M \restriction Y$ is directed.
    (3.2) There are sets $Y_0, Y_1 \in M$ such that $\langle Y, Y_0, Y_1\rangle$ is Δ-system (in $\mathbb{M}$) and $M \restriction Y = \{Y_0, Y_1\} \cup (M \restriction Y_0) \cup (M \restriction Y_1)$.
- A submorass $M \in [\mathbb{M}]^{<\lambda}$ covers a set of ordinals $O \in [\kappa^{+3}]^{<\lambda}$ if $\bigcup M \supseteq O$, and for each $\alpha \in O$:

(1) If $Y \in M$ and $\alpha < \sup^+ Y$ then $\min(Y \setminus \alpha) \in O$.
(2) If $Y_0, Y_1 \in M$ are of the same rank and $\alpha \in Y_0$ then $\pi_{Y_0,Y_1}(\alpha) \in O$.
(3) If the sets $Y, Y_0, Y_1 \in M$ form a Δ-system with witnessing ordinals α_0, α_1, then $\alpha_0, \alpha_1 \in O$.

In the next section we will reflect the morass $\mathbb{M}$ to cardinals below κ. By reflection we mean that the morass properties are being preserved by a small set of subsets. For this we need the following definition which is tailored to the very specific case we need.

Definition 2.7. Assume $\lambda < \kappa$ is a cardinal and $n < \omega$.

- A family of subsets $M \subseteq [\lambda^{+n+3}]^{\leq \lambda^{+n+1}}$ is a fake $\langle \lambda^{+n+2}, 1\rangle$-morass if:
  (1) $|M| < \lambda$.
  (2) M has a maximal element, i.e., there is a set $Z \in M$ such that for each $Y \in M$, $Y \subseteq Z$.
  (3) $\langle M, \subsetneq\rangle$ is well-founded. Define the M-rank function ρ_M by recursion:
  $$\rho_M(Y) = \sup\{\rho_M(X) + 1 \mid X \in M \restriction Y\}.$$
  (4) M is homogeneous, i.e., for each two sets $Y_0, Y_1 \in M$ of the same M-rank there is an order preserving bijection $\pi_{Y_0,Y_1} : Y_0 \to Y_1$ such that $M \restriction Y_1 = \{\pi''_{Y_0,Y_1} X \mid X \in M \restriction Y_0\}$.
  (5) M is locally almost directed, i.e., for each $Y \in M$ one of the following holds:
  (5.1) $M \restriction Y$ is directed.
  (5.2) There are sets $Y_0, Y_1 \in M$ and ordinals $\alpha_0 \in Y_0$, $\alpha_1 \in Y_1$ such that
  $$M \restriction Y = \{Y_0, Y_1\} \cup (M \restriction Y_0) \cup (M \restriction Y_1),$$
  and
  $$Y_0 \cap Y_1 = Y_0 \cap \alpha_0 = Y_1 \cap \alpha_1.$$
  The ordinals α_0 and α_1 will be called the witnessing ordinals for the Δ-system $\langle Y, Y_0, Y_1\rangle$.
- The fake $\langle \lambda^{+n+2}, 1\rangle$-morass M covers a set of ordinals $O \in [\lambda^{+n+3}]^{<\lambda}$ if $\bigcup M \supseteq O$, and for each $\alpha \in O$:
  (1) If $Y \in M$ and $\alpha < \sup^+ Y$, then $\min(Y \setminus \alpha) \in O$.
  (2) If $Y_0, Y_1 \in M$ are of the same M-rank and $\alpha \in Y_0$ then $\pi_{Y_0,Y_1}(\alpha) \in O$.
  (3) If the sets $Y, Y_0, Y_1 \in M$ form a Δ-system with witnessing ordinals α_0, α_1, then $\alpha_0, \alpha_1 \in O$.

3. Gap-3 forcing

Assume the GCH and let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of cardinals such that for each $n < \omega$ there is an elementary embedding $j_n : V \to M_n$ such that M_n is transitive, $\mathrm{crit}(j_n) = \kappa_n$, $M_n \supseteq M_n^{\kappa_n}$, and $j_n(\kappa_n) \geq \kappa_n^{+n+3}$. Let E_n be the $\langle \kappa_n, \kappa_n^{+n+3}\rangle$-extender derived from j_n. Without loss of generality assume that j_n is the natural embedding from V to $\mathrm{Ult}(V, E_n) \simeq M_n$. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Assume $\mathbb{M}$ is a neat simplified $\langle \kappa^{++}, 1\rangle$-morass such that $\{X \in \mathbb{M} \mid \mathrm{cf}(\rho_{\mathbb{M}}(X)) = \kappa^+\}$ is a stationary subset of $[\kappa^{+3}]^{\leq \kappa^+}$. Note that by the previous section we can force this morass while keeping all the assumptions on the cardinals κ_n.

We begin with the definition of good structures, continue with the universe for measures, and then proceed to the relevant measures.

Definition 3.1. • For each $n < \omega$ fix a cardinal $\chi_n < \kappa$ large relative to κ_n. Assume $k \leq n < \omega$. Define the following structure.

$$\mathscr{H}_{n,k} = \langle H(\chi_n^{+k}), \in, E_n, A, \xi \rangle_{A \in V_{\kappa_n+1},\ \xi < \kappa_n^{+k}}.$$

The important property of these structures, for our purpose, is that $\mathscr{H}_{n,k} \in \mathscr{H}_{n,k+1}$.

- Assume $k \leq n < \omega$ and $x \in \mathscr{H}_{n,k}$. The $\langle n, k\rangle$-type of x, denoted by $\mathrm{tp}_{n,k}(x)$, is defined to be the set of formulae with one free parameter holding in the structure $\mathscr{H}_{n,k}$ for the assignment x, i.e.,

$$\mathrm{tp}_{n,k}(x) = \{\ulcorner \phi(-) \urcorner \mid \mathscr{H}_{n,k} \vDash \phi(x)\}.$$

We will be interested in the types of elements in $[\kappa_n^{+n+3}]^{<\kappa_n} \cup \big[[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}\big]^{<\kappa_n}$. A type will be coded by an ordinal. I.e., $\mathrm{tp}_{n,k}(x) \in \kappa_n^{+k+1}$. Hence for $k < n$ there are constants in the language of the structure $\mathscr{H}_{n,k+1}$ for the $\mathscr{H}_{n,k}$-types.
- An ordinal $\alpha < \kappa_n^{+n+3}$ is called $\langle n, k\rangle$-good, or just k-good when n is clear if $\{\beta < \kappa_n^{+n+3} \mid \forall k' \leq k\ \ \mathrm{tp}_{n,k'}(\beta) = \mathrm{tp}_{n,k'}(\alpha)\}$ contains a club (We need only unboundedness of the last set in this work).
- An elementary substructure $N \prec \mathscr{H}_{n,k}$ codes an ordinal $\alpha < \kappa_n^{+n+3}$ if $\alpha = N \cap \kappa_n^{+n+3}$, $|N| = \kappa_n^{+n+2}$, and $N \supseteq N^{<\kappa_n}$. We use $\mathring{N}$ to denote the ordinal $N \cap \kappa_n^{+n+3}$.
- An elementary substructure $N \prec \mathscr{H}_{n,k}$ coding an ordinal is called k-good if $\mathring{N}$ is k-good. An elementary substructure $N \prec \mathscr{H}_{n,k}$ coding an ordinal is called good if it is k-good for some $k \leq n$.
- Assume $N_1 \prec \mathscr{H}_{n,k_1}$ and $N_2 \prec \mathscr{H}_{n,k_2}$ are good structures. We use the notation $N_1 \in^{\boldsymbol{\cdot}} N_2$ to mean $N_1 \cap \mathscr{H}_{n,k_2} \in N_2$.
- A family x of good structures codes an element of $[\kappa_n^{+n+3}]^{<\kappa_n}$ if:
  (1) $|x| < \kappa_n$.
  (2) N codes an ordinal $< \kappa_n^{+n+3}$ for each $N \in x$.
  (3) For each $N_1, N_2 \in x$ such that $N_1 \neq N_2$, $\mathring{N}_1 \neq \mathring{N}_2$.
  (4) If $N_1, N_2 \in x$ and $\mathring{N}_1 < \mathring{N}_2$, then $N_1 \in^{\boldsymbol{\cdot}} N_2$.
  We set $\mathring{x} = \{\mathring{N} \mid N \in x\}$.
- An elementary substructure $N \prec \mathscr{H}_{n,k}$ codes an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ if $|N| = \kappa_n^{+n+1}$ and $N \supseteq N^{<\kappa_n}$. We use $\mathring{N}$ to denote the subset $N \cap \kappa_n^{+n+3}$.
- A subset $a \in [\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ is called $\langle n, k\rangle$-good, or just k-good when n is clear if $\{b \in [\kappa_n^{+n+3}]^{\kappa_n^{+n+1}} \mid \forall k' \leq k\ \ \mathrm{tp}_{n,k'}(b) = \mathrm{tp}_{n,k'}(a)\}$ contains a club (We need only unboundedness of the last set in this work).
- An elementary substructure $N \prec \mathscr{H}_{n,k}$ coding an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ is called k-good if $\mathring{N}$ is k-good. An elementary substructure $N \prec \mathscr{H}_{n,k}$ coding an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ is called good if it is k-good for some $k \leq n$.
- A family x of good structures codes a $\langle \kappa_n^{+n+2}, 1\rangle$-fake morass if:
  (1) $\mathring{x} = \{\mathring{N} \mid N \in x\}$ is a $\langle \kappa_n^{+n+2}, 1\rangle$-fake morass.
  (2) N codes an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ for each $N \in x$.
  (3) For each $N_1, N_2 \in x$ such that $N_1 \neq N_2$, $\mathring{N}_1 \neq \mathring{N}_2$.
  (4) If $N_1, N_2 \in x$, and $\mathring{N}_1 \subsetneq \mathring{N}_2$, then $N_1 \in^{\boldsymbol{\cdot}} N_2$.
- Assume $M \in [\mathbb{M}]^{<\kappa_n}$ is a submorass covering the set $O \in [\kappa^{+3}]^{<\kappa_n}$. A function $a : M \cup O \to H(\chi_n^{+\omega})$ is called morass reflecting if:

(1) $a''O$ codes an element of $[\kappa_n^{+n+3}]^{<\kappa_n}$ which is covered by the $\langle \kappa_n^{+n+2}, 1\rangle$-fake morass coded by $a''M$.
(2) If $x, y \in M \cup O$ and $x \subsetneq y$ then $a(x) \in^{\cdot} a(y)$.
(3) If $A \in M$ and $\alpha \in A \cap O$ then $a(\alpha) \in^{\cdot} a(A)$.
(4) If $A \in M$, $\alpha \in A \cap O$, $\beta \in O$, and $\alpha < \beta$, then $a(A) \cap a(\alpha) \in^{\cdot} a(\beta)$.

Definition 3.2. Assume $d \in [\kappa^{+3}]^{<\kappa_n}$.

- The set $\mathrm{OB}_n(d)$ is composed of the order preserving functions $\nu : d \to \kappa_n \setminus \kappa_{n-1}$. (Consider κ_{-1} to be $\emptyset$).
- Assume $a : d \to H(\chi_n^{+\omega})$ is a function such that $\operatorname{ran} a$ codes an element of $[\kappa_n^{+n+3}]^{<\kappa_n}$.
  - The function $\mathring{a} : d \to \kappa_n^{+n+3}$ is defined by $\mathring{a}(\alpha) = a(\alpha) \cap \kappa_n^{+n+3}$.
  - The measure $E_n(a)$ is defined on $\mathrm{OB}_n(d)$ as follows:
$$\forall X \subseteq \mathrm{OB}_n(d) \ \big(X \in E_n(a) \iff \{\langle j_n(\alpha), \mathring{a}(\alpha)\rangle \mid \alpha \in d\} \in j_n(X)\big).$$
- Assume $e \subseteq d$ and $X \subseteq \mathrm{OB}_n(d)$. Then
$$X \restriction e = \{\nu \restriction e \mid \nu \in X\}.$$
  For finite products define: If $\langle d_n \mid l \leq n \leq m\rangle$ is $\subseteq$-increasing, $e \subseteq \bigcup_{l \leq n \leq m} d_n$, and $X \subseteq \prod_{l \leq n \leq m} \mathrm{OB}_n(d_n)$ then
$$X \restriction e = \{\langle \nu_l \restriction e, \dots, \nu_m \restriction e\rangle \mid \langle \nu_l, \dots, \nu_m\rangle \in X\}.$$

Definition 3.3. We define the forcing notion $\langle \mathbb{P}^*, \leq^*_{\mathbb{P}^*}\rangle$ and following it the forcing $\langle \mathbb{P}, \leq^*_{\mathbb{P}}\rangle$. The order $\leq^*_{\mathbb{P}}$ will be the Prikry order of the main forcing order $\leq_{\mathbb{P}}$ which will be defined later.

- A condition f is in the forcing notion $\mathbb{P}^*$ if $f : d \to {}^{<\omega}\kappa$ is a function such that:
  (1) $d \in [\kappa^{+3}]^{\leq\kappa}$.
  (2) For each $\alpha \in d$, $f(\alpha) = \langle f_0(\alpha), \dots, f_n(\alpha)\rangle \in {}^{<\omega}\kappa$ is an increasing sequence.
  The forcing notion $\mathbb{P}^*$ is equipped with the partial order $f \leq^*_{\mathbb{P}^*} g \iff f \supseteq g$. (Thus $\langle \mathbb{P}^*, \leq^*_{\mathbb{P}^*}\rangle$ is the Cohen forcing adding κ^{+3} subsets to κ^+).
- A condition $p = \langle f, \bar{a}, \bar{A}\rangle$ is in the forcing notion $\mathbb{P}$ if there is $l < \omega$ such that:
  (1) $f \in \mathbb{P}^*$.
  (2) $\bar{a} = \langle a_n : O_n \cup M_n \to H(\chi_n^{+\omega}) \mid l \leq n < \omega\rangle$ is a sequence of functions such that:
    (2.1) $\bigcup_{l \leq n < \omega} O_n = \operatorname{dom} f$.
    For each $l \leq n < \omega$:
    (2.2) $O_n \in [\kappa^{+3}]^{<\kappa_n}$ and $O_n \subseteq O_{n+1}$.
    (2.3) $M_n \in [\mathbb{M}]^{<\kappa_n}$ is a submorass covering O_n, $M_n \subseteq M_{n+1}$ and $\max M_n = \max M_{n+1}$.
    (2.4) The function a_n is morass reflecting.
  (3) Assume $x \in \bigcup_{l \leq n < \omega} \operatorname{dom} a_n$, let $l \leq n^* < \omega$ be minimal such that $x \in \operatorname{dom} a_{n^*}$, and assume that for each $n^* \leq n < \omega$, $a_n(x) \prec \mathscr{H}_{n,k_n}$. Then $\bigcup_{n^* \leq n < \omega} k_n = \omega$ and $\langle k_n \mid n^* \leq n < \omega\rangle$ is a non-decreasing sequence.
  (4) $\bar{A} = \langle A_n \mid l \leq n < \omega\rangle$ and for each $l \leq n < \omega$, $A_n \in E_n(a_n)$.
  We write l^p, f^p, $\bar{a}^p$, $\bar{A}^p$, a^p_n, A^p_n, O^p_n, O^p, M^p_n, M^p, $\max M^p$ and $\mathrm{Lev}_m(p)$, for l, f, $\bar{a}$, $\bar{A}$, a_n, A_n, O_n, $\bigcup_{l^p \leq n < \omega} O_n$, M_n, $\max M_n$ (for some $l \leq n < \omega$), $\bigcup_{l^p \leq n < \omega} M_n$, and $\prod_{l \leq n \leq l+m} A_n$, respectively.
- Let $p, q \in \mathbb{P}$. The condition p is a Prikry extension of q ($p \leq^*_{\mathbb{P}} q$) if:
  (1) $f^p \leq^*_{\mathbb{P}^*} f^q$.

(2) $l^p = l^q$ (we use l to denote the common value).
(3) For each $l \leq n < \omega$:
  (3.1) $a^p_n \supseteq a^q_n$.
  (3.2) $O^p_n \setminus O^q_n \subseteq \operatorname{dom} f^p \setminus \operatorname{dom} f^q$.
  (3.3) $A^p_n \restriction O^q_n \subseteq A^q_n$.

First we observe that $\leq^*_{\mathbb{P}}$ is finally κ-closed.

Claim 3.4. *(1) Assume $\langle p_\xi \mid \xi < \lambda < \kappa_{l^{p_0}} \rangle \subseteq \mathbb{P}$ is a $\leq^*$-decreasing sequence. Then there is a condition $p^* \in \mathbb{P}$ such that for each $\xi < \lambda$, $p^* \leq^* p_\xi$.*
(2) Assume $\langle p_\xi \mid \xi < \lambda < \kappa_{n^*} \rangle \subseteq \mathbb{P}$ is a $\leq^*$-decreasing sequence, and for each $\xi_0 < \xi_1 < \lambda$ and $l^p \leq n < n^*$, $a^{p_{\xi_0}}_n = a^{p_{\xi_1}}_n$ and $A^{p_{\xi_0}}_n = A^{p_{\xi_1}}_n$. Then there is a condition $p^* \in \mathbb{P}$ such that for each $\xi < \lambda$, $p^* \leq^* p_\xi$.

In [4] it is shown that every set from the morass and every ordinal less than κ^{+3} can be added to a condition. This is a very strong property and we were not able to reproduce it just from a morass. We suspect that the piste used in [4], which was called there walk, is necessary for this. We settle for less. The essential thing is to be able to add every ordinal less than κ^{+3} to a condition, and this is done in claim 3.6. The only restriction on addition of ordinal is that a set from the morass appearing in the condition contains this ordinal. Dealing with this restriction is easy, the coarest thing we can do is add a new maximal set to the condition, which we do in claim 3.5. We note that a major case from the original [4] proof of the stronger property, the addition of a Δ-system, has been moved here to claim 3.11.

Claim 3.5. *Assume $p \in \mathbb{P}$ is a condition, $Y \in \mathbb{M}$, and $\max M^p \subsetneq Y$. Then there is a Prikry extension $p^* \leq^* p$ such that $\max M^{p^*} = Y$.*

Proof. For each $l^p \leq n < \omega$ choose an elementary substructure $Y^{\centerdot}_n \prec \mathscr{H}_{n,n}$ coding an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ such that for each $X \in M^p_n$, $a_n(X) \in^{\centerdot} Y^{\centerdot}_n$. Let $p^* \leq^* p$ be a Prikry extension satisfying $f^{p^*} = f^p$, $\bar{A}^{p^*} = \bar{A}^p$, and for each $l^p \leq n < \omega$, $a^{p^*}_n = a^p_n \cup \{\langle Y, Y^{\centerdot}_n \rangle\}$. $\square$

Claim 3.6. *Assume $p \in \mathbb{P}$ is a condition and $\beta < \kappa^{+3}$. Then there is a Prikry extension $p^* \leq^* p$ such that $\beta \in O^{p^*}$.*

Proof. If $\beta \in O^p$ then there is nothing to do. Thus for the following assume that $\beta \notin O^p$. By claim 3.5 we can assume without loss of generality that $\beta \in \max M^p$. Set $l = l^p$. The proof is by recursion and is splitted into two cases. The first case is the non-recursive one and where the substantial work is being done. The second case invokes recursively the claim and the work there is to show that the recursion terminates at some point.

(1) Assume $\min(X \setminus \beta) \in O^p$ for each $X \in M^p$ such that $\beta \notin X$ and $\beta < \sup^+ X$: Less formally, in this case the requirements for the covering of β by M^p are satisfied, thus we only need to add β to O^p. This requires the extension of a^p_n in accordance with the current information in a^p_n. Thus we need to find structures $\beta^{\centerdot}_n$ such that if $\alpha < \beta$ and $\alpha \in O^p_n$, then $a^p_n(\alpha) \in^{\centerdot} \beta^{\centerdot}_n$; if $\beta < \gamma$ and $\gamma \in O^p_n$, then $\beta^{\centerdot}_n \in^{\centerdot} a^p_n(\gamma)$; and if $\beta \in X \in M^p_n$ then $\beta^{\centerdot}_n \in^{\centerdot} a^p_n(X)$. Moreover, in general we cannot add β alone due to the closure demands on O^p_n imposed by M^p_n. Thus in addition to β we need to add to O^p_n also $\pi_{X,X'}(\beta)$ whenever $X, X' \in M^p_n$ are of the same rank and $\beta \in X$. Luckily after $\beta^{\centerdot}_n$ was chosen the addition of the projections is easy: If $X, X' \in M^p_n$ then $X^{\centerdot}_n = a^p_n(X)$ and $X'^{\centerdot}_n = a^p_n(X')$ are

defined and there is an elementary embedding $\pi_{X^\bullet_n, X'^\bullet_n} : X^\bullet_n \to X'^\bullet_n$. If $\beta \in X$ then we will be done by sending $\pi_{X,X'}(\beta)$ to $\pi_{X^\bullet_n, X'^\bullet_n}(\beta^\bullet_n)$. Let us now work the above formally.

Let a set $X \in M^p$ be of minimal rank such that $\beta \in X$. Set $X^\bullet_n = a^p_n(X)$ for each $l \leq n < \omega$ such that $X \in M^p_n$. If $O^p \cap [\beta, \sup^+ X) = \emptyset$ then set $\gamma^\bullet_n = \mathscr{H}_{n,n}$ for each $l \leq n < \omega$. Otherwise set $\gamma = \min(O^p \setminus \beta)$ and $\gamma^\bullet_n = a^p_n(\gamma)$ for each $l \leq n < \omega$ such that $\gamma \in O^p_n$. Let $l \leq n^* < \omega$ be minimal such that $X^\bullet_{n^*}$ and $\gamma^\bullet_{n^*}$ are defined and are at least 2-good structures. For each $n^* \leq n < \omega$ work as follows. Let $k < n$ be maximal such that $\mathscr{H}_{n,k+1} \in X^\bullet_n \cap \gamma^\bullet_n$. Choose in $X^\bullet_n \cap \gamma^\bullet_n$ a $k+1$-good structure $\beta^\bullet_n$ coding an ordinal $< \kappa_n^{+n+3}$ such that for each $\alpha \in O^p_n \cap X \cap \beta$, $a^p_n(\alpha) \in^\bullet \beta^\bullet_n$, and for each $Y \in M^p_n \restriction X$, $a^p_n(Y) \cap a^p_n(\min(Y \setminus \beta)) \in^\bullet \beta^\bullet_n$.

Note that in constructing $\beta^\bullet_n$ we did not take into consideration explicitly ordinals which might be in $(O^p_n \cap [\min X, \beta)) \setminus X$. The reason is that if $\alpha \in (O^p_n \cap [\min X, \beta)) \setminus X$ then $\min(X \setminus \alpha) \in O^p_n$ hence α was taken into consideration implicitly.

Note also that by working with X of minimal rank above we get immediately that if $X \subsetneq Y \in M^p$ then $\beta^\bullet \in^\bullet a^p_n(Y)$.

The construction of the condition p^* for which $\langle \beta, \beta^\bullet_n \rangle \in a^{p^*}_n$ is done as follows. Let $p^* \leq^* p$ be a Prikry extension satisfying $f^{p^*} = f^p$, for each $l \leq n < n^*$, $a^{p^*}_n = a^p_n$ and $A^{p^*}_n = A^p_n$, and for each $n^* \leq n < \omega$,

$$a^{p^*}_n = a^p_n \cup \{\langle \pi_{X,X'}(\beta), \pi_{a^p_n(X), a^p_n(X')}(\beta^\bullet_n) \rangle \mid X' \in M^p_n,\ \rho(X') = \rho(X)\},$$

$A^{p^*}_n \in E(a^{p^*}_n)$ and $A^{p^*}_n \restriction \operatorname{dom} a^p_n \subseteq A^p_n$.

(2) Assume there is a set $X \in M^p$ such that $\beta \notin X$, $\beta < \sup^+ X$ and $\min(X \setminus \beta) \notin O^p$: What the above conditions means is there is at least one set X for which the covering demands for β are not satisfied, i.e., $\min(X \setminus \beta) \notin O$. Thus we should add $\min(X \setminus \beta)$ to O^p, which might add another ordinal, etc. This is a recursive operation, and we need to show that the recursiveness will halt at certain point. The following facts are used to show that indeed the recursive step halts.

(2.1) Assume the sets $Y_0, Y_1 \in M^p$ are of the same rank, $\beta < \sup^+ Y_0$ and $\beta < \sup^+ Y_1$. Then either $\min(Y_0 \setminus \beta) = \min(Y_1 \setminus \beta)$ or $\min(Y_i \setminus \beta) \in O^p$ for some $i < 2$.

Proof. Without loss of generality assume $\sup^+ Y_0 < \sup^+ Y_1$. Find in M^p a Δ-system of minimal rank $\langle Z, Z_0, Z_1 \rangle$ with witnessing ordinals $\alpha_i \in Z_i$ $(i < 2)$, such that $Z_i \supseteq Y_i$ $(i < 2)$. If $\beta \in [\sup^+(Y_1 \cap \alpha_0), \alpha_1)$ then $\min(Y_1 \setminus \beta) = \min(Y_1 \setminus \alpha_1) \in O^p$ and we are done. If $\beta \in [\sup^+(Y_0 \cap \alpha_0), \alpha_0)$ then $\min(Y_0 \setminus \beta) = \min(Y_0 \setminus \alpha_0) \in O^p$ and we are done. If $\beta < \min(\sup^+(Y_0 \cap \alpha_0), \sup^+(Y_1 \cap \alpha_0))$ then work as follows. If $\pi''_{Z_1,Z_0} Y_1 = Y_0$ then $Y_0 \cap \alpha_0 = Y_1 \cap \alpha_0$, hence $\min(Y_0 \setminus \beta) = \min(Y_1 \setminus \beta)$ and we are done. Otherwise set $Y'_1 = \pi''_{Z_1,Z_0} Y_1$ and $Y'_0 = Y_0$. Note that $\min(Y'_i \setminus \beta) = \min(Y_i \setminus \beta)$ for each $i < 2$. We are done by noting that by recursion either $\min(Y'_0 \setminus \beta) = \min(Y'_1 \setminus \beta)$ or $\min(Y'_i \setminus \beta) \in O^p$ for some < 2. $\square$

(2.2) Assume $Y \in M^p$, $\beta < \sup^+ Y$, and $\min(Y \setminus \beta) \notin O^p$. Then for each $Z \in M^p$ of rank $\geq \rho(Y)$ such that $\beta < \sup^+ Z$, either $\min(Z \setminus \beta) \in O^p$ or $\min(Z \setminus \beta) \leq \min(Y \setminus \beta)$.

Proof. If $\rho(Z) = \rho(Y)$ then we are done by the first fact. Thus assume $\rho(Z) > \rho(Y)$. If $Z \supsetneq Y$ then trivially $\min(Z \setminus \beta) \leq \min(Z \setminus \beta)$ and we are done. Hence assume $Z \not\supseteq Y$. Choose $Z' \in M^p$ such that $\rho(Z) = \rho(Z')$ and $Z' \supsetneq Y$. Trivially $\min(Z' \setminus \beta) \leq \min(Y \setminus \beta)$. By the first fact we can have one of three cases. If $\min(Z \setminus \beta) = \min(Z' \setminus \beta)$ or $\min(Z \setminus \beta) \in O^p$ then we are done. We are left with the case $\min(Z' \setminus \beta) \in O^p$ and $\min(Z \setminus \beta) \notin O^p$. Set $\tau = \min(Z' \setminus \beta)$. If $\tau < \min(Z \setminus \beta)$ then $\min(Z \setminus \beta) = \min(Z \setminus \tau) \in O^p$, which is a contradiction. Hence $\min(Z \setminus \beta) \leq \tau \leq \min(Y \setminus \beta)$ and we are done. $\square$

The upshot of the above two facts is:

(2.1) $O = \{\min(X \setminus \beta) \notin O^p \mid \beta \notin X \in M^p,\ \beta < \sup^+ X\}$ is finite.

(2.2) $\max\big\{ \min\{\rho(X) \mid \beta' \in X \in M^p\} \big| \beta' \in O\big\} < \min\{\rho(X) \mid \beta \in X \in M^p\}$.

Thus we can invoke the lemma by recursion for $|O|$-many times constructing a Prikry extension $p' \leq^* p$ for which $O \subseteq O^{p'}$. Now we can invoke the recursion again letting $p^* \leq^* p'$ be a Prikry extension such that $\beta \in O^{p^*}$.

$\square$

In fact the above proof yields a claim somewhat stronger than what was stated in claim 3.6:

Claim 3.7. *Assume $p \in \mathbb{P}$ is a condition, $l^p \leq n < \omega$, and $\beta < \kappa^{+3}$. Then there is a Prikry extension $p^* \leq^* p$ such that $\beta \in O^{p^*}$ and for each $l^p \leq m < n$, $a^{p^*}_m = a^p_m$.*

We are ready to define the forcing order $\leq_{\mathbb{P}}$ of $\mathbb{P}$.

Definition 3.8. • Assume $f \in \mathbb{P}^*$, $\nu \in \mathrm{OB}_n(d)$, where $d \in [\operatorname{dom} f]^{<\kappa_n}$. Define the condition $f_{\langle \nu \rangle} \in \mathbb{P}^*$ to be the function $g \in \mathbb{P}^*$ with domain $\operatorname{dom} f$ satisfying for each $\alpha \in \operatorname{dom} g$,

$$g(\alpha) = \begin{cases} f(\alpha) ^\frown \langle \nu(\alpha) \rangle & \alpha \in \operatorname{dom} \nu,\ \nu(\alpha) > \max f(\alpha), \\ f(\alpha) & \text{Otherwise.} \end{cases}$$

Assume $\langle \nu_l, \ldots, \nu_{m-1} \rangle \in \prod_{l \leq n < m} \mathrm{OB}_n(d_n)$ where $d_n \in [\operatorname{dom} f]^{<\kappa_n}$. Define the condition $f_{\langle \nu_l, \ldots, \nu_{m-1} \rangle} \in \mathbb{P}^*$ recursively as $(f_{\langle \nu_l, \ldots, \nu_{m-2} \rangle})_{\langle \nu_{m-1} \rangle}$.

- Assume $p \in \mathbb{P}$. By writing $\langle \nu_{l^p}, \ldots, \nu_{n-1} \rangle \in \bar{A}^p$ we mean that $\langle \nu_{l^p}, \ldots, \nu_{n-1} \rangle \in \mathrm{Lev}_{n-l^p-1}(p)$.
- Assume $p \in \mathbb{P}$ and $\langle \nu_{l^p}, \ldots, \nu_{m-1} \rangle \in \bar{A}^p$. By $\bar{a}^p_{\langle \nu_{l^p}, \ldots, \nu_{m-1} \rangle}$ and $\bar{A}^p_{\langle \nu_{l^p}, \ldots, \nu_{m-1} \rangle}$ we mean the sequences $\langle a^p_n \mid m \leq n < \omega \rangle$ and $\langle A^p_n \mid m \leq n < \omega \rangle$, respectively.
- Assume $p \in \mathbb{P}$ and $\langle \nu \rangle \in \bar{A}^p$. Define the condition $p_{\langle \nu \rangle} \in \mathbb{P}$ to be $\langle f^p_{\langle \nu \rangle}, \bar{a}^p_{\langle \nu \rangle}, \bar{A}^p_{\langle \nu \rangle} \rangle$.
- Assume $p \in \mathbb{P}$ and $\langle \nu_l, \ldots, \nu_{n-1} \rangle \in \bar{A}^p$. Define recursively $p_{\langle \nu_l, \ldots, \nu_{n-1} \rangle} \in \mathbb{P}$ to be the condition $(p_{\langle \nu_l, \ldots, \nu_{n-2} \rangle})_{\langle \nu_{n-1} \rangle} \in \mathbb{P}$.

The natural way to define the forcing order would have been to extend a condition p to a Prikry extension of $p_{\langle \nu_l, \ldots, \nu_{n-1} \rangle}$. Alas, this definition collapses κ^{++}. In order to restrict the length of the antichains, we identify conditions with different a's. This is done according to the types of the a's defined as follows.

- Assume $p, q \in \mathbb{P}$. We say that p is an extension of q ($p \leq_{\mathbb{P}} q$) if there is $\langle \nu_{l^q}, \ldots, \nu_{l^p-1} \rangle \in \bar{A}^q$, and a non-decreasing sequence $\langle k_n \leq n \mid l^p \leq n < \omega \rangle$, such that:
  (1) $f^p \leq^*_{\mathbb{P}^*} f^q_{\langle \nu_{l^q}, \ldots, \nu_{l^p-1} \rangle}$.
  (2) $\bigcup_{l^p \leq n < \omega} k_n = \omega$.

(3) For each $l^p \leq n < w$:
  (3.1) $\operatorname{dom} a^p_n \supseteq \operatorname{dom} a^q_n$.
  (3.2) $O^p_n \setminus O^q_n \subseteq \operatorname{dom} f^p \setminus \operatorname{dom} f^q$.
  (3.3) $A^p_n \restriction O^q_n \subseteq A^q_n$.
  (3.4) $\operatorname{tp}_{n,k_n}(\operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n)) = \operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{a}^q_n)$.

Observe that $\leq_{\mathbb{P}} \supseteq \leq^*_{\mathbb{P}}$. Note that $p <_{\mathbb{P}} q$ implies $l^p \geq l^q$ and not necessarily $l^p > l^q$. It might happen that $l^p = l^q$ if there is only a type change with no extension of the Prikry sequences f^q.

In view of claim 3.6 the following definition makes sense.

Definition 3.9. Assume $G \subset \mathbb{P}$ is generic. Define the function f^G by setting for each $\alpha < \kappa^{+3}$, $f^G(\alpha) = \bigcup\{f^p(\alpha) \mid p \in G,\ \alpha \in \operatorname{dom} f^p\} = \langle f^G_n(\alpha) \mid n < \omega\rangle$. For each $\alpha < \kappa^{+3}$ there is $k_\alpha < \omega$ such that $\langle f^G_{k_\alpha+n}(\alpha) \mid n < \omega\rangle \in \prod_{n<\omega} \kappa_n$. Define for each $\alpha < \kappa^{+3}$, $G^\alpha = \langle f^G_{k_\alpha+n}(\alpha) \mid n < \omega\rangle$.

It is immediate that the sequence $\langle G^\alpha \mid \alpha < \kappa^{+3}\rangle$ is increasing, thus we have:

Corollary 3.10. *(In $V[G]$) $2^\kappa \geq |(\kappa^{+3})^V|$.*

Note that Gitik's original forcing satisfies the κ^{++}-cc. We do not know how to get this stronger chain condition without having a piste on the morass.

Claim 3.11. *The forcing notion $\mathbb{P}$ satisfies the κ^{+3}-cc.*

Proof. Let $E = \{p_\xi \mid \xi < \kappa^{+3}\} \subseteq \mathbb{P}$ be a set of κ^{+3} conditions. Shrink E several times in succession until the following is obtained:

(1) There is $l < \omega$ such that for each $\xi < \kappa^{+3}$, $l^{p_\xi} = l$.
(2) There is $\rho < \kappa^{++}$ such that $\rho(\max M^{p_\xi}) = \rho$ for each $\xi < \kappa^{+3}$.
(3) The family $\{\max M^{p_\xi} \mid \xi < \kappa^{+3}\}$ is a strong Δ-system, where a Δ-system is called strong if for each two sets A, B in the Δ-system, there are ordinals α, β such that $A \cap \alpha = B \cap \beta$ and either $A \subseteq \beta$ or $B \subseteq \alpha$.
(4) The set $\{\operatorname{dom} f^{p_\xi} \mid \xi < \kappa^{+3}\}$ is a strong Δ-system.

For each $\xi_0 < \xi_1 < \kappa^{+3}$,

(5) $f^{p_{\xi_0}}$ and $f^{p_{\xi_1}}$ are compatible in $\mathbb{P}^*$.
(6) For each $l \leq n < \omega$, $\operatorname{ran} a^{p_{\xi_0}}_n = \operatorname{ran} a^{p_{\xi_1}}_n$.
(7) For each $l \leq n < \omega$, $\{\operatorname{ran}\nu \mid \nu \in A^{p_{\xi_0}}_n\} = \{\operatorname{ran}\nu \mid \nu \in A^{p_{\xi_1}}_n\}$.

Fix $\xi_0 < \xi_1 < \kappa^{+3}$. We claim that the conditions p_{ξ_0} and p_{ξ_1} are compatible. Set $Y_i = \max M^{p_{\xi_i}}$, where $i < 2$. Since $\rho(Y_0) = \rho(Y_1)$ there is an order preserving bijection $\pi : Y_0 \to Y_1$. The bijection π is a composition of order preserving bijections derived from Δ-systems. I.e., there are Δ-systems $\langle\langle X^{(k)}, X^{(k)}_0, X^{(k)}_1\rangle \mid k < n\rangle$ such that $Y_0 \subseteq X^{(k)}_0 \subsetneq X^{(k+1)}_0$ $(k < n)$, $Y_1 \subseteq X^{(n-1)}_1$, and $\pi = \pi_{X^{(n-1)}_0, X^{(n-1)}_1} \circ \cdots \circ \pi_{X^{(0)}_0, X^{(0)}_1}$. In the construction we will need that the reflection of the sets $X^{(k)}_0$ will be at least 2-good. This will be immediate for $l \geq 2$ except for the case of $X^{(0)}_0$ which might be Y_0 and already has reflection value. We fix this by setting $q_i = p_{\xi_i \langle \nu^i_0, \ldots, \nu^i_j\rangle}$, where $i < 2$ and $\operatorname{ran}\nu^0_0 = \operatorname{ran}\nu^1_0$, $\ldots$, $\operatorname{ran}\nu^0_j = \operatorname{ran}\nu^1_j$, so that $l^{q_i} \geq 2$ and $a^{q_i}_{l^{q_i}}(Y_i)$ is at least 2-good.

Set $p^0 = q_0$ and proceed by induction constructing a $\leq^*$-decreasing sequence $\langle p^k \mid k \leq n\rangle$, for which $\max M^{p^{k+1}} = X^{(k)}$ and $X^{(k)}_0, X^{(k)}_1 \in M^{p^{k+1}}$, as follows.

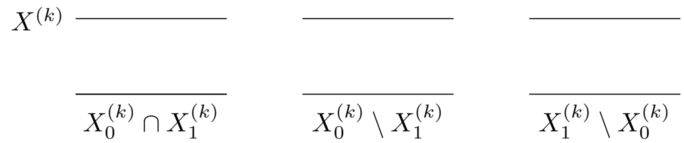

FIGURE 1. The two top levels of $M^{p^{k+1}}$ when $\beta_0 < \beta_1$.

Assume that p^k was constructed and construct p^{k+1} as follows. Let $\beta_0 \in X_0^{(k)}$ and $\beta_1 \in X_1^{(k)}$ be the Δ-system witnesses, i.e., $X_0^{(k)} \cap X_1^{(k)} = X_0^{(k)} \cap \beta_0 = X_1^{(k)} \cap \beta_1$. By claim 3.5 and claim 3.6 there is a Prikry extension $p' \leq^* p^k$ such that $X^{(k)} = \max M^{p'}$, $X_0^{(k)} \in M^{p'}$ and $\beta_0 \in O^{p'}$.

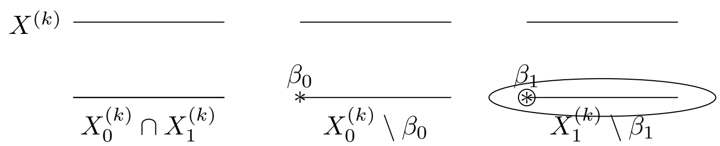

FIGURE 2. The two top levels of $M^{p'}$ when $\beta_0 < \beta_1$.

Set $\beta^{\bullet}_{0,n} = a_n^{p'}(\beta_0)$ for each $l \leq n < \omega$ such that $\beta_0 \in O_n^{p'}$, and for each $l \leq n < \omega$ set $X^{\bullet}_{0,n} = a_n^{p'}(X_0^{(k)})$ and $X^{\bullet}_n = a_n^{p'}(X^{(k)})$. In order to construct $X^{\bullet}_{1,n}$ and $\beta^{\bullet}_{1,n}$ we split the handling according to whether $\beta_0 < \beta_1$ or $\beta_1 < \beta_0$.

Assume $\beta_0 < \beta_1$: Let $l \leq n^* < \omega$ be minimal such that $X^{\bullet}_{n^*}$ is at least 2-good and $\beta^{\bullet}_{0,n^*}$ is at least 1-good. For each $n^* \leq n < \omega$ construct $X^{\bullet}_{1,n}$ and $\beta^{\bullet}_{1,n}$ as follows. Let $k < \omega$ be maximal such that $\mathscr{H}_{n,k+1} \in^{\bullet} X^{\bullet}_n$. Choose in $X^{\bullet}_n$ a $k+1$-good structure $\beta^{\bullet}_{1,n}$ coding an ordinal $< \kappa_n^{+n+3}$ such that $\beta^{\bullet}_{0,n} \in^{\bullet} \beta^{\bullet}_{1,n}$ and

$$\mathrm{tp}_{n,k+1}(\langle \mathring{\beta}^{\bullet}_{1,n}, \mathring{X}^{\bullet}_{0,n} \cap \mathring{\beta}^{\bullet}_{0,n} \rangle) = \mathrm{tp}_{n,k+1}(\langle \mathring{\beta}^{\bullet}_{0,n}, \mathring{X}^{\bullet}_{0,n} \cap \mathring{\beta}^{\bullet}_{0,n} \rangle).$$

Then choose in $X^{\bullet}_n$ a k-good structure $X^{\bullet}_{1,n}$ coding an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ such that $\beta^{\bullet}_{1,n} \in X^{\bullet}_{1,n}$ and $\mathrm{tp}_{n,k}(\langle \mathring{X}^{\bullet}_{1,n}, \mathring{\beta}^{\bullet}_{1,n} \rangle) = \mathrm{tp}_{n,k}(\langle \mathring{X}^{\bullet}_{0,n}, \mathring{\beta}^{\bullet}_{0,n} \rangle)$.

Assume $\beta_1 < \beta_0$:

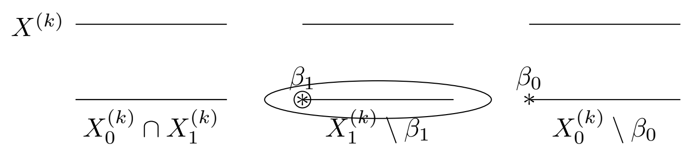

FIGURE 3. The two top levels of $M^{p'}$ when $\beta_1 < \beta_0$.

Since $X_1^{(k)} \notin M^{p'}$ it is immediate that $O^{p'} \cap [\beta_1, \beta_0) = \emptyset$. Let $l \leq n^* < \omega$ be minimal such that $X^{\bullet}_{n^*}$ and $\beta^{\bullet}_{0,n^*}$ are defined and are at least 2-good structures. For each $n^* \leq n < \omega$ construct $X^{\bullet}_{1,n}$ and $\beta^{\bullet}_{1,n}$ as follows. Let $k < \omega$ be maximal such that $\mathscr{H}_{n,k+1} \in X^{\bullet}_n \cap \beta^{\bullet}_{0,n}$. Choose in $X^{\bullet}_n \cap \beta^{\bullet}_{0,n}$ a $k+1$-good structure $\beta^{\bullet}_{1,n}$ coding an ordinal $< \kappa_n^{+n+3}$ such that

$$\mathrm{tp}_{n,k+1}(\langle \mathring{\beta}^{\bullet}_{1,n}, \mathring{X}^{\bullet}_{0,n} \cap \mathring{\beta}^{\bullet}_{0,n} \rangle) = \mathrm{tp}_{n,k+1}(\langle \mathring{\beta}^{\bullet}_{0,n}, \mathring{X}^{\bullet}_{0,n} \cap \mathring{\beta}^{\bullet}_{0,n} \rangle).$$

Then choose in $X^{\centerdot}_n \cap \beta_{\dot{0},n}$ a k-good structure $X_{\dot{1},n}$ coding an element of $[\kappa_n^{+n+3}]^{\kappa_n^{+n+1}}$ such that $\beta_{\dot{1},n} \in X_{\dot{1},n}$ and $\operatorname{tp}_{n,k}(\langle \mathring{X}_{\dot{1},n}, \mathring{\beta}_{\dot{1},n}\rangle) = \operatorname{tp}_{n,k}(\langle \mathring{X}_{\dot{0},n}, \mathring{\beta}_{\dot{0},n}\rangle)$.

Having constructed $\beta_{\dot{1},n}$ and $X_{\dot{1},n}$ we proceed to construct the condition p^{k+1} as follows. Set $f^{p^{k+1}} = f^{p'}$. For each $l \leq n < n^*$ set $a_n^{p^{k+1}} = a_n^{p'}$ and $A_n^{p^{k+1}} = A_n^{p'}$. For each $n^* \leq n < \omega$ let the function a_n be the closure under $\langle X^{(k)}, X_0^{(k)}, X_1^{(k)}\rangle$ of $M^{p'} \restriction X^{(k)}$, i.e.,

$$
\begin{aligned}
a_n^{p^{k+1}} = \{\langle X^{(k)}, X^{\centerdot}_n\rangle\} \cup a_n^{p^k} \cup \\
\{\langle \pi''_{X_0^{(k)}, X_1^{(k)}} X, \pi_{X_{\dot{0},n}, X_{\dot{1},n}}(a_n^{p'}(X))\rangle \mid X \in M_n^{p'} \restriction X^{(k)}\} \cup \\
\{\langle \pi_{X_0^{(k)}, X_1^{(k)}}(\beta), \pi_{X_{\dot{0},n}, X_{\dot{1},n}}(a_n^{p'}(\beta))\rangle \mid \beta \in O_n^{p'} \setminus X_1^{(k)}\}.
\end{aligned}
$$

For each $n^* \leq n < \omega$ choose $A_n^{p^{k+1}} \in E_n(a_n^{p^{k+1}})$ such that $A_n^{p^{k+1}} \restriction \operatorname{dom} f_n^{p'} \subseteq A_n^{p'}$.

At stage n the induction terminates and it is not hard to see that $p^n \leq q_1$. Note that in general we cannot expect $p^n \leq^* q_1$ since the bijection $\pi_{a_n(Y_0), a_n(Y_1)}$ will not necessarily bring us the exact values of $a_n^{q_1}$. However it will bring us to the same type of $\operatorname{ran} a_n^{q_1}$. $\square$

Assume χ is large enough and let $N \prec H_\chi$ be an elementary substructure such that $\mathbb{P} \in N$. A condition $p \in \mathbb{P}$ is $\langle N, \mathbb{P}\rangle$-generic [13] if for each dense open subset $D \in N$ of $\mathbb{P}$, $p \Vdash$ "$D \cap \underset{\sim}{G} \cap N \neq \emptyset$".

The forcing notion $\mathbb{P}$ is $\mathbb{M}$-proper [10] if for each $\mathbb{M}$-admissible elementary substructure $N \prec H_\chi$ such that $\mathbb{P} \in N$ and a condition $p \in \mathbb{P} \cap N$, there is a stronger condition $p^* \leq p$ which is $\langle N, \mathbb{P}\rangle$-generic. Since the set of $\mathbb{M}$-admissible substructures of H_χ is stationary, the argument for preservation of κ^{++} by an $\mathbb{M}$-proper forcing notion is the same as for proper forcing [13].

We compensate for having only κ^{+3}-cc by $\mathbb{M}$-properness.

Claim 3.12. *The forcing $\mathbb{P}$ is $\mathbb{M}$-proper.*

Proof. Let χ be large enough, $N \prec H_\chi$ be $\mathbb{M}$-admissible, $\operatorname{cf} \rho_{\mathbb{M}}(N \cap \kappa^{++}) = \kappa^+$ and $p, \mathbb{P} \in N$. Set $Y = N \cap \kappa^{+3}$. By claim 3.5 there is a Prikry extension $p^* \leq^* p$ such that $Y = \max M^{p^*}$. We claim that p^* is $\langle N, \mathbb{P}\rangle$-generic. In order to show this let $D \in N$ be a dense open subset of $\mathbb{P}$. We will be done by showing that $D \cap N$ is predense below p^*. That is for each condition $q \leq p^*$ we need to present a condition $r \in N \cap \mathbb{P}$ such that $q \parallel r$.

Thus let us fix some $q \leq p^*$. If needed extend q so that $a_{l^q}^q(\max M^q)$ will be at least $\langle l^q, 1\rangle$-good. Since $|M^q \cup O^q| \leq \kappa$ and $\operatorname{cf} \rho_{\mathbb{M}}(Y) = \kappa^+$, there is a set $X \in \mathbb{M} \cap N$ such that $M^q \cap N \subseteq \mathbb{M} \restriction X$ and $o^q \cap N \subseteq X$. For each $l^q \leq n < \omega$ pick an elementary substructure $X^{\centerdot}_n \prec \mathscr{H}_{n,n}$ such that for each $W \in M_n^q \cap N$, $a_n(W) \in^{\centerdot} X^{\centerdot}_n$, and for each $a \in O_n^q \cap N$, $a_n(\alpha) \in^{\centerdot} X^{\centerdot}_n$. Define the condition q_N as follows:

$$
\begin{aligned}
&f^{q_N} = f^q \restriction N, \\
&a_n^{q_N} = (a_n^q \restriction N) \cup \{\langle X, X^{\centerdot}_n\rangle\} \text{ for each } l^q \leq n < \omega,
\end{aligned}
$$

and

$$
A^{q_N} = A^q \restriction N.
$$

Then $q_N \in N$. By the density of the set D there is a condition $r' \leq q_N$ such that $r' \in D \cap N$. Note that since $Y \notin N$ the condition r' does not respect necessarily the restrictions imposed by the $a^q_n(Y)$'s. However, this can be fixed rather easily. For each $l^{r'} \le n < \omega$ let

$$t_n = \mathrm{tp}_{n,k_n}(\langle \operatorname{ran} a^{r'}_n \restriction \operatorname{dom} a^{q_N}_n, \operatorname{ran} a^{r'}_n \restriction (\operatorname{dom} a^{r'}_n \setminus \operatorname{dom} a^{q_N}_n)\rangle),$$

where $a^q_n(Y) \prec \mathscr{H}_{n,k_n+1}$. Then let $x_n \in a^q_n(Y)$ realize the type t_n, i.e.,

$$t_n = \mathrm{tp}_{n,k_n}(\langle \operatorname{ran} a^{r'}_n \restriction \operatorname{dom} a^{q_N}_n, x_n\rangle),$$

and let $r \le r'$ be a condition for which $\operatorname{ran} a^r_n = x_n \cup \operatorname{ran}(a^{r'}_n \restriction \operatorname{dom} a^{q_N}_n)$. The condition $s \le r, q$ is defined as follows. Set $f^s = f^r \cup f^q$,

$$\begin{aligned} a^s_n = a^q_n \restriction \big(\{Z \in M^q_n \mid \rho(Z) \ge \rho(Y)\} \cup (O^q_n \setminus O^r_n)\big) \cup a^r_n \cup \\ \{\langle \pi''_{Y,Y'}(W), \pi_{a^q_n(Y),a^q_n(Y')}(a^r_n(W))\rangle \mid W \in M^r_n,\ Y' \in M^q_n,\ \rho(Y') = \rho(Y)\} \cup \\ \{\langle \pi_{Y,Y'}(\alpha), \pi_{a^q_n(Y),a^q_n(Y')}(a^r_n(\alpha))\rangle \mid \alpha \in O^r_n,\ Y' \in M^q_n,\ \rho(Y') = \rho(Y)\} \end{aligned}$$

and choose $A^s_n \in E_n(a^s_n)$ such that $A^s_n \restriction \operatorname{dom} f^q_n \subseteq A^q_n$ and $A^s_n \restriction \operatorname{dom} f^r_n \subseteq A^r_n$. $\square$

Claims 3.13 to 3.17 are verbatim copies of the corresponding claims in [12]. They are repeated here for the sake of completeness. These claims contain the proofs of the Prikry property and the preservation of κ^+. Of course, the hidden point is the changed definition of the extension $p_{\langle\nu\rangle}$ in the transition from [12] to the current paper.

Let us specify two properties of the order $\le_{\mathbb{P}}$ which are not obvious at once and are at the core of the Prikry property of the forcing notion.

It can happen that two conditions satisfy both $p \le_{\mathbb{P}} q$ and $f^p \le^*_{\mathbb{P}^*} f^q$ but $p \not\le^*_{\mathbb{P}} q$. That is changing only the values of the a_n's while keeping their types is a non-Prikry extension. This allows the Prikry order to be closed.

Another fact is as follows. Suppose $p = \langle f, \bar{a}, \bar{A}\rangle$, $q = \langle f, \bar{b}, \bar{A}\rangle$, and $l^p = l^q = l$. Moreover, assume that $\mathrm{tp}_{n,k_n}(\operatorname{ran} \mathring{a}_n) = \mathrm{tp}_{n,k_n}(\operatorname{ran} \mathring{b}_n)$ for each $n^* \le n < \omega$, where $l < n^* < \omega$. Assume that for each $l \le n < n^*$, $\mathrm{tp}_{n,0}(\operatorname{ran} \mathring{a}_n) \ne \mathrm{tp}_{n,0}(\operatorname{ran} \mathring{b}_n)$ while $E_n(a_n) = E_n(b_n)$. According to the definition of the order $\le$ we have both $p \not\le q$ and $q \not\le p$. The basic observation is that $p \Vdash$ "$\check{q} \in \underset{\sim}{G}$" and $q \Vdash$ "$\check{p} \in \underset{\sim}{G}$". That is, while the conditions p and q are incomparable, from the forcing point of view they are equivalent. This is due to the following. Suppose $G \subseteq \mathbb{P}$ is generic and $p \in G$. Since $\{p_{\langle \nu_l,\dots,\nu_{n^*-1}\rangle} \mid \langle \nu_l, \dots, \nu_{n^*-1}\rangle \in \bar{A}\}$ is a maximal antichain below p, there is $\langle \nu_l, \dots, \nu_{n^*-1}\rangle \in \bar{A}$ such that $p_{\langle \nu_l,\dots,\nu_{n^*-1}\rangle} \in G$. By the definition of the order $\le_{\mathbb{P}}$ we have $p_{\langle \nu_l,\dots,\nu_{n^*-1}\rangle} \le q_{\langle \nu_l,\dots,\nu_{n^*-1}\rangle} \le q$. Thus $q \in G$ and we proved that $p \Vdash$ "$\check{q} \in \underset{\sim}{G}$". The same argument with p and q interchaged will show that $q \Vdash$ "$\check{p} \in \underset{\sim}{G}$".

The following claim is the crux of the Prikry property proof. It connects the forcing order $\le$ with the Prikry order $\le^*$. The best option would have been to have $p \le^* q_{\langle \nu_{l^q},\dots,\nu_{l^p-1}\rangle}$ if $p \le q$. This however fails in the current definition. (We can resurrect it but at the too high price of loosing the closedness of the Prikry order.) The following claim shows that the best option is almost achieved. Just instead of getting that $p \le^* q_{\langle \nu_{l^q},\dots,\nu_{l^p-1}\rangle}$, we have a stronger condition $p^* \le p$ for which $p^* \le^* q_{\langle \nu_{l^q},\dots,\nu_{l^{p^*}}\rangle}$.

Claim 3.13. *Assume $p, q \in \mathbb{P}$ are conditions such that $p \le q$. Then there is a stronger condition $p^* \le p$ and a sequence $\langle \nu_{l^q}, \dots, \nu_{l^{p^*}-1}\rangle \in \bar{A}^q$ such that $p^* \le^* q_{\langle \nu_{l^q},\dots,\nu_{l^{p^*}-1}\rangle}$.*

Proof. Let $\langle \nu_{l^q}, \ldots, \nu_{l^p-1}\rangle \in \bar{A}^q$ and $\langle k_n \leq n \mid l^p \leq n < \omega\rangle$ witness that p is an extension of q. Let $l^p \leq n^* < \omega$ be minimal such that $k_n > 0$, and choose $\langle \mu_{l^p}, \ldots, \mu_{n^*-1}\rangle \in \bar{A}^p$. Observe that $\langle \mu_{l^p} \restriction \operatorname{dom} f^q, \ldots, \mu_{n^*-1} \restriction \operatorname{dom} f^q\rangle \in \bar{A}^q$.

Construct the sequences $\bar{a}$ and $\bar{A}$ by doing the following for each $n^* \leq n < \omega$. Set $A_n = A^p_n$. Set $\tau = \operatorname{tp}_{n,k_n-1}(\operatorname{ran} \mathring{a}^p_n)$. The set $\{N \cap \mathscr{H}_{n,k_n-1} \mid N \in \operatorname{ran} a^p_n\}$ witnesses

$$\mathscr{H}_{n,k_n} \vDash \text{“}\exists x \subset \mathscr{H}_{n,k_n-1} \left(\mathring{x} \supseteq \operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n) \wedge \operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau\right)\text{”}.$$

Since $\operatorname{tp}_{n,k_n}(\operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n)) = \operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{a}^q_n)$,

$$\mathscr{H}_{n,k_n} \vDash \text{“}\exists x \subset \mathscr{H}_{n,k_n-1} \left(\mathring{x} \supseteq \operatorname{ran} \mathring{a}^q_n \wedge \operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau\right)\text{”}.$$

Now let $x \subseteq \mathscr{H}_{n,k_n-1}$ be a set satisfying $\mathring{x} \supseteq \operatorname{ran} \mathring{a}^q_n$ and $\operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau$. Set a_n to be the morass reflecting function from $\operatorname{dom} a^p_n$ to x.

Set $p^* = \langle f^p_{\langle \mu_{l^p}, \ldots, \mu_{n^*-1}\rangle}, \bar{a}, \bar{A}\rangle$. Then $p^* \leq p$ and

$$p^* \leq^* q_{\langle \nu_{l^q}, \ldots, \nu_{l^p-1}, \mu_{l^p} \restriction \operatorname{dom} f^q, \ldots, \mu_{n^*-1} \restriction \operatorname{dom} f^q\rangle}.$$

$\square$

The following simple fact is used in stage I of the proof of claim 3.15, which is the basic fact from which the Prikry property and the preservation of κ^+ are derived.

Lemma 3.14. *Assume $p \in \mathbb{P}$ is a condition and $q \leq^* p_{\langle \nu_{l^p}, \ldots, \nu_{m-1}\rangle}$. Then there is a Prikry extension $p^* \leq^* p$ such that $p^*_{\langle \nu_{l^p}, \ldots, \nu_{m-1}\rangle} \leq^* q$ and for each $l^p \leq n < m$, $a^{p^*}_n = a^p_n$ and $A^{p^*}_n = A^p_n$.*

Proof. Set $l = l^p$. Construct the sequences $\bar{a}$ and $\bar{A}$ as follows. For each $l \leq n < m$ set $a_n = a^p_n$ and $A_n = A^p_n$. For each $m \leq n < \omega$ set $a_n = a^q_n$ and $A_n = A^q_n$. Set $p^* = \langle f^q, \bar{a}, \bar{A}\rangle$. $\square$

Claim 3.15. *Assume $p \in \mathbb{P}$ is a condition and D is a dense open subset of $\mathbb{P}$. Then there is a Prikry extension $p^* \leq^* p$ and $l^p \leq n < \omega$ such that*

$$\forall \langle \nu_{l^p}, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p^*}\ p^*_{\langle \nu_{l^p}, \ldots, \nu_{n-1}\rangle} \in D.$$

Proof. Set $l = l^p$. The proof is done in two stages. In stage I we prove that for each $l \leq n < \omega$ there is a Prikry extension $p^* \leq^* p$ such that either (the good case)

$$\forall \langle \nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p^*}\ p^*_{\langle \nu_l, \ldots, \nu_{n-1}\rangle} \in D$$

or (the bad case)

$$\forall \langle \nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p^*}\ \forall q \leq^* p^*_{\langle \nu_l, \ldots, \nu_{n-1}\rangle}\ q \notin D.$$

In stage II we show that it is not possible to get the bad case for every $l \leq n < \omega$.

Stage I. Fix $l \leq n < \omega$ and let $\prec$ be a well ordering of $\operatorname{Lev}_{n-l}(p)$. We use the notation $\vec{\nu}$, $\vec{\mu}$ to denote elements of $\operatorname{Lev}_{n-l}(p)$. E.g., $\vec{\nu} = \langle \nu_l, \ldots, \nu_n\rangle$. We will construct by induction the $\leq^*$-decreasing sequence $\langle p^{\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{n-l}(p)\rangle$ so as to satisfy that if a condition $q \leq^* p^{\vec{\nu}}_{\langle \vec{\nu}\rangle}$ satisfies $q \in D$, then $p^{\vec{\nu}}_{\langle \vec{\nu}\rangle} \in D$. The induction is carried out as follows.

Assume that $\langle p^{\vec{\mu}} \mid \vec{\mu} \prec \vec{\nu}\rangle$ was constructed. Let p' be a condition such that for each $\vec{\mu} \prec \vec{\nu}$, $p' \leq^* p^{\vec{\mu}}$. If there is a Prikry extension $q \leq^* p'_{\langle \vec{\nu}\rangle}$ such that $q \in D$ then use lemma 3.14 to set $p^{\vec{\nu}} \leq^* p'$ to be a condition such that $p^{\vec{\nu}}_{\langle \vec{\nu}\rangle} \leq^* q$. Otherwise set $p^{\vec{\nu}} = p'$.

At the end of the induction let p^* be a condition such that for each $\vec{\nu} \in \mathrm{Lev}_{n-l}(p^*)$, $p^* \leq^* p^{\vec{\nu}}$. By removing a measure zero set from $\mathrm{Lev}_{n-l}(p^*)$ we get the conclusion.

Stage II. Begin with a condition $p \in \mathbb{P}$ and a dense open subset D. Set $p_l = p$ and by induction construct $p_{n+1} \leq^* p_n$ using stage I. If we get that

$$\forall \langle \nu_l, \ldots, \nu_{n-1} \rangle \in \bar{A}^{p_{n+1}} \ p_{n+1 \langle \nu_l, \ldots, \nu_{n-1} \rangle} \in D,$$

then we are done by setting $p^* = p_{n+1}$. Otherwise we have that

$$\forall \langle \nu_l, \ldots, \nu_{n-1} \rangle \in \bar{A}^{p_{n+1}} \ \forall q \leq^* p_{n+1 \langle \nu_l, \ldots, \nu_{n-1} \rangle} \ q \notin D.$$

and the induction continues.

We claim that at some $n < \omega$ the induction had to stop, which proves the claim. Towards a contradiction assume that the induction did not stop. Thus we have a $\leq^*$-decreasing sequence of conditions $\langle p_n \mid l \leq n < \omega \rangle$ such that for each $l \leq n < \omega$,

$$\forall \langle \nu_l, \ldots, \nu_{n-1} \rangle \in \bar{A}^{p_{n+1}} \ \forall q \leq^* p_{n+1 \langle \nu_l, \ldots, \nu_{n-1} \rangle} \ q \notin D.$$

Let p^* be a condition such that for each $l \leq n < \omega$, $p^* \leq^* p_n$. Let $q \in D$ be a condition such that $q \leq p^*$. By lemm 3.13 there is $q^* \leq q$ such that $q^* \leq^* p^*_{\langle \nu_l, \ldots, \nu_{n-1} \rangle}$, where $\langle \nu_l, \ldots, \nu_{n-1} \rangle \in \bar{A}^{p^*}$. Since D is open and $q \in D$, $q^* \in D$. By the construction of p^*, $q^* \leq^* p_{n+1 \langle \nu_l, \ldots, \nu_{n-1} \rangle}$. By the construction of p_{n+1} this means that $q^* \notin D$. Contradiction. $\square$

The triple $\langle \mathbb{P}, \leq, \leq^* \rangle$ is said to be of Prikry type if for each condition $p \in \mathbb{P}$ and formula σ in the $\mathbb{P}$-forcing language there is a Prikry extension $p^* \leq^* p$ deciding σ. The Prikry property is immediately derived from the previous lemma:

Corollary 3.16. *The forcing $\langle \mathbb{P}, \leq, \leq^* \rangle$ is of Prikry type.*

Claim 3.17. *The cardinal κ^+ is preserved in a $\mathbb{P}$-generic extension.*

Proof. Since κ is of cofinality ω it is enough to show that each sequence in κ^+ of length less than κ is bounded. Thus assume $\lambda < \kappa$ and $p \Vdash$ "$\dot{f} : \check{\lambda} \to (\kappa^+)_V$". Set $l = l^p$. We can assume that $\kappa_l > \lambda$. We will exhibit a condition $p_\lambda \leq^* p$ forcing that $\dot{f}$ is bounded in κ^+. For each $\zeta < \lambda$ set $D_\zeta = \{ q \leq p \mid \exists \xi < \kappa^+ \ q \Vdash \text{“}\dot{f}(\check{\zeta}) = \check{\xi}\text{”} \}$. Using claim 3.15 and claim 3.4 construct by induction a $\leq^*$-decreasing sequence $\langle p_\zeta \mid \zeta \leq \lambda \rangle$ and a sequence $\langle m_\zeta < \lambda \mid \zeta < \lambda \rangle$ satisfying $p_0 \leq^* p$ and for each $\zeta < \lambda$,

$$\forall \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle \in \bar{A}^{p_\zeta} \ \exists \xi < \kappa^+ \ p_{\zeta \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle} \Vdash \text{“}\dot{f}(\check{\zeta}) = \check{\xi}\text{”}.$$

Thus for each $\zeta < \lambda$,

$$\forall \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle \in \bar{A}^{p_\lambda} \ \exists \xi < \kappa^+ \ p_{\lambda \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle} \Vdash \text{“}\dot{f}(\check{\zeta}) = \check{\xi}\text{”}.$$

For each $\zeta < \lambda$ define the function $F_\zeta : \mathrm{Lev}_{m_\zeta - l}(p_\lambda) \to \kappa^+$ so that

$$\forall \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle \in \bar{A}^{p_\lambda} \ p_{\lambda \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle} \Vdash \text{“}\dot{f}(\check{\zeta}) = \check{F}_\zeta(\nu_l, \ldots, \nu_{m_\zeta})\text{”}.$$

Set $\mu = \sup\{ F_n(\nu_l, \ldots, \nu_{m_\zeta}) \mid \zeta < \lambda, \ \langle \nu_l, \ldots, \nu_{m_\zeta} \rangle \in \mathrm{Lev}_{m_\zeta - l}(p_\lambda) \}$. By its definition, $p_\lambda \Vdash$ "$\operatorname{ran} \dot{f} \subseteq \check{\mu}$". Since the sup in the definition of μ is being taken over a set of size smaller than κ^+ and κ^+ is regular, we get $\mu < \kappa^+$. $\square$

Combining the above claims we get:

Theorem (M. Gitik [4]). *Assume GCH and let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of cardinals such that for each $n < \omega$ there is a $\langle \kappa_n, \kappa_n^{+n+3}\rangle$ extender. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Then there is a cardinal preserving generic extension adding no new bounded subsets to κ such that $2^\kappa = \kappa^{+3}$.*

Proof. Work in a universe where a stationary neat simplified $\langle \kappa^{++}, 1\rangle$-morass exists. (Just force one if needed.) Then force with $\mathbb{P}$.

- The Prikry property 3.16 of $\langle \mathbb{P}, \leq, \leq^*\rangle$, together with the closure 3.4 of $\langle \mathbb{P}, \leq^*\rangle$ yield that V and $V[G]$ have the same bounded subset of κ, and thus that κ is preserved. By the κ^{+3}-cc 3.11 all the cardinals above κ^{++} are preserved, by the $\mathbb{M}$-properness 3.12 the cardinal κ^{++} is preserved, and by 3.17 the cardinal κ^+ is preserved. Thus all cardinals are preserved.
- On the one hand, the κ^{+3}-cc together with $|\mathbb{P}| = \kappa^{+3}$ imply $2^\kappa \leq \kappa^{+3}$ in the generic extension. On the other hand, 3.10 gives $2^\kappa \geq \kappa^{+3}$.

□

References

[1] Keith Devlin. *Constructibility*, volume 6 of *Perspectives in Mathematical Logic*. Springer, Berlin, 1984. `http://projecteuclid.org/euclid.pl/1235419477`.

[2] Moti Gitik. Simpler short extenders forcing — gap 3. `http://www.math.tau.ac.il/~gitik/gap3.pdf`.

[3] Moti Gitik. Blowing up the power of a singular cardinal. *Annals of Pure and Applied Logic*, 80(1):17–33, July 1996. doi:10.1016/0168-0072(95)00046-1.

[4] Moti Gitik. Blowing up power of a singular cardinal–wider gaps. *Annals of Pure and Applied Logic*, 116(1–3):1–38, August 2002. doi:10.1016/S0168-0072(01)00098-7.

[5] Moti Gitik. On gaps under GCH type assumptions. *Annals of Pure and Applied Logic*, 119(1–3):1–18, February 2003. doi:10.1016/S0168-0072(02)00031-3.

[6] Moti Gitik. Short extender gap-3 forcing notion. TAU Set theory seminar circulated notes, May 2008.

[7] Moti Gitik and Menachem Magidor. The singular continuum hypothesis revisited. In Haim Judah, Winfried Just, and W. Hugh Woodin, editors, *Set theory of the continuum*, volume 26 of *Mathematical Sciences Research Institute publications*, pages 243–279. Springer, 1992.

[8] Moti Gitik and William J. Mitchell. Indiscernible sequences for extenders, and the singular cardinal hypothesis. *Annals of Pure and Applied Logic*, 82(3):273–316, December 1996. doi:10.1016/S0168-0072(96)00007-3.

[9] Piotr Koszmider. Semimorasses and nonreflection at singular cardinals. *Annals of pure and applied logic*, 72:1–23, 1995. doi:10.1016/0168-0072(93)E0068-Y.

[10] Piotr Koszmider. On strong chains of uncountable functions. *Israel Journal of Mathematics*, 118(1):289–315, December 2000. doi:10.1007/BF02803525.

[11] Carmi Merimovich. Gitik's short extender gap-2 forcing notion. TAU Set theory seminar circulated notes, February 2008.

[12] Carmi Merimovich. The short extender gap two forcing is of Prikry type. *Archive for Mathematical logic*, 48(8):737–747, December 2009. doi:10.1007/s00153-009-0147-1.

[13] Saharon Shelah. *Proper and improper forcing*, volume 5 of *Perspectives in Mathematical Logic*. Springer, 1998. `http://projecteuclid.org/euclid.pl/1235419814`.

[14] Dan Velleman. Simplified morasses. *Journal of Symbolic Logic*, 49(1):257–271, March 1984. `http://projecteuclid.org/euclid.jsl/1183741492`.

[15] Dan Velleman. Simplified morasses with linear limits. *Journal of Symbolic Logic*, 49(4):1001–1021, December 1984. `http://projecteuclid.org/euclid.jsl/1183741684`.

[16] Dan Velleman. Souslin trees constructed from morasses. In J. Baumgartner, D. Martin, and S. Shelah, editors, *Axiomatic Set Theory*, volume 31 of *Contemporary Mathematics*, pages 219–241. American Mathematical Society, Providence, RI, 1984.

[17] Dan Velleman. Simplified gap-2 morasses. *Annals of Pure and Applied Logic*, 34:171–208, 1987. doi:10.1016/0168-0072(87)90070-4.

[18] William S. Zwicker. $\mathscr{P}_\kappa\,\lambda$ Combinatorics I: Statoinary coding sets rationalize the club filter. In J. Baumgartner, D. Martin, and S. Shelah, editors, *Axiomatic Set Theory*, volume 31 of *Contemporary Mathematics*, pages 243–259. American Mathematical Society, Providence, RI, 1984.

Department of Computer Science, Tel-Aviv Academic College, Rabenu Yeroham St., Tel-Aviv 68182, Israel

E-mail address: carmi@cs.mta.ac.il