THE SHORT EXTENDERS GAP TWO FORCING IS OF PRIKRY TYPE

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Abstract. We show that Gitik's short extender gap-2 forcing is of Prikry type.

1. Introduction

In [5] a forcing notion blowing up the powerset of a cardinal κ carrying an extender together with changing κ's cofinality to ω in one step was introduced. The size of the powerset in the generic extension was set to the size of the extender. It was felt at the time that this is the optimal assumption. That is, if one begins with a model for κ of cofinality ω with large powerset, then in the core model one should find an extender on κ of the powerset size. Going this way [6] and [3] assumed $2^\kappa > \lambda$ and found, quite unexpectedly, *two* possibilities. One possibility was indeed that in the core model the cardinal κ carries an extender of the size 2^κ. The other possibility, however, was that in the core model the cardinal κ is a singular cardinal of cofinality ω, and there is an increasing sequence of cardinals κ_n with limit κ each carrying a rather short extender. In the sequence of papers [1, 2, 3] Gitik showed that indeed this other possibility can be used to blow up the powerset of κ.

Gitik presented his forcing notions in the TAU set theory seminar of the year 2007 from which the notes [7, 4] grew out. We thank the participants of the seminar Eilon Belinski, Omer Ben-Naria, Assaf Ferber, Assaf Rinot, and Liad Tal. Of course we thank Moti Gitik for the organization, presentation, and for being rather patient with the enormous amount of questions he had to answer by phone, email, and in person.

In this paper we reprove the following.

Theorem (M. Gitik [1]**).** *Assume $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of cardinals such that for each $n < \omega$ there is a $\langle \kappa_n, \kappa_n^{+n+2} \rangle$ extender. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Then there is a cardinal preserving generic extension adding no new bounded subsets to κ such that $2^\kappa = \kappa^{++}$.*

In case somehow it was not clear until now, we stress that the forcing notion presented is due to Gitik. The new feature we present is that the forcing notion is of Prikry type with the Prikry order being closed enough so it is useful. We present all details, thus the paper is self contained assuming one knows forcing and large cardinals theory. Our plan is to fit in the wider gap short extender forcing notions into this framework hopefully gaining the Prikry property.

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2. Gap-2 forcing

Assume $\kappa = \bigcup_{n<\omega} \kappa_n$, where $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of cardinals. Furthermore, assume that for each $n < \omega$ there is an elementary embedding $j_n : V \to M_n$ such that M_n is transitive, $\operatorname{crit}(j_n) = \kappa_n$, $M_n \supseteq M_n^{\kappa_n}$, and $j_n(\kappa_n) \geq \kappa_n^{+n+2}$. Let E_n be the $\langle \kappa_n, \kappa_n^{+n+2} \rangle$-extender derived from j_n. Without loss of generality assume that j_n is the natural embedding from V to $\operatorname{Ult}(V, E_n) \simeq M_n$.

We write S^λ_μ to denote the set $\{\xi < \lambda \mid \operatorname{cf} \xi = \mu\}$.

Definition 2.1. The following list of points leads to the definition of the forcing notion $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and its Prikry order $\leq^*_{\mathbb{P}}$. We begin with the definition of the universe for measures, continue with structures and then get to the relevant measures.

- Assume $d \in [S^{\kappa^{++}}_{\kappa^+}]^{<\kappa_n}$. The set $\operatorname{OB}_n(d)$ is composed of the order preserving functions $\nu : d \to \kappa_n \setminus \kappa_{n-1}$. (Consider κ_{-1} to be $\emptyset$).
- For the duration of the current section fix for each $n < \omega$ a cardinal $\chi_n < \kappa$ large relative to κ_n. Assume $k \leq n < \omega$. Define the following structure.
$$\mathscr{H}_{n,k} = \langle H(\chi_n^{+k}), \in, E_n, A, \xi \rangle_{A \in V_{\kappa_n+1},\ \xi < \kappa_n^{+k}}.$$
Instead of using χ_n^{+k} in the above definition any sequence $\langle \chi_{n,k} \mid k < n \rangle$ having the property $H(\chi_{n,k}) \in H(\chi_{n,k+1})$ could have been used.
- An elementary submodel $N \prec \mathscr{H}_{n,k}$ codes an ordinal $\alpha < \kappa_n^{+n+2}$ if $\alpha = N \cap \kappa_n^{+n+2}$, $|N| = \kappa_n^{+n+1}$, and $N \supseteq N^{<\kappa_n}$. We use $\mathring{N}$ to denote the ordinal $N \cap \kappa_n^{+n+2}$.
- A family of elementary submodels x codes an element of $[\kappa_n^{+n+2}]^{<\kappa_n}$ if:
  (1) $|x| < \kappa_n$.
  (2) For each $N \in x$ there is $k \leq n$ such that $N \prec \mathscr{H}_{n,k}$ codes an ordinal $< \kappa_n^{+n+2}$.
  (3) For each $N_1, N_2 \in x$ such that $N_1 \neq N_2$, $\mathring{N}_1 \neq \mathring{N}_2$.
  (4) If $N_1, N_2 \in x$, $N_2 \prec \mathscr{H}_{n,k+1}$, and $\mathring{N}_1 < \mathring{N}_2$, then $N_1 \cap \mathscr{H}_{n,k} \in N_2$.

  We set $\mathring{x} = \{\mathring{N} \mid N \in x\}$.
- Assume $d \in [S^{\kappa^{++}}_{\kappa^+}]^{<\kappa_n}$ and $a : d \to H(\chi_n^{+\omega})$ is a function such that $\operatorname{ran} a$ codes an element of $[\kappa_n^{+n+2}]^{<\kappa_n}$.
  - The function $\mathring{a} : d \to \kappa_n^{+n+2}$ is defined by $\mathring{a}(\alpha) = a(\alpha) \cap \kappa_n^{+n+2}$.
  - The measure $E_n(a)$ is defined on $\operatorname{OB}_n(d)$ as follows:
$$\forall X \subseteq \operatorname{OB}_n(d) \ \big(X \in E_n(a) \iff \{\langle j_n(\alpha), \mathring{a}(\alpha) \rangle \mid \alpha \in d\} \in j_n(X) \big).$$
- Assume $e \subseteq d \in [S^{\kappa^{++}}_{\kappa^+}]^{\leq\kappa}$ and $X \subseteq \operatorname{OB}_n(d)$. Then
$$X \restriction e = \{\nu \restriction e \mid \nu \in X\}.$$
For finite products define: If $\langle d_n \mid l \leq n \leq m \rangle$ is $\subseteq$-increasing, $e \subseteq \bigcup_{l \leq n \leq m} d_n$, and $X \subseteq \prod_{l \leq n \leq m} \operatorname{OB}_n(d_n)$ then
$$X \restriction e = \{\langle \nu_l \restriction e, \dots, \nu_m \restriction e \rangle \mid \langle \nu_l, \dots, \nu_m \rangle \in X\}.$$

We define now the Cohen part $\mathbb{P}^*$ of the forcing notion $\mathbb{P}$. It will consists of κ initial segments of Prikry sequences to be generated.

- A condition f is in the forcing notion $\mathbb{P}^*$ if $f : d \to {}^{<\omega}\kappa$ is a function such that:
  (1) $d \in [S^{\kappa^{++}}_{\kappa^+}]^{\leq\kappa}$.
  (2) For each $\alpha \in d$, $f(\alpha) = \langle f_{n_1}(\alpha), \dots, f_{n_2-1}(\alpha) \rangle \in \prod_{n_1 \leq n < n_2} \kappa_n$ is an increasing sequence, where $n_1 \leq n_2 < \omega$ and n_1 and n_2 depend on α.

  The forcing notion $\mathbb{P}^*$ is equipped with the partial order $f \leq^*_{\mathbb{P}^*} g \iff f \supseteq g$. (Thus $\langle \mathbb{P}^*, \leq^* \rangle$ is the Cohen forcing adding κ^{++} subsets to κ^+).

Finally we define the forcing notion $\mathbb{P}$ and its two orders $\leq^*_{\mathbb{P}}$ and $\leq_{\mathbb{P}}$.

- A condition $p = \langle f, \bar{a}, \bar{A}\rangle$ is in the forcing notion $\mathbb{P}$ if there is $l < \omega$ such that:
  (1) $f \in \mathbb{P}^*$.
  (2) $\bar{a} = \langle a_n : d_n \to H(\chi_n^{+\omega}) \mid l \leq n < \omega\rangle$ is a sequence of functions such that:
    (2.1) $\bigcup_{l \leq n < \omega} d_n = \operatorname{dom} f$.
    (2.2) $\langle d_n \mid l \leq n < \omega\rangle$ is $\subseteq$-increasing.
    (2.3) For each $l \leq n < \omega$, $|d_n| < \kappa_n$.
  (3) (3.1) For each $l \leq n < \omega$, a_n codes an increasing sequence $\langle \mathring{a}_n(\alpha) \mid \alpha \in d_n\rangle$ of ordinals in κ_n^{+n+2}.
    (3.2) Assume $\alpha \in \operatorname{dom} f$, and let $l \leq n^* < \omega$ be minimal such that $\alpha \in d_{n^*}$. Furthermore, assume that for each $n^* \leq n < \omega$, $a_n(\alpha) \prec \mathscr{H}_{n,k_n}$. Then $\bigcup_{n^* \leq n < \omega} k_n = \omega$, and the sequence $\langle k_n \mid n^* \leq n < \omega\rangle$ is non-decreasing.
  (4) $\bar{A} = \langle A_n \mid l \leq n < \omega\rangle$ and for each $l \leq n < \omega$, $A_n \in E_n(a_n)$.

  We write l^p, f^p, $\bar{a}^p$, $\bar{A}^p$, a^p_n, A^p_n, and $\operatorname{Lev}_m(p)$, for l, f, $\bar{a}$, $\bar{A}$, a_n, A_n, and $\prod_{l \leq n \leq l+m} A_n$, respectively.
- Let $p, q \in \mathbb{P}$. The condition p is a Prikry extension of q ($p \leq^*_{\mathbb{P}} q$) if:
  (1) $f^p \leq^*_{\mathbb{P}^*} f^q$.
  (2) $l^p = l^q$ (we use l to denote the common value).
  (3) For each $l \leq n < \omega$:
    (3.1) $a^p_n \supseteq a^q_n$.
    (3.2) $\operatorname{dom} a^p_n \setminus \operatorname{dom} a^q_n \subseteq \operatorname{dom} f^p \setminus \operatorname{dom} f^q$.
    (3.3) $A^p_n \restriction \operatorname{dom} a^q_n \subseteq A^q_n$.

  Note that the Prikry order is quite closed as stated in claim 2.2.
- Assume $f \in \mathbb{P}^*$, $\nu \in \operatorname{OB}_n(d)$, where $d \in [\operatorname{dom} f]^{<\kappa_n}$. Define the condition $f_{\langle\nu\rangle} \in \mathbb{P}^*$ to be the function $g \in \mathbb{P}^*$ with domain $\operatorname{dom} f$ satisfying for each $\alpha \in \operatorname{dom} g$,

$$g(\alpha) = \begin{cases} f(\alpha)^\frown\langle\nu(\alpha)\rangle & \alpha \in \operatorname{dom}\nu,\ \nu(\alpha) > \max f(\alpha), \\ f(\alpha) & \text{Otherwise.} \end{cases}$$

  Assume $\langle \nu_l, \ldots, \nu_{m-1}\rangle \in \prod_{l \leq n < m} \operatorname{OB}_n(d_n)$ where $d_n \in [\operatorname{dom} f]^{<\kappa_n}$. Define the condition $f_{\langle \nu_l, \ldots, \nu_{m-1}\rangle} \in \mathbb{P}^*$ recursively as $(f_{\langle \nu_l, \ldots, \nu_{m-2}\rangle})_{\langle \nu_{m-1}\rangle}$.
- Assume $p \in \mathbb{P}$. By writing $\langle \nu_{l^p}, \ldots, \nu_{n-1}\rangle \in \bar{A}^p$ we mean that $\langle \nu_{l^p}, \ldots, \nu_{n-1}\rangle \in \operatorname{Lev}_{n-l^p-1}(p)$.
- Assume $p \in \mathbb{P}$ and $\langle \nu_{l^p}, \ldots, \nu_{m-1}\rangle \in \bar{A}^p$. By $\bar{a}^p_{\langle \nu_{l^p}, \ldots, \nu_{m-1}\rangle}$ and $\bar{A}^p_{\langle \nu_{l^p}, \ldots, \nu_{m-1}\rangle}$ we mean the sequences $\langle a^p_n \mid m \leq n < \omega\rangle$ and $\langle A^p_n \mid m \leq n < \omega\rangle$, respectively.
- Assume $p \in \mathbb{P}$ and $\langle\nu\rangle \in \bar{A}^p$. Define the condition $p_{\langle\nu\rangle} \in \mathbb{P}$ to be $\langle f^p_{\langle\nu\rangle}, \bar{a}^p_{\langle\nu\rangle}, \bar{A}^p_{\langle\nu\rangle}\rangle$.
- Assume $p \in \mathbb{P}$ and $\langle \nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^p$. Define recursively $p_{\langle \nu_l, \ldots, \nu_{n-1}\rangle} \in \mathbb{P}$ to be the condition $(p_{\langle \nu_l, \ldots, \nu_{n-2}\rangle})_{\langle \nu_{n-1}\rangle} \in \mathbb{P}$.

The natural way to define the forcing order would have been to extend a condition p to a Prikry extension of $p_{\langle \nu_l, \ldots, \nu_{n-1}\rangle}$. Alas, this definition collapses κ^{++}. In order to restrict the length of the anti-chains, we identify conditions with different a's. This is done according to the types of the a's defined as follows.

- Assume $k \leq n < \omega$ and $x \in \mathscr{H}_{n,k}$. The $\langle n, k\rangle$-type of x is defined to be the set of formulae with one free parameter holding in the structure $\mathscr{H}_{n,k}$ for the assignment x, i.e.,

$$\operatorname{tp}_{n,k}(x) = \{\ulcorner \phi(-) \urcorner \mid \mathscr{H}_{n,k} \vDash \phi(x)\}.$$

We will be interested in the types of sets in $[\kappa_n^{+n+2}]^{<\kappa_n}$. A type will be coded by an ordinal. I.e., $\operatorname{tp}_{n,k}(x) \in \kappa_n^{+k+1}$. Hence for $k < n$ there are constants in the language of the structure $\mathscr{H}_{n,k+1}$ for the $\mathscr{H}_{n,k}$-types.

- Assume $p, q \in \mathbb{P}$. We say that p is an extension of q ($p \leq_{\mathbb{P}} q$) if there is $\langle \nu_{l^q}, \dots, \nu_{l^p-1} \rangle \in \bar{A}^q$, and a non-decreasing sequence $\langle k_n \leq n \mid l^p \leq n < \omega \rangle$, such that:
  (1) $f^p \leq^*_{\mathbb{P}^*} f^q_{\langle \nu_{l^q}, \dots, \nu_{l^p-1} \rangle}$.
  (2) $\bigcup_{l^p \leq n < \omega} k_n = \omega$.
  (3) For each $l^p \leq n < w$:
    (3.1) $\operatorname{dom} a^p_n \supseteq \operatorname{dom} a^q_n$.
    (3.2) $\operatorname{dom} a^p_n \setminus \operatorname{dom} a^q_n \subseteq \operatorname{dom} f^p \setminus \operatorname{dom} f^q$.
    (3.3) $A^p_n \restriction \operatorname{dom} a^q_n \subseteq A^q_n$.
    (3.4) $\operatorname{tp}_{n,k_n}(\operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n)) = \operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{a}^q_n)$.

Let us specify two properties of the order $\leq_{\mathbb{P}}$ which are not obvious at once.

It can happen that two conditions satisfy both $p \leq_{\mathbb{P}} q$ and $f^p \leq^*_{\mathbb{P}^*} f^q$ but $p \not\leq^*_{\mathbb{P}} q$. That is changing only the values of the a_n's while keeping their types is a non-Prikry extension. This allows the Prikry order to be closed.

Another fact is as follows. Suppose $p = \langle f, \bar{a}, \bar{A} \rangle$, $q = \langle f, \bar{b}, \bar{A} \rangle$, and $l^p = l^q = l$. Moreover, assume that $\operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{a}_n) = \operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{b}_n)$ for each $n^* \leq n < \omega$, where $l < n^* < \omega$. Assume that for each $l \leq n < n^*$, $\operatorname{tp}_{n,0}(\operatorname{ran} \mathring{a}_n) \neq \operatorname{tp}_{n,0}(\operatorname{ran} \mathring{b}_n)$ while $E_n(a_n) = E_n(b_n)$. According to the definition of the order $\leq$ we have both $p \not\leq q$ and $q \not\leq p$. The basic observation of the current work is that $p \Vdash$ "$\check{q} \in \underset{\sim}{G}$" and $q \Vdash$ "$\check{p} \in \underset{\sim}{G}$". That is, while the conditions p and q are incomparable, from the forcing point of view they are equivalent. This is due to the following. Suppose $G \subseteq P$ is generic and $p \in G$. Since $\{p_{\langle \nu_l, \dots, \nu_{n^*-1} \rangle} \mid \langle \nu_l, \dots, \nu_{n^*-1} \rangle \in \bar{A}\}$ is a maximal antichain below p, there is $\langle \nu_l, \dots, \nu_{n^*-1} \rangle \in \bar{A}$ such that $p_{\langle \nu_l, \dots, \nu_{n^*-1} \rangle} \in G$. By the definition of the order $\leq_{\mathbb{P}}$ we have $p_{\langle \nu_l, \dots, \nu_{n^*-1} \rangle} \leq q_{\langle \nu_l, \dots, \nu_{n^*-1} \rangle} \leq q$. Thus $q \in G$ and we proved that $p \Vdash$ "$\check{q} \in \underset{\sim}{G}$". The same argument with p and q interchaged will show that $q \Vdash$ "$\check{p} \in \underset{\sim}{G}$".

We state the exact closure properties of the Prikry order.

Claim 2.2. *(1) Assume $\langle p_\xi \mid \xi < \lambda < \kappa_{l^{p_0}} \rangle$ is a $\leq^*$-decreasing sequence. Then there is a condition p^* such that for each $\xi < \lambda$, $p^* \leq^* p_\xi$.*
(2) Assume $\langle p_\xi \mid \xi < \lambda < \kappa_{n^*} \rangle$ is a $\leq^*$-decreasing sequence, and for each $\xi_0 < \xi_1 < \lambda$ and $l^p \leq n < n^*$, $a^{p_{\xi_0}}_n = a^{p_{\xi_1}}_n$ and $A^{p_{\xi_0}}_n = A^{p_{\xi_1}}_n$. Then there is a condition p^* such that for each $\xi < \lambda$, $p^* \leq^* p_\xi$.

The following claim shows that in the generic extension there is an ω-sequence for each $\alpha \in S^{\kappa^{++}}_{\kappa^+}$. It is a bit stronger than what is actually needed since it also shows that we can control the initial segments of κ-many sequences at once.

Claim 2.3. *Assume $p \in \mathbb{P}$ is a condition and $g \leq^* f^p$. Then there is a Prikry extension $p^* \leq^* p$ such that $f^{p^*} = g$.*

Proof. Set $l = l^p$. The proof is done is three stages. In stage I we show how to extend f^p to $f^p \cup \{\langle \alpha, g(\alpha) \rangle\}$ ensuring on the way that this can be done without changing a_m and A_m ($l^p \leq m < n^*$) for any $l \leq n^* < \omega$. In stage II we show how to extend f^p with less than κ-many ordinals. Finally, in stage III we show how extend f^p to an arbitrary g.

Stage I. Assume $g \setminus f^p = \langle \alpha, g(\alpha) \rangle$. Set $f^{p^*} = g$ and fix some $l \leq n^* < \omega$. The construction of $\bar{a}^{p^*}$ and $\bar{A}^{p^*}$ is done according to the whereabouts of α:

(1) For each $\beta \in \operatorname{dom} f^p$, $\alpha > \beta$: The construction is by induction. Begin by setting for each $l \leq n < n^*$, $a^{p^*}_n = a^p_n$ and $A^{p^*}_n = A^p_n$.

The inductive step is done as follows. Assume $a^{p^*}_{n-1}$ and $A^{p^*}_{n-1}$, where $n^* \leq n < \omega$, have been constructed. Choose an elementary submodel $N \prec \mathscr{H}_{n,n}$ coding an ordinal in κ_n^{+n+2} such that for each $\beta \in \operatorname{dom} a^p_n$, $a^p_n(\beta) \cap \mathscr{H}_{n,n-1} \in N$. Then set $a^{p^*}_n(\alpha) = N$, and $a^{p^*}_n \restriction \operatorname{dom} a^p_n = a^p_n$. Now choose $A^{p^*}_n \in E_n(a^{p^*}_n)$ such that $A^{p^*}_n \restriction \operatorname{dom} a^p_n \subseteq A^p_n$.

(2) There is $\gamma \in \operatorname{dom} f^p$ such that $\alpha < \gamma$: Let $\gamma \in \operatorname{dom} f^p$ be minimal such that $\alpha < \gamma$. Let $n^* \leq m < \omega$ be minimal such that $\max g(\alpha) < \kappa_m$, $\gamma \in \operatorname{dom} a^p_m$, and $a^p_m(\gamma) \prec \mathscr{H}_{n,k+2}$. The construction of $\bar{a}^{p^*}$ and $\bar{A}^{p^*}$ is by induction. Begin by setting for each $l \leq n < m$, $a^{p^*}_n = a^p_n$ and $A^{p^*}_n = A^p_n$.

The inductive step is done as follows. Assume that $a^{p^*}_{n-1}$ and $A^{p^*}_{n-1}$, where $m \leq n < \omega$, have been constructed. Choose an elementary submodel $N \prec \mathscr{H}_{n,k+1}$ coding an ordinal in κ_n^{+n+2} such that $N \in a^p_n(\gamma)$, and for each $\beta \in \operatorname{dom} a^p_n \cap \gamma$, $a^p_n(\beta) \cap \mathscr{H}_{n,k} \in N$. Then set $a^{p^*}_n(\alpha) = N$, and $a^{p^*}_n \restriction a^p_n = a^p_n$. Now choose $A^{p^*}_n \in E_n(a^{p^*}_n)$ such that $A^{p^*}_n \restriction \operatorname{dom} a^p_n \subseteq A^p_n$.

Stage II. Let $\langle \alpha_\xi \mid \xi < \lambda < \kappa \rangle$ be an enumeration of $\operatorname{dom} g \setminus \operatorname{dom} f^p$. Let $l \leq n^* < \omega$ be minimal such that $\kappa_{n^*} > \lambda$. Construct by induction, using stage I and 2.2, the $\leq^*$-decreasing sequence of conditions $\langle p_\xi \mid \xi \leq \lambda \rangle$ satisfying:

(1) $p_0 = p$.
(2) For each $\xi < \lambda$, $f^{p_{\xi+1}} = f^{p_\xi} \cup \{\langle \alpha_\xi, g(\alpha_\xi) \rangle\}$.
(3) For each $\xi_0 < \xi_1 < \lambda$ and $l \leq n < n^*$, $a^{p_{\xi_0}}_n = a^{p_{\xi_1}}_n$ and $A^{p_{\xi_0}}_n = A^{p_{\xi_1}}_n$.

We are done by setting $p^* = p_\lambda$.

Stage III. Let $\langle \alpha_\xi \mid \xi < \kappa \rangle$ be an enumeration of $\operatorname{dom} g \setminus \operatorname{dom} f^p$. For each $l \leq n < \omega$ let $g_n \subseteq g$ be a function such that $|g_n| < \kappa_n$, $\{\operatorname{dom} g_n \mid l \leq n < \omega\}$ is a set of mutually disjoint sets, and $g = \bigcup_{l \leq n < \omega} g_n$. Construct by induction, using stage II and claim 2.2, a $\leq^*$-decreasing sequence $\langle p_n \mid l < n \leq \omega \rangle$ such that $p_l = p$, and for each $l \leq n < \omega$, $f^{p_{n+1}} = f^{p_n} \cup g_n$. We are done by setting $p^* = p_\omega$. $\square$

In view of claim 2.3 the following definition makes sense.

Definition 2.4. Assume $G \subset \mathbb{P}$ is generic. For each $\alpha \in S^{\kappa^{++}}_{\kappa^+}$ set

$$G^{\alpha *} = \bigcup \{ f^p(\alpha) \mid p \in G,\ \alpha \in \operatorname{dom} f^p \}.$$

Let G^α be a shift of $G^{\alpha *}$ satisfying $G^\alpha \in \prod_{n<\omega} \kappa_n$.

It is immediate that $\langle G^\alpha \mid \alpha \in S^{\kappa^{++}}_{\kappa^+} \rangle$ is increasing, thus we have:

Corollary 2.5. *(In $V[G]$)* $2^\kappa \geq |(\kappa^{++})^V|$.

The following claim is the crux of the matter. It connects the forcing order $\leq$ with the Prikry order $\leq^*$. The best option would have been to have $p \leq^* q_{\langle \nu_{l^q}, \ldots, \nu_{l^p - 1} \rangle}$ if $p \leq q$. This however fails in the current definition. We can resurrect it but loose the closedness of the Prikry order. The following claim shows that we almost have the best option. Just instead of getting that $p \leq^* q_{\langle \nu_{l^q}, \dots, \nu_{l^p-1} \rangle}$, we have a stronger condition $p^* \leq p$ for which $p^* \leq^* q_{\langle \nu_{l^q}, \dots, \nu_{l^{p^*}} \rangle}$.

Claim 2.6. *Assume $p, q \in \mathbb{P}$ are conditions such that $p \leq q$. Then there is a stronger condition $p^* \leq p$ and a sequence $\langle \nu_{l^q}, \dots, \nu_{l^{p^*}-1} \rangle \in \bar{A}^q$ such that $p^* \leq^* q_{\langle \nu_{l^q}, \dots, \nu_{l^{p^*}-1} \rangle}$.*

Proof. Let $\langle \nu_{l^q}, \dots, \nu_{l^p-1} \rangle \in \bar{A}^q$ and $\langle k_n \leq n \mid l^p \leq n < \omega \rangle$ witness that p is an extension of q. Let $l^p \leq n^* < \omega$ be minimal such that $k_n > 0$, and choose $\langle \mu_{l^p}, \dots, \mu_{n^*-1} \rangle \in \bar{A}^p$. Observe that $\langle \mu_{l^p} \restriction \operatorname{dom} f^q, \dots, \mu_{n^*-1} \restriction \operatorname{dom} f^q \rangle \in \bar{A}^q$.

Construct $\bar{a}$ and $\bar{A}$ by doing the following for each $n^* \leq n < \omega$. Set $A_n = A^p_n$. Set $\tau = \operatorname{tp}_{n,k_n-1}(\operatorname{ran} \mathring{a}^p_n)$. The set $\{N \cap \mathscr{H}_{n,k_n-1} \mid N \in \operatorname{ran} a^p_n\}$ witnesses

$$\mathscr{H}_{n,k_n} \models \exists x \subset \mathscr{H}_{n,k_n-1} \ \big(\mathring{x} \supseteq \operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n) \wedge \operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau\big).$$

Since $\operatorname{tp}_{n,k_n}(\operatorname{ran}(\mathring{a}^p_n \restriction \operatorname{dom} a^q_n)) = \operatorname{tp}_{n,k_n}(\operatorname{ran} \mathring{a}^q_n)$,

$$\mathscr{H}_{n,k_n} \models \exists x \subset \mathscr{H}_{n,k_n-1} \ \big(\mathring{x} \supseteq \operatorname{ran} \mathring{a}^q_n \wedge \operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau\big).$$

Now let $x \subseteq \mathscr{H}_{n,k_n-1}$ be a set satisfying $\mathring{x} \supseteq \operatorname{ran} \mathring{a}^q_n$ and $\operatorname{tp}_{n,k_n-1}(\mathring{x}) = \tau$. Set a_n to be the function with domain $\operatorname{dom} a^p_n$ and $\operatorname{ran} a_n = x$ satisfying for each $\alpha, \beta \in \operatorname{dom} a^p_n$, if $\alpha < \beta$ then $\mathring{a}_n(\alpha) < \mathring{a}_n(\beta)$.

Set $p^* = \langle f^p_{\langle \mu_{l^p}, \dots, \mu_{n^*-1} \rangle}, \bar{a}, \bar{A} \rangle$. Then $p^* \leq p$ and

$$p^* \leq^* q_{\langle \nu_{l^q}, \dots, \nu_{l^p-1}, \mu_{l^p} \restriction \operatorname{dom} f^q, \dots, \mu_{n^*-1} \restriction \operatorname{dom} f^q \rangle}.$$

□

The following technical lemma is used in stage I of the proof of 2.8.

Lemma 2.7. *Assume $p \in \mathbb{P}$ is a condition and $q \leq^* p_{\langle \nu_{l^p}, \dots, \nu_{m-1} \rangle}$. Then there is a Prikry extension $p^* \leq^* p$ such that $p^*_{\langle \nu_{l^p}, \dots, \nu_{m-1} \rangle} \leq^* q$ and for each $l^p \leq n < m$, $a^{p^*}_n = a^p_n$ and $A^{p^*}_n = A^p_n$.*

Proof. Set $l = l^p$. Construct $\bar{a}$ and $\bar{A}$ as follows. For each $l \leq n < m$ set $a_n = a^p_n$ and $A_n = A^p_n$. For each $m \leq n < \omega$ set $a_n = a^q_n$ and $A_n = A^q_n$. Set $p^* = \langle f^q, \bar{a}, \bar{A} \rangle$. □

Claim 2.8. *Assume $p \in \mathbb{P}$ is a condition and D is a dense open subset of $\mathbb{P}$. Then there is a Prikry extension $p^* \leq^* p$ and $l^p \leq n < \omega$ such that*

$$\forall \langle \nu_{l^p}, \dots, \nu_{n-1} \rangle \in \bar{A}^{p^*} \ p^*_{\langle \nu_{l^p}, \dots, \nu_{n-1} \rangle} \in D.$$

Proof. Set $l = l^p$. The proof is done in two stages. In stage I we prove that for each $l \leq n < \omega$ there is a Prikry extension $p^* \leq^* p$ such that either (the good case)

$$\forall \langle \nu_l, \dots, \nu_{n-1} \rangle \in \bar{A}^{p^*} \ p^*_{\langle \nu_l, \dots, \nu_{n-1} \rangle} \in D$$

or (the bad case)

$$\forall \langle \nu_l, \dots, \nu_{n-1} \rangle \in \bar{A}^{p^*} \ \forall q \leq^* p^*_{\langle \nu_l, \dots, \nu_{n-1} \rangle} \ q \notin D.$$

In stage II we show that it is not possible to get the bad case for every $l \leq n < \omega$.

Stage I. Fix $l \leq n < \omega$ and let $\prec$ be a well ordering of $\operatorname{Lev}_{n-l}(p)$. We use the notation $\vec{\nu}$, $\vec{\mu}$ to denote elements of $\operatorname{Lev}_{n-l}(p)$. E.g., $\vec{\nu} = \langle \nu_l, \dots, \nu_n \rangle$. We will construct by induction the $\leq^*$-decreasing sequence $\langle p^{\vec{\nu}} \mid \vec{\nu} \in \operatorname{Lev}_{n-l}(p) \rangle$ so as to satisfy that if a condition $q \leq^* p^{\vec{\nu}}_{\langle\vec{\nu}\rangle}$ satisfies $q \in D$, then $p^{\vec{\nu}}_{\langle\vec{\nu}\rangle} \in D$. The induction is carried out as follows.

Assume that $\langle p^{\vec{\mu}} \mid \vec{\mu} \prec \vec{\nu}\rangle$ was constructed. Let p' be a condition such that for each $\vec{\mu} \prec \vec{\nu}$, $p' \leq^* p^{\vec{\mu}}$. If there is a Prikry extension $q \leq^* p'_{\langle\vec{\nu}\rangle}$ such that $q \in D$ then use 2.7 to set $p^{\vec{\nu}} \leq^* p'$ to be a condition such that $p^{\vec{\nu}}_{\langle\vec{\nu}\rangle} \leq^* q$. Otherwise set $p^{\vec{\nu}} = p'$.

At the end of the induction let p^* be a condition such that for each $\vec{\nu} \in \mathrm{Lev}_{n-l}(p^*)$, $p^* \leq^* p^{\vec{\nu}}$. By removing a measure zero set from $\mathrm{Lev}_{n-l}(p^*)$ we get the conclusion.

Stage II. Begin with a condition $p \in \mathbb{P}$ and a dense open subset D. Set $p_l = p$ and by induction construct $p_{n+1} \leq^* p_n$ using stage I. If we get that

$$\forall\langle\nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p_{n+1}}\ p_{n+1\langle\nu_l,\ldots,\nu_{n-1}\rangle} \in D,$$

then we are done by setting $p^* = p_{n+1}$. Otherwise we have that

$$\forall\langle\nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p_{n+1}}\ \forall q \leq^* p_{n+1\langle\nu_l,\ldots,\nu_{n-1}\rangle}\ q \notin D.$$

and the induction continues.

We claim that at some $n < \omega$ the induction had to stop, which proves the claim. Towards a contradiction assume that the induction did not stop. Thus we have a $\leq^*$-decreasing sequence of conditions $\langle p_n \mid l \leq n < \omega\rangle$ such that for each $l \leq n < \omega$,

$$\forall\langle\nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p_{n+1}}\ \forall q \leq^* p_{n+1\langle\nu_l,\ldots,\nu_{n-1}\rangle}\ q \notin D.$$

Let p^* be a condition such that for each $l \leq n < \omega$, $p^* \leq^* p_n$. Let $q \in D$ be a condition such that $q \leq p^*$. By 2.6 there is $q^* \leq q$ such that $q^* \leq^* p^*_{\langle\nu_l,\ldots,\nu_{n-1}\rangle}$, where $\langle\nu_l, \ldots, \nu_{n-1}\rangle \in \bar{A}^{p^*}$. Since D is open and $q \in D$, $q^* \in D$. By the construction of p^*, $q^* \leq^* p_{n+1\langle\nu_l,\ldots,\nu_{n-1}\rangle}$. By the construction of p_{n+1} this means that $q^* \notin D$. Contradiction. □

The triple $\langle\mathbb{P}, \leq, \leq^*\rangle$ is said to be of Prikry type if for each condition $p \in \mathbb{P}$ and formula σ in the $\mathbb{P}$-forcing language there is a Prikry extension $p^* \leq^* p$ deciding σ. The Prikry property is immediately derived from the previous lemma:

Corollary 2.9. *The forcing $\langle\mathbb{P}, \leq, \leq^*\rangle$ is of Prikry type.*

As usual in this family of forcing notions, a special care should be given to κ^+.

Claim 2.10. *The cardinal κ^+ is preserved in a $\mathbb{P}$-generic extension.*

Proof. Since κ is of cofinality ω we need to show that every sequence in κ^+ of length less than κ is bounded. Thus assume $\lambda < \kappa$ and $p \Vdash$ "$\dot{f} : \check{\lambda} \to (\kappa^+)_V$". Set $l = l^p$. We can assume that $\kappa_l > \lambda$. We will exhibit a condition $p^* \leq^* p$ forcing that $\dot{f}$ is bounded in κ^+. For each $\zeta < \lambda$ set $D_\zeta = \{q \leq p \mid \exists\xi < \kappa^+\ q \Vdash \text{"}\dot{f}(\check{\zeta}) = \check{\xi}\text{"}\}$. Using claim 2.8 and claim 2.2 construct by induction a $\leq^*$-sequence $\langle p_\zeta \mid \zeta \leq \lambda\rangle$ and a sequence $\langle m_\zeta < \omega \mid \zeta < \lambda\rangle$ satisfying $p_0 \leq^* p$ and for each $\zeta < \lambda$,

$$\forall\langle\nu_l, \ldots, \nu_{m_\zeta}\rangle \in \bar{A}^{p_\zeta}\ \exists\xi < \kappa^+\ p_{\zeta\langle\nu_l,\ldots,\nu_{m_\zeta}\rangle} \Vdash \text{"}\dot{f}(\check{\zeta}) = \check{\xi}\text{"}.$$

Thus for each $\zeta < \lambda$,

$$\forall\langle\nu_l, \ldots, \nu_{m_\zeta}\rangle \in \bar{A}^{p_\lambda}\ \exists\xi < \kappa^+\ p_{\lambda\langle\nu_l,\ldots,\nu_{m_\zeta}\rangle} \Vdash \text{"}\dot{f}(\check{\zeta}) = \check{\xi}\text{"}.$$

For each $\zeta < \lambda$ define the function $F_\zeta : \mathrm{Lev}_{m_\zeta - l}(p_\lambda) \to \kappa^+$ so that

$$\forall\langle\nu_l, \ldots, \nu_{m_\zeta}\rangle \in \bar{A}^{p_\lambda}\ p_{\lambda\langle\nu_l,\ldots,\nu_{m_\zeta}\rangle} \Vdash \text{"}\dot{f}(\check{\zeta}) = \check{F}_\zeta(\nu_l, \ldots, \nu_{m_\zeta})\text{"}.$$

Set $\mu = \sup\{F_\zeta(\nu_l, \dots, \nu_{m_\zeta}) \mid \zeta < \lambda,\ \langle \nu_l, \dots, \nu_{m_\zeta} \rangle \in \mathrm{Lev}_{m_\zeta - l}(p_\lambda)\}$. By its definition, $p_\lambda \Vdash$ "$\mathrm{ran}\, \dot{f} \subseteq \check{\mu}$". Since the sup in the definition of μ is being taken over a set of size smaller than κ^+ and κ^+ is regular, we get $\mu < \kappa^+$. □

Recall that a family of sets $\mathscr{D}$ is called a Δ-system if there is a set d such that for each two different sets $D_1, D_2 \in \mathscr{D}$ in the family, $D_1 \cap D_2 = d$. The set d is called the kernel of the Δ-system $\mathscr{D}$. Existence of Δ-systems is the basis for proving the chain-condition property in many forcing notions. The current forcing notion also needs a Δ-system argument in order to show it satisfies the κ^{++}-cc. However a property somewhat stronger than just $D_1 \cap D_2 = d$ is used. Thus a family $\mathscr{D}$ of sets of ordinals is said to be a strong Δ-system if it is a Δ-system, and in addition, for each two different sets $D_1, D_2 \in \mathscr{D}$, there are two ordinals $\alpha_1 \in D_1$ and $\alpha_2 \in D_2$ such that $D_1 \cap D_2 = D_1 \cap \alpha_1 = D_2 \cap \alpha_2$. Now let $\{d_\xi \in \mathscr{P}_{\kappa^+}(\kappa^{++}) \mid \xi < \kappa^{++}\}$ be a family of subsets of κ^{++}. By following the proof of the Δ-lemma we can see that the above family contains a strong Δ-system of size κ^{++}. It is worth mentioning that, for example, the family $\{d_\xi \in \mathscr{P}_{\kappa^+}(\kappa^{+3}) \mid \xi < \kappa^{++}\}$ contains a Δ-system but not necessarily a strong Δ-system, and this is the reason the higher gap forcing notions get complicated: A rather complicated structure having the strong Δ-system property is being used there.

Claim 2.11. *The forcing $\mathbb{P}$ satisfies the κ^{++}-cc.*

Proof. Let $X = \{p_\xi \mid \xi < \kappa^{++}\} \subseteq \mathbb{P}$ be a set of κ^{++} conditions. For each $\xi < \kappa^{++}$ and $l^{p_\xi} \leq n < \omega$ set $S^\xi_n = \{\mathrm{ran}\, \nu \mid \nu \in A^{p_\xi}_n\}$. For each $\xi < \kappa^{++}$, $l^{p_\xi} \leq n < \omega$, and $\mu \in S^\xi_n$ set $\mu^\xi_n \in A^{p_\xi}_n$ to be such that $\mu = \mathrm{ran}\, \mu^\xi_n$.

Shrink X several times in succession, calling the shrunk set still X, so as to satisfy the following:

(1) The set $\{\mathrm{dom}\, f^{p_\xi} \mid \xi < \kappa^{++}\}$ is a strong Δ-system with kernel d. For each $\xi < \kappa^{++}$ set $d_\xi \in [\mathrm{dom}\, f^{p_\xi}]^{\leq\kappa}$ to be a set disjoint from d satisfying $\mathrm{dom}\, f^{p_\xi} = d \cup d_\xi$.

For each $\xi_0 < \xi_1 < \kappa^{++}$:

(2) $\sup d < \min d_{\xi_0} < \sup d_{\xi_0} < \min d_{\xi_1}$.
(3) $l^{p_{\xi_0}} = l^{p_{\xi_1}}$. Let l be the common value of the l^{p_ξ}'s.
(4) For each $l \leq n < \omega$, $S^{\xi_0}_n = S^{\xi_1}_n$. Let S_n be the common value of the S^ξ_n's.
(5) For each $l \leq n < \omega$, $\mathrm{ran}\, a^{\xi_0}_n = \mathrm{ran}\, a^{\xi_1}_n$.
(6) For each $\alpha \in d$, $f^{p_{\xi_0}}(\alpha) = f^{p_{\xi_1}}(\alpha)$. (Thus $f^{p_{\xi_0}}$ and $f^{p_{\xi_1}}$ are compatible in the forcing $\mathbb{P}^*$).

Pick some $\xi_0 < \xi_1 < \kappa^{++}$. We claim that p_{ξ_0} and p_{ξ_1} are compatible. We will show this by constructing the stronger conditions $q_i \leq p_{\xi_i}$ $(i < 2)$, and then constructing a condition $r \leq q_0, q_1$.

For each $i < 2$ construct the conditions $q_i \leq p_{\xi_i}$ as follows. Set $\alpha_i = \min d_{\xi_i}$. Choose $l \leq n^* < \omega$ such that for each $i < 2$, $\alpha_i \in \mathrm{dom}\, a^{p_{\xi_i}}_{n^*}$ and $a^{p_{\xi_i}}_{n^*}(\alpha_i) \prec \mathscr{H}_{n^*,k+2}$ for some $k \leq n^* - 2$. Now choose for each $l \leq n < n^*$, $\mu^{\xi_i}_n$ such that $\mathrm{ran}\, \mu^{\xi_0}_n = \mathrm{ran}\, \mu^{\xi_1}_n$. Set $q_i = p_{\xi_i \langle \mu^{\xi_i}_l, \dots, \mu^{\xi_i}_{n^*-1} \rangle}$.

Set $f = f^{q_1} \cup f^{q_2}$. We will construct by induction the sequences $\bar{a}$ and $\bar{A}$ so that $\langle f, \bar{a}, \bar{A} \rangle \in \mathbb{P}$ is a condition as follows.

By induction assume that for $n^* \leq n < \omega$, a_{n-1} and A_{n-1} have been constructed. Let $\tau = \mathrm{tp}_{n,k+1}(\mathrm{ran}\, \mathring{a}^{q_0}_n) = \mathrm{tp}_{n,k+1}(\mathrm{ran}\, \mathring{a}^{q_1}_n)$ and $x = \mathrm{ran}(a^{q_0}_n \restriction d) = \mathrm{ran}(a^{q_1}_n \restriction d)$. The set $\{N \cap \mathscr{H}_{n,k+1} \mid N \in \mathrm{ran}\, a^{q_0}_n\}$ witnesses that $\mathscr{H}_{n,k+2} \vDash$ "$\exists y \subset \mathscr{H}_{n,k+1}\ \mathring{y} \supseteq$

$\mathring{x}$, $\mathrm{tp}_{n,k+1}(\mathring{y}) = \tau$". Observe that $\mathring{x} \in a^{q_1}_n(\alpha_1)$. Thus by elementarity $a^{q_1}_n(\alpha_1) \models$ "$\exists y \subset \mathscr{H}_{n,k+1}\ \mathring{y} \supseteq \mathring{x}$, $\mathrm{tp}_{n,k+1}(\mathring{y}) = \tau$". Thus we can choose $y \in a_1(\alpha_1)$ such that $\mathring{y} \supseteq \mathring{x}$ and $\mathrm{tp}_{n,k+1}(\mathring{y}) = \tau$. Let a_n be the function from $\mathrm{dom}\, a^{q_0}_n \cup \mathrm{dom}\, a^{q_1}_n$ to $y \cup \mathrm{ran}(a^{q_1}_n)$ satisfying $a_n \restriction \mathrm{dom}\, a^{q_1}_n = a^{q_1}_n$, $\mathrm{ran}\, a_n \restriction (\mathrm{dom}\, a^{q_0}_n \setminus d) = y$, and for each $\alpha, \beta \in (\mathrm{dom}\, a^{q_0}_n) \setminus d$, if $\alpha < \beta$ then $\mathring{a}_n(\alpha) < \mathring{a}_n(\beta)$. Choose a set $A_n \in E_n(a_n)$ such that for each $i < 2$, $A_n \restriction \mathrm{dom}\, a^{q_i}_n \subseteq A^{q_i}_n$.

When the construction of $\bar{a}$ and $\bar{A}$ is completed set $r = \langle f, \bar{a}, \bar{A}\rangle$. It is not hard to verify that $r \leq q_0, q_1$. □

Combining the above claims we get:

Theorem (M. Gitik [1]). *Assume $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of cardinals such that for each $n < \omega$ there is a $\langle \kappa_n, \kappa_n^{+n+2}\rangle$ extender. Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Then there is a cardinal preserving generic extension adding no new bounded subsets to κ such that $2^\kappa = \kappa^{++}$.*

Proof.
- The Prikry property 2.9 of $\langle \mathbb{P}, \leq, \leq^*\rangle$, together with the closure 2.2 of $\langle \mathbb{P}, \leq^*\rangle$ yield that V and $V[G]$ have the same bounded subset of κ, and thus that κ is preserved. By the κ^{++}-cc 2.11 all the cardinals above κ^+ are preserved, and by 2.10, the cardinal κ^+ is also preserved. Thus all cardinals are preserved.
- On the one hand, the κ^{++}-cc together with $|\mathbb{P}| = \kappa^{++}$ imply $2^\kappa \leq \kappa^{++}$ in the generic extension. On the other hand, 2.5 gives $2^\kappa \geq \kappa^{++}$.

□

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